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SYMMETRY IN THE SEQUENCE OF APPROXIMATION COEFFICIENTS

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ABSTRACT. Let $\{a_n\}_1^\infty$ and $\{\theta_n\}_0^\infty$ be the sequences of partial quotients and approximation coefficients for the continued fraction expansion of an irrational number. We will provide a function f such that $a_{n+1} = f(\theta_{n\pm 1}, \theta_n)$. In tandem with a formula due to Dajani and Kraaikamp, we will write $\theta_{n\pm 1}$ as a function of $(\theta_{n\mp 1}, \theta_n)$, revealing an elegant symmetry in this classical sequence and allowing for its recovery from a pair of consecutive terms.

1. Introduction

Given an irrational number r and a rational number written as the unique quotient $\frac{p}{q}$ of the two integers p and q with $\gcd(p,q) = 1$ and $q > 0$, our fundamental object of interest from diophantine approximation is the **approximation coefficient** $\theta(r, \frac{p}{q}) := q^2 \Big| r - \frac{p}{q} \Big|$. Small approximation coefficients suggest high quality approximations, combining accuracy with simplicity. For instance, the error in approximating π using the fraction $\frac{355}{113} = 3.14159203539823008849557522124$ is smaller than the error of its decimal expansion to the fifth digit, $3.14159 = \frac{314159}{100000}$. Since the former rational also has a much smaller denominator, it is of far greater quality than the latter. Indeed $\theta\big(\pi, \frac{355}{113}\big) < 0.0341$, whereas $\theta\big(\pi, \frac{314159}{100000}\big) > 26535$.

We obtain the high quality approximations for r by using the euclidean algorithm to write r as an infinite continued fraction:

$$r = a_0 + [a_1, a_2, ...] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} ,$$

where the **partial quotients** $a_0 = a_0(r) \in \mathbb{Z}$ and $a_n = a_n(r) \in \mathbb{N} := \mathbb{Z} \cap [1, \infty)$ for all $n \geq 1$ are uniquely determined by r. This expansion also provides us with the infinite sequence of rational numbers

$$\frac{p_0}{q_0} := \frac{a_0}{1}, \quad \frac{p_n}{q_n} := a_0 + [a_1, ..., a_n], \quad n \geq 1,$$

tending to r known as the **convergents** of r. Define the approximation coefficient of the n$^{\text{th}}$ convergent of r by

$$\theta_n := \theta\left(r, \frac{p_n}{q_n}\right) = q_n^2 \Big| r - \tfrac{p_n}{q_n} \Big|$$

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and refer to the sequence $\{\theta_n\}_0^\infty$ as the **sequence of approximation coefficients**. Since adding an integer to a fraction does not change its denominator, the number $x_0 := r - a_0$ shares the same sequences $\{a_n\}_1^\infty$ and $\{\theta_n\}_0^\infty$ as r, allowing us to restrict our attention solely to the unit interval. Throughout this paper, we fix an initial seed $x_0 \in (0,1) - \mathbb{Q}$ and let $\{a_n\}_1^\infty$ and $\{\theta_n\}_0^\infty$ be its sequences of partial quotients and approximation coefficients. While the rest of this section is not a prerequisite, the following results illustrate some of the key properties for this classical sequence and are given for motivation as well as for sake of completeness.

For all $n \geq 0$, it is well known [2, Theorem 4.6] that

$$\left|x_0 - \tfrac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

We conclude that $\theta_n < 1$ for all $n \geq 0$. Conversely, Legendre [2, Theorem 5.12] proved that if $\theta(x_0, \frac{p}{q}) < \frac{1}{2}$, then $\frac{p}{q}$ is a convergent of x_0. In 1891, Hurwitz proved that there exist infinitely many pairs of integers p and q, such that $\theta(x_0, \frac{p}{q}) < \frac{1}{\sqrt{5}} \approx 0.4472$ and that this constant, known as the **Hurwitz Constant**, is sharp. Therefore, all irrational numbers possess infinitely many high quality approximations using rational numbers whose associated approximation coefficients are less than $\frac{1}{\sqrt{5}}$. Using Legendre's result, we see that all these high quality approximations must belong to the sequence of continued fraction convergents for x_0.

