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# A NOTE ON A BRUNN-MINKOWSKI INEQUALITY FOR THE GAUSSIAN MEASURE

PIOTR NAYAR AND TOMASZ TKOCZ

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ABSTRACT. We give counterexamples related to a Gaussian Brunn-Minkowski inequality and the (B) conjecture.

## 1. Introduction and notation

Let $\gamma_n$ be the standard Gaussian distribution on $\mathbb{R}^n$, i.e. the measure with the density

$$g_n(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2},$$

where $|\cdot|$ stands for the standard Euclidean norm. A powerful tool in convex geometry is the Brunn-Minkowski inequality for Lebesgue measure (see [Sch] for more information). Concerning the Gaussian measure, the following question has recently been posed.

**Question** (R. Gardner and A. Zvavitch, [GZ])**.** Let $0 < \lambda < 1$ and let $A$ and $B$ be closed convex sets in $\mathbb{R}^n$ such that $o \in A \cap B$. Is it true that

$$\text{(GBM)} \qquad \gamma_n(\lambda A + (1-\lambda)B)^{1/n} \geq \lambda\gamma_n(A)^{1/n} + (1-\lambda)\gamma_n(B)^{1/n}?$$

A counterexample is given in this note. However, we believe that this question has an affirmative answer in the case of $o$-symmetric convex sets, i.e. the sets satisfying $K = -K$.

In [CFM] it is proved that for an $o$-symmetric convex set $K$ in $\mathbb{R}^n$ the function

$$(1.1) \qquad \mathbb{R} \ni t \mapsto \gamma_n(e^t K)$$

is log-concave. This was conjectured by W. Banaszczyk and was popularized by R. Latała [Lat]. It turns out that the (B) conjecture cannot be extended to the class of sets which are not necessarily $o$-symmetric yet contain the origin, as one of the sets provided in our counterexample shows.

As for the notation, we frequently use the function

$$T(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} \mathrm{d}t.$$

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## 2. Counterexamples

Now we construct the convex sets $A, B \subset \mathbb{R}^2$ containing the origin such that inequality (GBM) does not hold. Later on we show that for the set $B$ the (B) conjecture is not true.

Fix $\alpha \in (0, \pi/2)$ and $\varepsilon > 0$. Take

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 \mid y \geq |x| \tan\alpha\}, \\ B = B_\varepsilon &= \{(x,y) \in \mathbb{R}^2 \mid y \geq |x| \tan\alpha - \varepsilon\} = A - (0,\varepsilon). \end{aligned}$$

Clearly, $A, B$ are convex and $0 \in A \cap B$. Moreover, from the convexity of $A$ we have $\lambda A + (1-\lambda)A = A$, and therefore

$$\lambda A + (1-\lambda)B = \lambda A + (1-\lambda)(A - (0,\varepsilon)) = A - (1-\lambda)(0,\varepsilon).$$

Observe that

$$\begin{aligned} \gamma_2(A) &= \frac{1}{2} - \frac{\alpha}{\pi}, \\ \gamma_2(B) &= 2\int_0^{+\infty} T(x\tan\alpha - \varepsilon)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x, \\ \gamma_2(\lambda A + (1-\lambda)B) &= 2\int_0^{+\infty} T(x\tan\alpha - \varepsilon(1-\lambda))\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x \end{aligned}$$

and that these expressions are analytic functions of $\varepsilon$. We will expand these functions in $\varepsilon$ up to the order 2. Let

$$a_k = \int_0^{+\infty} T^{(k)}(x\tan\alpha)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,\mathrm{d}x,$$

for $k = 0,1,2$, where $T^{(k)}$ is the $k$-th derivative of $T$ (we adopt the standard notation $T^{(0)} = T$). We get

$$\begin{aligned} \gamma_2(A) &= 2a_0, \\ \gamma_2(B) &= 2a_0 - 2\varepsilon a_1 + \varepsilon^2 a_2 + o(\varepsilon^2), \\ \gamma_2(\lambda A + (1-\lambda)B) &= 2a_0 - 2\varepsilon(1-\lambda)a_1 + \varepsilon^2(1-\lambda)^2 a_2 + o(\varepsilon^2). \end{aligned}$$

Thus

$$\sqrt{\gamma_2(B)} = \sqrt{2a_0} - \frac{a_1}{\sqrt{2a_0}}\varepsilon + \left(\frac{a_2}{2\sqrt{2a_0}} - \frac{a_1^2}{2(2a_0)^{3/2}}\right)\varepsilon^2 + o(\varepsilon^2).$$

