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ERGODIC PROPERTIES OF VIANA-LIKE MAPS WITH SINGULARITIES IN THE BASE DYNAMICS

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Abstract. We consider two examples of Viana maps for which the base dynamics has singularities (discontinuities or critical points) and show the existence of a unique absolutely continuous invariant probability measure and related ergodic properties such as stretched exponential decay of correlations and stretched exponential large deviations.

1. Introduction

If μ is an invariant measure for a map $F : M \to M$, its *basin* is the set of all points $x \in M$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{F^i(x)} \overset{\text{weak}^*}{\longrightarrow} \mu, \qquad \text{as } n \to \infty.$$

Assuming M is endowed with a Riemannian structure and a volume form extended to the Borel sets in M (Lebesgue measure), we say that an invariant probability measure μ is a *Sinai-Ruelle-Bowen (SRB) measure* if its basin has positive Lebesgue measure. It follows from Birkhoff's ergodic theorem that ergodic absolutely continuous invariant probability measures (acip) are necessarily SRB measures. Here, absolute continuity is always considered with respect to Lebesgue measure. We are interested in studying some statistical features of ergodic acip's for certain classes of dynamical systems.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ denote Banach spaces of real valued measurable functions defined on M. We denote the *correlation* of functions $\varphi \in \mathcal{B}_1$ and $\psi \in \mathcal{B}_2$ with respect to a measure μ as

$$\mathrm{Cor}_\mu(\varphi, \psi \circ F^n) := \left| \int \varphi\,(\psi \circ F^n)\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu \right|.$$

We say that we have *decay of correlations*, with respect to the measure μ, for observables in $\mathcal{B}_1$ *against* observables in $\mathcal{B}_2$ if, for every $\varphi \in \mathcal{B}_1$ and every $\psi \in \mathcal{B}_2$,

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we have

$$\mathrm{Cor}_\mu(\varphi, \psi \circ F^n) \to 0, \qquad \text{as } n \to \infty.$$

Given $\varphi \in \mathcal{B}_1$ and $\epsilon > 0$ we define the *large deviation* of φ at time n as

$$\mathrm{LD}_\mu(\varphi, \epsilon, n) := \mu\left(\left|\frac{1}{n}\sum_{i=0}^{n-1}\varphi \circ F^i - \int \varphi d\mu\right| > \epsilon\right).$$

By Birkhoff's ergodic theorem the quantity $\mathrm{LD}_\mu(\varphi, \epsilon, n) \to 0$, as $n \to \infty$, and a relevant question also in this case is the rate of this decay. In our main results we shall consider $\mathcal{B}_1 = \mathcal{H}_\gamma$, the space of *Hölder continuous functions* with Hölder constant $\gamma > 0$. The Hölder norm of an observable $\varphi \in \mathcal{H}_\gamma$ is given by

$$\|\varphi\|_{\mathcal{H}_\gamma} = \|\varphi\|_\infty + \sup_{y_1 \neq y_2} \frac{|\varphi(y_1) - \varphi(y_2)|}{\mathrm{dist}(y_1, y_2)^\gamma}.$$

In such cases we shall take $\mathcal{B}_2 = L^\infty(\mu)$. In part of the proof of Theorem A we shall also consider $\mathcal{B}_1$ as the space of *bounded variation functions* and $\mathcal{B}_2$ as $L^1(\mu)$.

The purpose of this paper is to apply the theories developed in [ALP], [G2], and [AFLV] to two examples of Viana maps studied in [S1] and [S2], and to deduce the existence of a unique acip and estimates for the Decay of Correlations and Large Deviations with respect to that measure. The Central Limit Theorem, the Almost Sure Invariance Principle, the Local Limit Theorem and the Berry-Esseen Theorem will also be deduced for our systems. Due to the technical nature of some of these concepts we introduce them in Appendix A. For another recent proof of the existence of an acip for the systems studied in the present paper, see [AS].

We shall consider skew-product maps similar to Viana maps: in one case with β-transformations as the base dynamics in the circle S^1, and in another case with a quadratic map as the base dynamics in an interval I. In the following let $Q_a(x) = a - x^2$, $x \in \mathbb{R}$, be a Misiurewicz-Thurston quadratic map; i.e., the parameter $a \in (0, 2)$ is chosen such that the critical point of Q_a is pre-periodic (but not periodic). The full quadratic map Q_2 is excluded since we look at perturbations of the parameter a. Furthermore, we assume that Q_a is non-renormalizable.

1.1. **β-transformations in the base dynamics.** In [S1], the map under consideration is of the form $F_1 : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$:

$$F_1(\theta, x) = (\beta\theta \bmod 1, Q_a(x) + \alpha \sin(2\pi\theta)),$$

where β is a real number greater than or equal to some lower bound $\beta_a < 2$ (depending on the parameter a of the quadratic map Q_a). This map is similar to the maps studied in [V] and [BST] but allowing a discontinuity in the base dynamics. If $p < 0$ denotes the negative fixed point of Q_a, it is easy to check that there is an open interval $J \supset [p, -p]$ such that $F_1(S^1 \times J) \subset S^1 \times J$ whenever $\alpha > 0$ is sufficiently small.