We may restate Hurwitz's theorem as the sharp inequality $\liminf\limits_{n\to\infty}\{\theta_n\} \leq \frac{1}{\sqrt{5}}$. In general, we use the value of $\liminf\limits_{n\to\infty}\{\theta_n\}$ to measure how well x_0 can be approximated by rational numbers. The set of values taken by $\liminf\limits_{n\to\infty}\big\{\theta_n(x_0)\big\}$, as x_0 varies in the set of all irrational numbers in the interval, is called the **Lagrange Spectrum**, and those irrational numbers x_0 which construe the spectrum, that is, for which $\liminf\limits_{n\to\infty}\big\{\theta_n(x_0)\big\} > 0$, are called **badly approximable numbers**. It is known [2, Theorem 7.3] that x_0 is badly approximable if and only if its sequence of partial quotients $\{a_n\}_1^\infty$ is bounded. For more details about the Lagrange Spectrum, refer to [3].

In 1895, Vahlen [4, Corollary 5.1.13] proved that for all $n \geq 1$ we have the sharp inequality

$$\min\{\theta_{n-1}, \theta_n\} < \frac{1}{2}, \tag{1}$$

and in 1903, Borel [4, Theorem 5.1.5] proved the sharp inequality

$$\min\{\theta_{n-1}, \theta_n, \theta_{n+1}\} < 5^{-.5}.$$

More recent improvements include the sharp inequalities

$$\min\big\{\theta_{n-1}, \theta_n, \theta_{n+1}\big\} < (a_{n+1}^2 + 4)^{-.5}$$

and

$$\max\big\{\theta_{n-1}, \theta_n, \theta_{n+1}\big\} > (a_{n+1}^2 + 4)^{-.5},$$

due to Bagemihl and McLaughlin [1] and Tong [7]. Therefore, this sequence exhibits a bounding symmetry on a triple of consecutive terms, which stems from its internal connection with the sequence of partial quotient.

For instance, we write

$$\pi - 3 = \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \dots\,.}}}} = [7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \dots]$$

The first ten convergents $\left\{\frac{p_n}{q_n}\right\}_0^9$ are

$$\left\{\frac{0}{1}, \frac{1}{7}, \frac{15}{106}, \frac{16}{113}, \frac{4687}{33102}, \frac{4703}{33215}, \frac{9390}{66317}, \frac{14093}{99532}, \frac{37576}{265381}, \frac{51669}{364913}\right\},$$

and the best upper bounds for $\{\theta_n\}_0^9$ using a four digit decimal expansion are

$$\{0.1416, 0.0612, 0.9351, 0.0034, 0.6237, 0.3641, 0.5363, 0.2885, 0.6045, 0.2134\}.$$

In particular, the small approximation coefficient $\theta_3 = 0.0034$ helps explain why the rational number $\frac{355}{113} = 3 + \frac{16}{113}$, first discovered by Archimedes (c. 287–212 BC), was a popular approximation for π throughout antiquity.

2. Preliminary results

In 1921, Perron [6] proved that

$$\frac{1}{\theta_{n-1}} = [a_{n+1}, a_{n+2}, \dots] + a_n + [a_{n-1}, a_{n-2}, \dots, a_1], \quad n \geq 1, \tag{2}$$

where we take $[\emptyset] := 0$ when $n = 1$. Thus, as far as the flow of information goes, the entire sequence of partial quotients is needed in order to generate a single member in the sequence of approximation coefficients. In 1978, Jurkat and Peyerimhoff [5] showed that for all irrational numbers and for all $n \geq 1$, the point (θ_{n-1}, θ_n) lies in the interior of the triangle with vertices $(0, 0), (0, 1)$ and $(1, 0)$. As a result, we have

$$\theta_{n-1} + \theta_n < 1, \tag{3}$$

which is an improvement of Vahlen's result (1). In addition, they proved that a_{n+1} can be written as a function of (θ_{n-1}, θ_n) but came up short of providing a simple expression that applies to all cases. Combining this observation with the pair of symmetric identities

$$\theta_{n+1} = \theta_{n-1} + a_{n+1}\sqrt{1 - 4\theta_{n-1}\theta_n} - a_{n+1}^2\theta_n, \quad n \geq 1,$$

and

$$\theta_{n-1} = \theta_{n+1} + a_{n+1}\sqrt{1 - 4\theta_{n+1}\theta_n} - a_{n+1}^2\theta_n, \quad n \geq 1,$$

due to Dajani and Kraaikamp [4, Proposition 5.3.6], allows us to recover the tail of the sequence of approximation coefficients from a pair of consecutive terms.