Taking $\varepsilon(1-\lambda)$ instead of $\varepsilon$ we obtain

$$\begin{aligned} \sqrt{\gamma_2(\lambda A + (1-\lambda)B)} &= \sqrt{2a_0} - \frac{a_1}{\sqrt{2a_0}}(1-\lambda)\varepsilon \\ &\quad + \left(\frac{a_2}{2\sqrt{2a_0}} - \frac{a_1^2}{2(2a_0)^{3/2}}\right)(1-\lambda)^2\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

Since

$$\begin{aligned} &\sqrt{\gamma_2(\lambda A + (1-\lambda)B)} - \lambda\sqrt{\gamma_2(A)} - (1-\lambda)\sqrt{\gamma_2(B)} \\ &\quad = -\lambda(1-\lambda)\frac{1}{2(2a_0)^{3/2}}(2a_0a_2 - a_1^2)\varepsilon^2 + o(\varepsilon^2), \end{aligned}$$

we will have a counterexample if we find $\alpha \in (0, \pi/2)$ such that

$$2a_0a_2 - a_1^2 > 0.$$

Recall that $a_0 = \frac{1}{2}\gamma_2(A) = \frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{\pi}\right)$. The integrals that define the $a_k$'s can be calculated. Namely,

$$\begin{aligned} a_1 &= \int_0^\infty T'(x\tan\alpha)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = -\frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{\mathbb{R}} e^{-(1+\tan^2\alpha)x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}} \\ &= -\frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{1+\tan^2\alpha}}, \\ a_2 &= \int_0^\infty T''(x\tan\alpha)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_0^\infty (x\tan\alpha)e^{-(1+\tan^2\alpha)x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}} \\ &= \frac{1}{2\pi}\frac{\tan\alpha}{1+\tan^2\alpha}. \end{aligned}$$

Therefore,

$$\begin{aligned} 2a_0a_2 - a_1^2 &= 2\left(\frac{1}{2}\left(\frac{1}{2} - \frac{\alpha}{\pi}\right)\cdot\frac{1}{2\pi}\frac{\tan\alpha}{1+\tan^2\alpha}\right) - \frac{1}{2\pi}\cdot\frac{1}{4(1+\tan^2\alpha)} \\ &= \frac{1}{8\pi}\frac{1}{1+\tan^2\alpha}\left(\tan\alpha\left(2 - \frac{4\alpha}{\pi}\right) - 1\right), \end{aligned}$$

which is positive for $\alpha$ close to $\pi/2$.

Now we turn our attention to the (B) conjecture. We are going to check that for the set $B = B_\varepsilon$ the function $\mathbb{R} \ni t \mapsto \gamma_n(e^t B)$ is not log-concave, provided that $\varepsilon$ is sufficiently small. Since

$$e^t B = \{(x,y) \in \mathbb{R}^2 \mid y \geq \tan\alpha|x| - \varepsilon e^t\},$$

we get

$$\begin{aligned} \ln\gamma_2(e^t B) &= \ln\left(2\int_0^\infty T(x\tan\alpha - e^t\varepsilon)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x\right) \\ &= \ln\left(2\int_0^\infty T(x\tan\alpha)\frac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x\right) - \varepsilon e^t\frac{\int_0^\infty T'(x\tan\alpha)e^{-x^2/2}\mathrm{d}x}{\int_0^\infty T(x\tan\alpha)e^{-x^2/2}\mathrm{d}x} + o(\varepsilon). \end{aligned}$$

This produces the desired counterexample for sufficiently small $\varepsilon$ as the function $t \mapsto \beta e^t$, where

$$\beta = -\frac{\int_0^\infty T'(x\tan\alpha)e^{-x^2/2}\mathrm{d}x}{\int_0^\infty T(x\tan\alpha)e^{-x^2/2}\mathrm{d}x} > 0,$$

is convex. □

*Remark.* The set $B_\varepsilon$ which serves as a counterexample to the (B) conjecture in the nonsymmetric case works when the parameter $\alpha = 0$ as well (and $\varepsilon$ is sufficiently small). Since $B_\varepsilon$ is simply a halfspace in this case, it shows that the symmetry of $K$ is required for log-concavity of (1.1) even in the one-dimensional case.

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Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland

*E-mail address*: nayar@mimuw.edu.pl

Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland

*E-mail address*: t.tkocz@mimuw.edu.pl