For sufficiently small $\alpha > 0$, it is shown in [S1] that the map F_1 is non-uniformly expanding, and furthermore that F_1 admits a unique acip for *almost all* $\beta \geq \beta_a$. In this paper we improve this result by showing that a unique acip exists, in fact, for *all* $\beta \geq \beta_a$. In addition, we obtain several statistical properties for this acip, such as stretched exponential decay of correlations and stretched exponential large deviations.

Theorem A. *For all small enough $\alpha > 0$ and all $\beta \geq \beta_a$, the map $F_1 : S^1 \times J \to S^1 \times J$ admits a unique acip μ whose basin has full Lebesgue measure in $S^1 \times J$. Moreover,*

(1) *there exist $C, \tau > 0$ such that $Cor_\mu(\varphi, \psi \circ F_1^n) \leq Ce^{-\tau n^{1/3}}$ for all $\varphi \in \mathcal{H}_\gamma$ and all $\psi \in L^\infty(\mu)$ with $\|\varphi\|_{\mathcal{H}_\gamma}, \|\psi\|_{L^\infty(\mu)} \leq 1$;*
(2) *for all $\epsilon > 0$ and all $\varphi \in \mathcal{H}_\gamma$ there exist $\tau' = \tau'(\tau, \varphi, \epsilon) > 0$ and $C' = C'(\varphi, \epsilon) > 0$ such that $LD_\mu(\varphi, \epsilon, n) \leq C'e^{-\tau' n^{1/7}}$;*
(3) *the Central Limit Theorem, the vector-valued Almost Sure Invariance Principle, the Local Limit Theorem and the Berry-Esseen Theorem hold for certain Hölder observables.*

1.2. **Quadratic maps in the base dynamics.** Let $Q_b(\theta) = b - \theta^2$, $\theta \in \mathbb{R}$ and $b \in (0, 2]$, be another Misiurewicz-Thurston map and set $I = [Q_b^2(0), Q_b(0)]$. The map studied in [S2] is of the form $F_2 : I \times \mathbb{R} \to I \times \mathbb{R}$:

$$F_2(\theta, x) = (Q_b^k(\theta), Q_a(x) + \alpha s(\theta)),$$

where $k \geq 1$ is an integer and $s : I \to [-1, 1]$ is a coupling function which is a priori not fixed. Again, if $p < 0$ denotes the negative fixed point of Q_a, it is easy to check that there is an open interval $J \supset [p, -p]$ such that $F_2(I \times J) \subset I \times J$ whenever $\alpha > 0$ is sufficiently small.

In [S2] it is shown that there is an integer $k_0 \geq 1$ and a family of (non-constant) coupling functions s which are C^2 outside a finite number of singularities such that for each such coupling function s, all $k \geq k_0$, and all sufficiently small α the map F_2 is non-uniformly expanding. In fact, the only singularities for s are square root singularities. Without loss of generality we assume that the map Q_b is non-renormalizable, from which it follows that Q_b^k has a unique acip for all $k \geq 1$. (Otherwise we can restrict the map F_2 to a smaller region $\tilde{I} \times \mathbb{R}$ such that $Q_b^k : \tilde{I} \to \tilde{I}$ and Q_b^k admits a unique acip.) In this paper we will show furthermore that F_2 admits a unique acip with the same statistical properties as for the map F_1.

Theorem B. *For small enough $\alpha > 0$, the map $F_2 : I \times J \to I \times J$ admits a unique acip μ whose basin has full Lebesgue measure in $I \times J$. Moreover, there exists $\tau > 0$ such that for all $0 < \zeta < 1/9$:*

(1) *there exists $C > 0$ such that $Cor_\mu(\varphi, \psi \circ F_2^n) \leq Ce^{-\tau n^\zeta}$ for all $\varphi \in \mathcal{H}_\gamma$ and all $\psi \in L^\infty(\mu)$ with $\|\varphi\|_{\mathcal{H}_\gamma}, \|\psi\|_{L^\infty(\mu)} \leq 1$;*
(2) *for all $\epsilon > 0$ and all $\varphi \in \mathcal{H}_\gamma$ there are $\tau' = \tau'(\tau, \varphi, \epsilon) > 0$ and $C' = C'(\varphi, \epsilon) > 0$ such that $LD_\mu(\varphi, \epsilon, n) \leq C'e^{-\tau' n^{\zeta'}}$, where $\zeta' = \zeta/(\zeta + 2)$;*
(3) *the Central Limit Theorem, the vector-valued Almost Sure Invariance Principle, the Local Limit Theorem and the Berry-Esseen Theorem hold for certain Hölder observables.*