We abbreviate the last two equations to the single working formula

$$\theta_{n\pm1} = \theta_{n\mp1} + a_{n+1}\sqrt{1 - 4\theta_{n\mp1}\theta_n} - a_{n+1}^2\theta_n, \quad n \geq 1. \tag{4}$$

Our goal, obtained in Theorem 3, is to provide a real valued function f such that $a_{n+1} = f(\theta_{n\pm1}, \theta_n)$. This will enable us, as expressed in Corollary 4, to eliminate a_{n+1} from formula (4) without disrupting its elegant symmetry. This will enable us to recover the *entire* sequence $\{\theta_n\}_0^\infty$ from a pair of consecutive terms.

3. Symbolic dynamics

The continued fraction expansion is a symbolic representation of irrational numbers in the unit interval as an infinite sequence of positive integers. Let $\lfloor\cdot\rfloor$ be the **floor** function, whose value on a real number r is the largest integer smaller than or equal to r. Then we obtain this expansion for the initial seed $x_0 \in (0,1) - \mathbb{Q}$ by using the following infinite iteration process:

(1) Let $n := 1$.
(2) Set the **reminder** of x_0 at time n to be $r_n := \frac{1}{x_{n-1}} \in (1, \infty)$.
(3) Define the **digit** and **future** of x_0 at time n to be the integer part and fractional part of r_n respectively, that is, $a_n := \lfloor r_n \rfloor \in \mathbb{N}$ and $x_n := r_n - a_n \in (0,1) - \mathbb{Q}$. Increase n by one and go to step (2).

Using this iteration scheme, we obtain

$$x_0 = \frac{1}{r_1} = \frac{1}{a_1 + x_1} = \frac{1}{a_1 + \cfrac{1}{r_2}} = \frac{1}{a_1 + \cfrac{1}{a_2 + x_2}} = \frac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{r_3}}} = \ldots ;$$

hence, the quantity a_n is no other than the n^{th} partial quotient of x_0. We relabel a_n as the digit for x_0 at time n in order to emphasize the underlying dynamical structure at hand and write

$$x_0 = [r_1] = [a_1, r_2] = [a_1, a_2, r_3] = \ldots . \tag{5}$$

The quantity $x_n = r_n - a_n$ is the value of x_{n-1} under the **Gauss Map**

$$T : \big((0,1) - \mathbb{Q}\big) \to \big((0,1) - \mathbb{Q}\big), \quad T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor. \tag{6}$$

This map is realized as a left shift operator on the set of infinite sequences of digits, i.e.

$$[a_n, a_{n+1}, r_{n+2}] = x_{n-1} \overset{T}{\mapsto} x_n = [a_{n+1}, r_{n+2}], \quad n \geq 1.$$

We preserve the n digits that the map T^n erases from this symbolic representation of x_0 by defining the **past** of x_0 at time $n \geq 1$ to be

$$y_n := -a_n - [a_{n-1}, a_{n-2}, \ldots, a_1] < -1. \tag{7}$$

The **natural extension map**

$$\mathcal{T}(x,y) := \left(\frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad \frac{1}{y} - \left\lfloor \frac{1}{x} \right\rfloor \right) = \left(T(x), \quad \frac{1}{y} - \left\lfloor \frac{1}{x} \right\rfloor \right)$$

is well defined whenever x is an irrational number and $y < -1$, providing us with the relationship

$$(x_{n+1}, y_{n+1}) = \mathcal{T}(x_n, y_n), \quad n \geq 1.$$

Since x_n is uniquely determined by $\{a_n\}_{n+1}^{\infty}$ and y_n is uniquely determined by $\{a_n\}_1^n$, this map can be thought of as one tick of the clock in the symbolic representation of x_0 using the sequence $\{a_n\}_1^{\infty}$:

$$[[a_1, a_2, \ldots, a_n | a_{n+1}, a_{n+2}, \ldots]] \overset{\mathcal{T}}{\mapsto} [[a_1, a_2, \ldots, a_n, a_{n+1} | a_{n+2}, \ldots]],$$

advancing the present time denoted by $|$ one step into the future.

4. Dynamic pairs vs. Jager pairs

Using the dynamical terminology of the last section, we restate Perron's result (2) as

$$\theta_{n-1} = \frac{1}{x_n - y_n}, \qquad n \geq 1. \tag{8}$$

Define the region $\Omega := (0,1) \times (-\infty, -1) \subset \mathbb{R}^2$ and the map

$$\Psi : \Omega \to \mathbb{R}^2, \quad \Psi(x,y) := \left(\frac{1}{x-y}, -\frac{xy}{x-y} \right), \tag{9}$$

which is clearly well-defined and continuous. Since for all $n \geq 1$, we have $(x_n, y_n) \in \Omega$, we use formulas (6) and (8) to obtain

$$\frac{1}{\theta_n} = x_{n+1} - y_{n+1} = \left(\frac{1}{x_n} - a_{n+1} \right) - \left(\frac{1}{y_n} - a_{n+1} \right) = -\frac{x_n - y_n}{x_n y_n}, \qquad n \geq 0,$$

so that

$$\Psi(x_n, y_n) = (\theta_{n-1}, \theta_n). \tag{10}$$

We call (θ_{n-1}, θ_n) the **Jagger Pair** of x_0 at time n. We also denote the image $\Psi(\Omega)$ by Γ. Then

Proposition 1. *The set Γ is the open region interior to the triangle in $\mathbb{R}^2$ with vertices $(0,0), (1,0)$ and $(1,0)$.*

Proof. For every positive integer $k \geq 2$, define the open region $\Omega_k := (\frac{1}{k}, 1) \times (-k, -1)$, whose boundary contains the open line segments $(\frac{1}{k}, 1) \times \{-1\}, \{1\} \times (-k, -1), \left(\frac{1}{k}, 1\right) \times \{-k\}$ and $\{\frac{1}{k}\} \times (-k, -1)$. Since Ψ is continuous, $\Gamma_k := \Psi(\Omega_k)$ is the open region interior to the image of the boundary for Ω_k under Ψ, which we will now find explicitly.

From definition (9) of Ψ, we have

$$x = \frac{1}{u} + y \tag{11}$$

and

$$v = -\frac{xy}{x-y} = -uxy. \tag{12}$$

Set $y := -1$ and $x \in \left(\frac{1}{k}, 1\right)$ so that, by definition (9) of Ψ, we have $u = \frac{1}{x-y} = \frac{1}{x+1} \in \left(\frac{1}{2}, \frac{k}{k+1}\right)$. Formulas (11) and (12) now yield $v = u\left(\frac{1}{u} - 1\right) = 1 - u \in \left(\frac{1}{2}, \frac{1}{k+1}\right)$. Conclude that Ψ maps the open line segment $(\frac{1}{k}, 1) \times \{-1\}$ in the xy-plane to the open line segment between the points $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{k}{k+1}, \frac{1}{k+1}\right)$ in the uv-plane.

Set $x := 1$ and $y \in (-k, -1)$ so that, by definition (9) of Ψ, we have $u = \frac{1}{1-y} \in \left(\frac{1}{2}, \frac{1}{k+1}\right)$. Formulas (11) and (12) now yield $v = u\left(\frac{1}{u} - 1\right) = 1 - u \in \left(\frac{1}{2}, \frac{k}{k+1}\right)$. Conclude that Ψ maps the open line segment $\{1\} \times (-k, -1)$ in the xy-plane to the open line segment between the points $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{k+1}, \frac{k}{k+1}\right)$ in the uv-plane.

Set $y := -k$ and $x \in \left(\frac{1}{k}, 1\right)$, so that, by definition (9) of Ψ, we have $u = \frac{1}{x-y} = \frac{1}{x+k} \in \left(\frac{1}{k+1}, \frac{k}{k^2+1}\right)$. Formulas (11) and (12) now yield $v = -u\left(\frac{1}{u} - k\right)(-k) = k - k^2u \in \left(\frac{k}{k^2+1}, \frac{k}{k+1}\right)$. Conclude that Ψ maps the open line segment $(0,1) \times \{-k\}$ in the xy-plane to the open line segment between the points $\left(\frac{1}{k+1}, \frac{k}{k+1}\right)$ and $\left(\frac{k}{k^2+1}, \frac{k}{k^2+1}\right)$ in the uv-plane.