1.3. **Strategy.** To prove the first two items of Theorem A and Theorem B we will apply the result in [G2] which shows the existence of a *Young tower* or *Gibbs-Markov structure* for the maps F_1 and F_2, with stretched exponential tail estimates for the expansion and slow recurrence tails. These objects will be defined precisely in Section 2, and in Section 3 we obtain stretched exponential bounds on these tails. From the existence of such a tower the decay of correlations conclusions as stated in the two theorems then follows. The conclusions on the large deviations are an immediate consequence of [AFLV, Theorem D(2)]. Finally, in Section 4 we obtain the topological transitivity of the maps, which assures the uniqueness of the acip in both cases.

The third items of Theorem A and Theorem B follow from the existence of a Gibbs-Markov structure (where it is sufficient to have polynomial tail estimates; see Appendix A).

2. Non-uniform expansion and slow recurrence

Let M be equal to $M_1 = S^1 \times J$ or $M_2 = I \times J$ and $F : M \to M$ be equal to F_1 or F_2, respectively. Let $\mathcal{S}$ be some closed set of zero Lebesgue measure of singularities/criticalities such that $F : M \setminus \mathcal{S} \to M$ is a C^2 local diffeomorphism. We say that F is *non-degenerate* close to $\mathcal{S}$ if there are constants $B > 1$ and $\xi > 0$ such that the following three conditions hold. For all $y \in M \setminus \mathcal{S}$ and $v \in T_yM \setminus \{0\}$, we have

(S1) $\dfrac{1}{B}\operatorname{dist}(y, \mathcal{S})^{\xi} \le \dfrac{\|DF(y)v\|}{\|v\|} \le B \operatorname{dist}(y, \mathcal{S})^{-\xi}$;

and for every $y_1, y_2 \in M \setminus \mathcal{S}$ with $\operatorname{dist}(y_1, y_2) < \operatorname{dist}(y_1, \mathcal{S})/2$, we have

(S2) $\big|\log \|DF(y_1)^{-1}\| - \log \|DF(y_2)^{-1}\|\,\big| \le B \dfrac{\operatorname{dist}(y_1, y_2)}{\operatorname{dist}(y_1, \mathcal{S})^{\xi}}$,

(S3) $\big|\log |\det DF(y_1)^{-1}| - \log |\det DF(y_2)^{-1}|\,\big| \le B \dfrac{\operatorname{dist}(y_1, y_2)}{\operatorname{dist}(y_1, \mathcal{S})^{\xi}}$.

The critical or singular set $\mathcal{S}$ for the map F_1 is the set

$$\big\{\{0\} \times J\big\} \cup \big\{S^1 \times \{0\}\big\},$$

and the singular set $\mathcal{S}$ for the map F_2 is the set

$$\Big\{ \bigcup_{1 \le i \le m} \{b_i\} \times J \Big\} \cup \big\{I \times \{0\}\big\},$$

where the points $\{b_1, ..., b_m\}$ consist of the critical points of Q_b^k and the points where the coupling function s is not C^2. It is straightforward to check that the map $F_1 : M_1 \to M_1$ satisfies the non-degeneracy conditions (S1)–(S3), where the constant ξ can be chosen equal to 1. Regarding the map $F_2 : M_2 \to M_2$, since the coupling function s has only square root singularities, one easily checks that the non-degeneracy conditions (S1)–(S3) hold for F_2 with $\xi = 2$.

The main result in [S1] and [S2], respectively, shows that the map F ($F = F_1$ or F_2) is *non-uniformly expanding*; i.e., there is some constant $c > 0$ such that for Lebesgue almost every $y \in M$,

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|DF(F^j(y))^{-1}\|^{-1} \ge c > 0. \tag{1}$$

This implies that the *expansion time* function

$$\mathcal{E}(y) = \min \left\{ N \ge 1 \ : \ \frac{1}{n} \sum_{j=0}^{n-1} \log \|DF(F^j(y))^{-1}\|^{-1} \ge c/2, \text{ for all } n \ge N \right\}$$

is defined and finite for Lebesgue almost every $y \in M$. Given $\delta > 0$ we define the δ-*truncated distance* from $y \in M$ to $\mathcal{S}$ as $\operatorname{dist}_\delta(y, \mathcal{S}) = \operatorname{dist}(y, \mathcal{S})$ if $\operatorname{dist}(y, \mathcal{S}) \le \delta$

and $\operatorname{dist}_\delta(y, \mathcal{S}) = 1$ otherwise. In the next section we will see that F has *slow recurrence to the critical set* $\mathcal{S}$; i.e., given any $\epsilon > 0$ there is $\delta > 0$ such that

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(F^j(y), \mathcal{S}) \leq \epsilon \tag{2}$$

for Lebesgue almost every $y \in M$. It follows that the *recurrence time* function

$$\mathcal{R}_{\epsilon,\delta}(y) = \min \left\{ N \geq 1 \;:\; \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(F^j(y), \mathcal{S}) \leq 2\epsilon, \text{ for all } n \geq N \right\} \tag{3}$$

is defined and finite for a.e. $y \in M$.