Finally, set $x := \frac{1}{k}$ and $y \in (-k, -1)$, so that, by definition (9) of Ψ, we have $u \in \left(\frac{k}{k^2+1}, \frac{k}{k+1}\right)$. Formulas (11) and (12) now yield $v = \frac{u}{k}\left(\frac{1}{u} - \frac{1}{k}\right) = \frac{1}{k} - \frac{u}{k^2} \in \left(\frac{k}{k^2+1}, \frac{k}{k+1}\right)$. Conclude that Ψ maps the open line segment $\left\{\frac{1}{k}\right\} \times (-k, -1)$ in the xy-plane to the open line segment between the points $\left(\frac{k}{k^2+1}, \frac{k}{k^2+1}\right)$ and $\left(\frac{k}{k+1}, \frac{1}{k+1}\right)$ in the uv-plane. From the continuity of Ψ, we have

$$\Gamma = \Psi(\Omega) = \Psi\left(\bigcup_2^\infty \Omega_k\right) = \bigcup_2^\infty \Psi(\Omega_k) = \bigcup_2^\infty \Gamma_k.$$

Therefore, we conclude the desired result after letting $k \to \infty$. $\square$

Note that since $(\theta_{n-1}, \theta_n) = \Psi(x_n, y_n) \in \Gamma$, this observation is in accordance with formula (3).

Lemma 2. *The map $\Psi : \Omega \to \Gamma$ is a homeomorphism with inverse:*

$$\Psi^{-1}(u,v) := \left(\frac{1-\sqrt{1-4uv}}{2u}, -\frac{1+\sqrt{1-4uv}}{2u}\right). \tag{13}$$

Proof. First, we will show that Ψ is a bijection. Since the map Ψ is surjective onto its image Γ, we need only show injectiveness. Let $(x_1, y_1), (x_2, y_2)$ be two points in Ω such that

$$\left(\frac{1}{x_1-y_1}, -\frac{x_1y_1}{(x_1-y_1)}\right) = \Psi(x_1,y_1) = \Psi(x_2,y_2) = \left(\frac{1}{x_2-y_2}, -\frac{x_2y_2}{(x_2-y_2)}\right).$$

By equating the first and then the second components of the exterior terms, we obtain that

$$x_1 - y_1 = x_2 - y_2 \tag{14}$$

and then that $x_1y_1 = x_2y_2$. Therefore,

$$(x_1+y_1)^2 = (x_1-y_1)^2 + 4x_1y_1 = (x_2-y_2)^2 + 4x_2y_2 = (x_2+y_2)^2.$$

Since both these points are in Ω, they must lie below the line $x + y = 0$; hence $x_1 + y_1 = x_2 + y_2 < 0$. Another application of condition (14) now proves that $x_1 = x_2$ and $y_1 = y_2$; hence Ψ is injective.

Since both Ψ and Ψ^{-1} are clearly continuous, it is left to prove that Ψ^{-1} is well-defined and that it is the inverse for Ψ. Given $(u_0, v_0) \in \Gamma$, set

$$(x_0, y_0) := \Psi^{-1}(u_0, v_0) = \left(\frac{1-\sqrt{1-4u_0v_0}}{2u_0}, -\frac{1+\sqrt{1-4u_0v_0}}{2u_0}\right).$$

From Proposition 1, we know that Γ lies entirely underneath the line $u + v = 1$ in the uv plane. The only point of intersection for this line and the hyperbola $4uv = 1$ is the point $(u,v) = \left(\frac{1}{2}, \frac{1}{2}\right)$; hence Γ must lie underneath this hyperbola as well. We conclude that $4u_0v_0 < 1$, so that both x_0 and y_0 must be real. Another implication of the inequality $u+v<1$ is that $4uv<4u-4u^2$; hence $1-4uv > 4u^2-4u+1=(2u-1)^2$. Conclude that $1+\sqrt{1-4uv}>2u$, so we must have $y_0=-\frac{1+\sqrt{1-4u_0v_0}}{2u_0}<-1$.

To prove that $x_0\in(0,1)$, we first observe that $\sqrt{1-4u_0v_0}<1$ implies that $1-\sqrt{1-4u_0v_0}>0$; hence x_0 is positive. If we further assume by contradiction that $x_0\ge 1$, then the definition of Ψ^{-1} (13) will imply the inequality

$$\begin{aligned}1+\sqrt{1-4u_0v_0} &\le x_0\left(1+\sqrt{1-4u_0v_0}\right)\\ &=\frac{1}{2u_0}\left(1-\sqrt{1-4u_0v_0}\right)\left(1+\sqrt{1-4u_0v_0}\right)=2v_0,\end{aligned}$$

so that we obtain the inequality

$$4v_0^2-4v_0+1=(2v_0-1)^2\ge 1-4u_0v_0.$$

After the appropriate cancellations and rearrangements, we obtain the inequality $u_0+v_0\ge 1$, which is in contradiction to Proposition 1. Conclude that $(x_0,y_0)\in\Omega$ and $\Psi^{-1}:\Gamma\to\Omega$ is well-defined.

Finally, we will show that Ψ^{-1} is the inverse for Ψ. Let $(u,v)\in\Gamma$ and set $(x,y):=\Psi^{-1}(u,v)\in\Omega$. Using the definitions (9) and (13) of Ψ and Ψ^{-1}, the first component of $\Psi(x,y)$ is

$$\frac{1}{x-y}=\left(\frac{1-\sqrt{1-4uv}}{2u}+\frac{1+\sqrt{1-4uv}}{2u}\right)^{-1}=\left(\frac{2}{2u}\right)^{-1}=u,$$

and its second component is

$$-\frac{xy}{(x-y)}=-u(xy)=u\left(\frac{1}{4u^2}\left(1-\sqrt{1-4uv}^2\right)\right)=\frac{1}{4u}\cdot 4uv=v;$$

hence Ψ^{-1} is the right inverse for Ψ. Since Ψ is a bijection, we conclude it is the (two-sided) inverse for Ψ, completing the proof. $\square$

5. Result

Theorem 3. *Let x_0 be an irrational number in the unit interval and let $n\in\mathbb{N}$. If a_{n+1} is the digit at time $n+1$ in the continued fraction expansion for x_0 and if $(\theta_{n-1},\theta_n,\theta_{n+1})$ are the approximation coefficients for x_0 at time $n-1,n$ and $n+1$, then*

$$a_{n+1}=\left\lfloor\frac{1+\sqrt{1-4\theta_{n-1}\theta_n}}{2\theta_n}\right\rfloor=\left\lfloor\frac{1+\sqrt{1-4\theta_{n+1}\theta_n}}{2\theta_n}\right\rfloor. \tag{15}$$

Proof. Let (x_n,y_n) be the dynamic pair of x_0 at time n. Formula (10), the fact that Ψ is a homeomorphism and definition (13) of Ψ^{-1} yield

$$(x_n,y_n)=\Psi^{-1}(\theta_{n-1},\theta_n)=\left(\frac{1-\sqrt{1-4\theta_{n-1}\theta_n}}{2\theta_{n-1}},-\frac{1+\sqrt{1-4\theta_{n-1}\theta_n}}{2\theta_{n-1}}\right). \tag{16}$$

Using formula (5), we write $x_n=[a_{n+1},r_{n+2}]=\frac{1}{a_{n+1}+[r_{n+2}]}$, so that the first components in the exterior terms of formula (16) equate to

$$a_{n+1}+[r_{n+2}]=\frac{2\theta_{n-1}}{1-\sqrt{1-4\theta_{n-1}\theta_n}}=\frac{1+\sqrt{1-4\theta_{n-1}\theta_n}}{2\theta_n}.$$

But since $[r_{n+2}] = x_{n+1} < 1$, we have

$$a_{n+1} = \lfloor a_{n+1} + [r_{n+2}] \rfloor = \left\lfloor \frac{1+\sqrt{1-4\theta_{n-1}\theta_n}}{2\theta_n} \right\rfloor,$$

which is the first equality in (15).

Next, we equate the second components in the exterior terms of formula (16), which, after using formula (7), yields

$$a_n + [a_{n-1}, ..., a_1] = \frac{1+\sqrt{1-4\theta_{n-1}\theta_n}}{2\theta_{n-1}}.$$

But since $[a_{n-1}, ..., a_1] < 1$, we conclude that

$$a_n = \lfloor a_n + [a_{n-1}, ..., a_1] \rfloor = \left\lfloor \frac{1+\sqrt{1-4\theta_{n-1}\theta_n}}{2\theta_{n-1}} \right\rfloor.$$

Adding one to all indices establishes the equality of the exterior terms in (15) and completes the proof. □

As a direct consequence of this theorem and formula (4), we obtain:

Corollary 4. *Assuming the hypothesis of the theorem, we have*

$$\theta_{n\pm1} = \theta_{n\mp1} + \left\lfloor \frac{1+\sqrt{1-4\theta_{n\mp1}\theta_n}}{2\theta_n} \right\rfloor \sqrt{1-4\theta_{n\mp1}\theta_n} - \left\lfloor \frac{1+\sqrt{1-4\theta_{n\mp1}\theta_n}}{2\theta_n} \right\rfloor^2 \theta_n.$$

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