According to the results in [G2], in order to prove Theorem A and Theorem B, it is left to show that all the iterates of the map F are topologically transitive on the attractor $\Lambda = \bigcap_{n \geq 0} F^n(M)$ and that there exist constants $\tau, \zeta > 0$ such that for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|\{y \in M \;:\; \mathcal{E}(y) > n \text{ or } \mathcal{R}_{\epsilon,\delta}(y) > n\}| \leq \mathcal{O}(e^{-\tau n^\zeta}), \tag{4}$$

where $|\,.\,|$ stands for Lebesgue measure. By the technique in [G2], if (4) is satisfied for some constants $\tau, \zeta > 0$, then the same constants also appear in the decay of correlations.

3. Stretched exponential bounds

The main part of the proof of the theorems is to establish the stretched exponential bound in (4). We divide the singular sets of F_1 and F_2 into two parts. One part will contain the singularities for which it is enough to study the base dynamics, and the other part will contain the critical points due to the quadratic map Q_a. More precisely, when considering F_1 let $\mathcal{S}_h = \{0\} \times J$ and $\mathcal{S}_v = S^1 \times \{0\}$ (the indices h and v stand for *horizontal* and *vertical*, respectively). When considering F_2 let $\mathcal{S}_h = \bigcup_{1 \leq i \leq m} \{b_i\} \times J$ and $\mathcal{S}_v = I \times \{0\}$.

Let $\mathcal{R}_{\epsilon,\delta,h}$ and $\mathcal{R}_{\epsilon,\delta,v}$ be defined in the same way as the set $\mathcal{R}_{\epsilon,\delta}$ (see (3)), but with $\mathcal{S}$ in its definition replaced by $\mathcal{S}_h$ and $\mathcal{S}_v$, respectively. Obviously, we have

$$\begin{aligned} &\{y \in M \;:\; \mathcal{E}(y) > n \text{ or } \mathcal{R}_{\epsilon,\delta}(y) > n\} \\ &\qquad \subset \{y \in M \;:\; \mathcal{R}_{\epsilon,\delta,h}(y) > n\} \cup \{y \in M \;:\; \mathcal{E}(y) > n \text{ or } \mathcal{R}_{\epsilon,\delta,v}(y) > n\}. \end{aligned}$$

Hence, in order to show (4) it is sufficient to show that there exist constants $\tau, \zeta > 0$ such that for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|\{y \in M \;:\; \mathcal{R}_{\epsilon,\delta,h}(y) > n\}| \leq \mathcal{O}(e^{-\tau n^\zeta}) \tag{5}$$

and

$$|\{y \in M \;:\; \mathcal{E}(y) > n \text{ or } \mathcal{R}_{\epsilon,\delta,v}(y) > n\}| \leq \mathcal{O}(e^{-\tau n^\zeta}). \tag{6}$$

3.1. **Bounds for the fiber maps.**

The main calculations here are done in [S1] and [S2] where the positivity of the Lyapunov exponents is shown. We can essentially follow Section 6.2.1 in [AA], which establishes tail estimates of the expansion and recurrence time function for the maps studied in [V] (also in their case, the essential part of the argument is done in the proof of positive Lyapunov exponents; see [V]).

We will treat the maps F_1 and F_2 simultaneously. In order to apply the results in [S2], we first have to conjugate the function F_2 to a function denoted by $\tilde{F}_2$. The conjugation function $\Phi : I \times J \to [-1,1] \times J$ is of the form $\Phi(\theta, x) = (\varphi(\theta), x)$, $(\theta, x) \in I \times J$, where $\varphi : I \to [-1,1]$ is analytic outside a finite number of singularities. φ is obtained by integrating the density of the acip for Q_b, from which it follows that the singularities of φ are of square root type. The conjugation function Φ is explained in detail in [S2, p. 2684]. The conjugated map $\tilde{F}_2 = \Phi \circ F_2 \circ \Phi^{-1}$ has the form $\tilde{F}_2 : [-1,1] \times J \to [-1,1] \times J$:

$$\tilde{F}_2(\theta, x) = (g(\theta), Q_a(x) + \alpha h(\theta)),$$

where $g = \varphi \circ Q_b^k \circ \varphi^{-1} : [-1,1] \to [-1,1]$ is analytic and uniformly expanding outside a finite set of singularities and $h : [-1,1] \to [-1,1]$ is C^2 (extendable to a neighborhood of $[-1,1]$) with first derivative bounded away from 0. In the setting of the map F_1 let the base dynamics $\theta \mapsto d\theta \bmod 1$ also be denoted by g. Depending on the context, in the following let M denote either M_1 or $[-1,1] \times J$, and let the map F stand for either F_1 or $\tilde{F}_2$, respectively. We define inductively $f_n(\theta, x)$, $(\theta, x) \in M$. $f_1(\theta, x)$ is equal to $Q_a(x) + \alpha \sin(2\pi\theta)$ for $F = F_1$ and equal to $Q_a(x) + \alpha h(\theta)$ for $F = \tilde{F}_2$. For $n \geq 2$, f_n is defined by the equation $F^n(\theta, x) = (g^n(\theta), f_n(\theta, x))$. In order to get the bound (6) of the tail of the expansion and recurrence time function, we have to study the returns of $f_n(\theta, x)$ to 0.

Henceforth, we consider only points $(\theta, x) \in M$ whose orbits do not hit the critical set $\mathcal{S}_v$. This is no restriction since the set of those points has full Lebesgue measure. For $r \geq 0$, set

$$J(r) = \{x \in I \ : \ |x| \leq \sqrt{\alpha} e^{-r}\},$$

and for each integer $j \geq 0$, define

$$r_j(\theta, x) = \min\{r \geq 0 \ : \ f_j(\theta, x) \notin J(r)\}.$$

In [S1] and [S2], for some given constant $0 < \kappa < 1/4$, one considers

$$G = \left\{0 \leq j < n \ : \ r_j(\theta, x) \geq \left(\frac{1}{2} - 2\kappa\right) \log \frac{1}{\alpha}\right\}.$$

Fix some integer $n \geq 1$ sufficiently large (only depending on $\alpha > 0$). From the estimates in [S1, equation (14)] and [S2, equation (23)], we deduce that if we take

$$B_2(n) = \{(\theta, x) \in M \ : \ \text{there is } 1 \leq j < n \text{ with } f_j(\theta, x) \in J(\sqrt{n})\},$$

then there is a constant $\tau_2 > 0$ such that

$$|B_2(n)| \leq \text{const}\, e^{-\tau_2 \sqrt{n}}.$$

Furthermore, there exists a constant $c > 0$ (depending only on the quadratic map Q_a, and not on α) such that

$$\log |\partial_x f_n(\theta, x)| \geq cn - \sum_{j \in G} r_j(\theta, x), \quad \text{for } (\theta, x) \notin B_2(n); \tag{7}$$

see [S1, equation (15)], [S2, equation (24)], and [V, pp. 75–76]. Let

$$B_1(n) = \left\{(\theta, x) \in M \ : \ \sum_{j \in G} r_j(\theta, x) \geq \frac{c}{2} n\right\}.$$

It is shown in [S1, equation (16)] and [S2, equation (25)] that there is $\tau_1 > 0$ such that

$$|B_1(n)| \leq \text{const}\, e^{-\tau_1\sqrt{n}}.$$

Since the base dynamics of F is uniformly expanding, we obtain immediately that

$$|\{y \in M \ : \ \mathcal{E}(y) > n\}| \leq |B_1(n) \cup B_2(n)| \leq \mathcal{O}(e^{-\tau\sqrt{n}}), \tag{8}$$

where $\tau = \min\{\tau_1, \tau_2\}$. Note that while the base dynamics g of $\tilde{F}_2$ is uniformly expanding, the base dynamics Q_b^k of F_2 is not. However $Q_b^{nk} = \varphi^{-1} \circ g^n \circ \varphi$ and, by the properties of the density of the acip for Q_b (see, e.g., [S2]), it follows that the derivative of φ is uniformly bounded away from zero (on its support) and the derivative of φ^{-1} is strictly positive outside a finite number of critical points of order 2. Hence, there exists $\lambda > 1$ such that, for each $n \geq 1$, $|D_\theta Q_b^{nk}(\theta)| \geq \lambda^n$ for all θ outside an exceptional set whose size is decreasing exponentially in n. Combined with (9) below, it follows that the tail estimate (8) of the expansion time function does not only hold for the maps F_1 and $\tilde{F}_2$ but also for the map F_2.

From the arguments in [S1], [S2], and [V] it is obvious that the constant c in the definition of $B_1(n)$ can be chosen arbitrarily small. Observe that in the set $B_1(n)$ we are only concerned about the returns of $f_n(\theta, x)$ to the set $J((1/2 - 2\kappa)\log(1/\alpha))$. Hence, setting

$$\delta = \frac{|J((1/2 - 2\kappa)\log(1/\alpha))|}{2} = \alpha^{1-2\kappa}$$

and writing ϵ instead of $c/2$, we obtain

$$\sum_{j=0}^{n-1} -\log \text{dist}_\delta(F^j(\theta, x), \mathcal{S}_v) = \sum_{j \in G} r_j(\theta, x) \leq \epsilon n,$$

for all $(\theta, x) \notin B_1(n) \cup B_2(n)$. Considering the map F_2 this implies that

$$\sum_{j=0}^{n-1} -\log \text{dist}_\delta(F_2^j(\theta, x), \mathcal{S}_v) \leq \epsilon n,$$

for all $(\theta, x) \notin \Phi^{-1}(B_1(n) \cup B_2(n))$, where $\Phi = (\varphi, \text{id})$ is the conjugating function described above. Since the derivative of φ^{-1} is bounded from above (see, e.g., [S2]), we obtain

$$|\Phi^{-1}(B_1(n) \cup B_2(n))| \leq \|D\varphi^{-1}\|_\infty |B_1(n) \cup B_2(n)| \leq \text{const}\, e^{-\tau\sqrt{n}}. \tag{9}$$

We conclude that for the maps F_1 and F_2 we have

$$|\{y \in M \ : \ \mathcal{R}_{\epsilon,\delta,v}(y) > n\}| \leq \mathcal{O}(e^{-\tau\sqrt{n}}).$$

Altogether we have proved for F_1 and F_2 the stretched exponential bounds required in (6) where the constant ζ can be taken equally to 1/2.

3.2. **Bounds for the base dynamics.** Note that to prove the decay on $|\{y \in M \ : \ \mathcal{R}_{\epsilon,\delta,h}(y) > n\}|$ we only have to consider the base dynamics. For the sake of notation we make this more precise. Consider the projection of $\mathcal{S}_h$ to the first coordinate. We denote this projection again by $\mathcal{S}_h$; i.e., for the base dynamics g_1 of the map F_1 the critical set $\mathcal{S}_h$ is equal to $0 \in S^1$, and for the base dynamics g_2

of the map F_2 the critical set $\mathcal{S}_h$ is equal to $\{b_1, ..., b_m\} \subset I$. For $y = (\theta, x) \in M_i$, $i = 1, 2$, we have

$$\mathcal{R}_{\epsilon,\delta,h}(y) = \mathcal{R}_{\epsilon,\delta,h}(\theta)$$
$$:= \min \left\{ N \geq 1 \ : \ \frac{1}{n} \sum_{j=0}^{n-1} - \log \mathrm{dist}_\delta(g_i^j(\theta), \mathcal{S}_h) \leq 2\epsilon, \text{ for all } n \geq N \right\},$$

where dist_δ is defined as above but restricted to S^1 or I, respectively. It follows that $|\{y \in M \ : \ \mathcal{R}_{\epsilon,\delta,h}(y) > n\}|$ is equal to $|\{\theta \in S^1 \ : \ \mathcal{R}_{\epsilon,\delta,h}(\theta) > n\}||J|$ or $|\{\theta \in I \ : \ \mathcal{R}_{\epsilon,\delta,h}(\theta) > n\}||J|$, respectively.

To establish the desired tail estimates of the recurrence time function for the base dynamics, we follow the strategy of [AFLV, Theorem 4.2]. We begin by introducing some auxiliary functions. For $\delta > 0$ sufficiently small, let

$$\phi(\theta) = \begin{cases} -\log \mathrm{dist}(\theta, \mathcal{S}_h) & \text{if } \mathrm{dist}(\theta, \mathcal{S}_h) < \delta \ , \\ \frac{\log \delta}{\delta}(\mathrm{dist}(\theta, \mathcal{S}_h) - 2\delta) & \text{if } \delta \leq \mathrm{dist}(\theta, \mathcal{S}_h) < 2\delta \ , \\ 0 & \text{if } \mathrm{dist}(\theta, \mathcal{S}_h) \geq 2\delta, \end{cases}$$

where θ is in S^1 or I, respectively. Observe that ϕ has discontinuities at the singular set $\mathcal{S}_h$. Let ν_1 and ν_2 denote the unique acip for g_1 and g_2, respectively. (The density of ν_2 is in $L^p(m)$, for all $1 \leq p < 2$; see, e.g., [S2].) We can choose $\delta > 0$ so small that, for $i = 1, 2$,

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log \mathrm{dist}_\delta(g_i^j(\theta), \mathcal{S}_h) \leq \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(g_i^j(\theta)) = \int \phi d\nu_i \leq \epsilon \,,$$

for ν_i-a.e. θ. For all $k > 0$ we let

$$A_k := \{\theta : \phi(\theta) \geq k\}$$

and define

$$\phi_k(\theta) := \begin{cases} k, & \text{if } \theta \in A_k, \\ \phi(\theta), & \text{otherwise.} \end{cases}$$

The functions ϕ_k and the sets A_k correspond to the functions $\phi_{2,k}$ and $A_{2,k}$ in [AFLV, Section 5], respectively.

3.2.1. *β-transformations.* We consider first the setting in the case of the map F_1. Let $\mathcal{B}$ denote the space of functions φ on S^1 with bounded variation and set

$$\|\varphi\|_\mathcal{B} := V_{S^1}\varphi + \|\varphi\|_{L^1(m)},$$

where m denotes the Lebesgue measure on S^1. By [AFLV, Appendix C.1 and Corollary H (2)] we get that for all $\varphi \in \mathcal{B}$ and for every $\epsilon > 0$ there exist $\tau(\varphi, \epsilon) > 0$ and $C(\varphi, \epsilon)$ such that

$$LD_{\nu_1}(\varphi, \epsilon, n) = \nu_1 \left(\left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ g_1^i - \int \varphi d\nu_1 \right| > \epsilon \right) \leq C(\varphi, \epsilon) e^{-\tau(\varphi,\epsilon)n}. \tag{10}$$

Furthermore, by [AFLV, Proposition 2.5 and Lemma 2.6], we get

$$\tau(\varphi, \epsilon) \geq \epsilon^2 (8(\|\varphi\|_\infty + C'\|\varphi\|_\mathcal{B})^2)^{-1}.$$

The constant C' is equal to $2\sum_{i\geq 0}\xi(i)$, where $\xi(i)$ is an upper bound for the decay of correlation for observables in $\mathcal{B}$ against $L^1(\nu_1)$ (whose corresponding norms are ≤ 1). By [AFLV, Appendix C.1 and Corollary H (1)], this decay is exponential and, hence, C' is finite. Regarding the constant $C(\varphi,\epsilon)$, we derive from the proof of [AFLV, Proposition 2.5] that $C(\varphi,\epsilon)\leq 2e^{\epsilon(4\|\varphi\|_\infty)^{-1}}$. Since the function ϕ is not of bounded variation we cannot apply (10) directly to ϕ. However, the functions ϕ_k are of bounded variation and, according to [AFLV, equation (5.1)], we have

$$LD_{\nu_1}(\phi,2\epsilon,n)\leq LD_{\nu_1}(\phi_k,\epsilon,n)+n\nu_1(A_k).$$

Since the density of ν_1 is bounded from above, this immediately implies $\nu_1(A_k)\leq \mathrm{const}\,|A_k|\leq \mathrm{const}\,e^{-k}$. Altogether we obtain

$$LD_{\nu_1}(\phi,2\epsilon,n)\leq 2e^{\epsilon(4\|\phi_k\|_\infty)^{-1}}e^{-\epsilon^2(8(\|\phi_k\|_\infty+C'\|\phi_k\|_{\mathcal{B}})^2)^{-1}n}+\mathrm{const}\,ne^{-k}.$$

Observe that $\|\phi_k\|_\infty=k$, $V_{S^1}\phi_k=2k$ and $\|\phi_k\|_{L^1(m)}$ is bounded from above by a constant independent on k. We derive that there is a constant C independent on k and ϵ such that

$$LD_{\nu_1}(\phi,2\epsilon,n)\leq C(e^{-\epsilon^2C^{-1}k^{-2}n}+ne^{-k}).$$

Choosing $k=n^{1/3}$, we get $LD_{\nu_1}(\phi,2\epsilon,n)\leq \mathcal{O}(e^{-\tau n^{1/3}})$ for some constant $\tau=\tau(\epsilon)>0$. Since $-\log\mathrm{dist}_\delta\leq\phi$, the density of ν_1 is bounded away from zero (see, e.g. [AFLV, Appendix C.1]), and $\int\phi d\nu_1\leq\epsilon$, we finally obtain

$$|\{\theta\in S^1\ :\ \mathcal{R}_{\epsilon,\delta,h}(\theta)>n\}|\leq \mathrm{const}\,LD_{\nu_1}(\phi,2\epsilon,n)\leq\mathcal{O}(e^{-\tau n^{1/3}}),$$

which implies (5) where $\zeta=1/3$.

3.2.2. *Quadratic maps.* It is only left to consider the case of the map F_2. By, e.g., [KN], (g_2,ν_2) has exponential decay of correlations for functions of bounded variation only against $L^p(\nu_2)$, $p>2$. Thus, we cannot apply the argument above for the map g_1 which gives a sharper result. However, by [G2], it follows that (g_2,ν_2) has exponential decay of correlations for Hölder observables against $L^\infty(\nu_2)$. In order to apply [G2], we need the existence of a tower with exponentially small tails. But this follows from [Y1]. Moreover, by [S2], it follows that the density of ν_2 is in $L^p(m)$, for any $1\leq p<2$, and uniformly bounded away from 0 on its support. Thus, [AFLV, Proposition 4.1 (2)] implies that there exists $\tau>0$ such that for any $0<\zeta<1/9$ and any $\epsilon>0$ sufficiently small one has

$$|\{\theta\in I\ :\ \mathcal{R}_{\epsilon,\delta,h}(\theta)>n\}|\leq \mathrm{const}\,LD_{\nu_2}(\phi,2\epsilon,n)\leq\mathcal{O}(e^{-\tau n^{\zeta}}).$$

This concludes the proof of (5) in the case of the map F_2.

Remark 1. For (g_2,ν_2), [KN] and [MN2] show exponentially large deviation estimates for observables of bounded variation and for Hölder observables, respectively. Thus, regarding the argument above for the map g_1, one might expect to get in (5) a constant ζ close to $1/3$. However, the constants in [KN] and [MN2] for the exponentially large deviation are not as explicit as in (10), which makes it difficult to apply their results to our setting.

4. Topological transitivity

Denote by Λ the attractor $\bigcap_{n\geq 0} F^n(M)$. We say that F is *topologically transitive* on the attractor Λ if, for every non-empty open subsets U and V of Λ, there exists n such that $F^{-n}(U)\cap V$ contains a non-empty open set. For the maps F_i, $i=1,2$, we have, by the same argument as in [AV, Lemma 6.1], that its attractor Λ_i coincides with $F_i^2(M_i)$. In fact, for the latter use we note that the argument in [AV] shows that even if D is an interval with its boundary points sufficiently close to $Q_a^2(0)$ and $Q_a(0)$, respectively, then $F_1^2(S^1\times D)=\Lambda_1$ and $F_2^2(I\times D)=\Lambda_2$.

The essential part for showing topological transitivity is done in [A]. By Sections 3.1 and 3.2 we know that $F_i: M_i\to M_i$, $i=1,2$, is non-uniformly expanding and slowly recurrent to the critical set. Hence, we can apply Lemma 4.3 in [A] and get that there is a constant $\delta>0$ only dependent on the constant c from the non-uniform expansion (see equation (7)) and on the constant β from the non-degeneracy condition such that the following holds. For every $\epsilon>0$ there exists $n_1=n_1(\epsilon)>0$ such that for any ball $B\subset M_i$ of radius ϵ there is an integer $n\leq n_1$ such that $F_i^n(B)$ contains a ball of radius δ (of course $F_i^n: B\to M_i$ might not be injective). Recall that the constant c and, thus, also the constant δ do not depend on α. The following argument is similar to that in [AV, p. 29]. Recall that we defined $I=[Q_b^2(0),Q_b(0)]$. Since Q_a and Q_b are non-renormalizable, it follows that the supports of the acip's for Q_a and Q_b are equal to $[Q_a^2(0),Q_a(0)]$ and I, respectively. Since the critical points of Q_a and Q_b are eventually mapped into repelling periodic points and since Q_a and Q_b are conjugated to uniformly expanding maps (see, e.g., [S2, Proposition 2.2]), it follows that there is an integer $n_2=n_2(\delta)>0$ such that if $V\subset[Q_a^2(0),Q_a(0)]$ and $V'\subset I$ are intervals of length δ, then $Q_a^{n_2}(V)=[Q_a^2(0),Q_a(0)]$ and $Q_b^{n_2}(V')=I$. Recall that if D is an interval with its boundary points sufficiently close to $Q_a^2(0)$ and $Q_a(0)$, respectively, then $F_1^2(S^1\times D)=\Lambda_1$ and $F_2^2(I\times D)=\Lambda_2$. Since F_1 and F_2 depend continuously on α, it follows that if $\theta\in S^1$, $\theta'\in I$, and V,V' are intervals of length δ satisfying $\theta\times V\subset\Lambda_1$ and $\theta'\times V'\subset\Lambda_2$, then for α sufficiently small we have

$$F_1^{n_1+2}(\theta\times V)=\{g^{n_1+2}(\theta)\}\times\mathbb{R}\cap\Lambda_1$$

and

$$F_2^{n_1+2}(\theta'\times V')=\{Q_b^{(n_1+2)k}(\theta')\}\times\mathbb{R}\cap\Lambda_2.$$

Altogether, we derive that for each $\epsilon>0$ there is an integer $n_0=n_0(\epsilon)$ such that if $B\subset\Lambda_i$, $i=1,2$, is a ball of radius ϵ, then $F_i^{n_0}(B)=\Lambda_i$. Thus, we conclude that F_i is topologically transitive on Λ_i. Obviously, this argument also works for arbitrary iterates of F_i.

Appendix A. Limit theorems

Here we define the statistical properties of dynamical systems; see properties A.1–A.4 below, which are mentioned in the last items of Theorems A and B. A sufficient condition on our maps F_1 and F_2 in order to obtain the following four properties A.1–A.4 is the existence of a Gibbs-Markov structure with tail estimates decaying at least as fast as $n^{-\alpha}$ for some $\alpha>2$. (For F_1 and F_2 this condition is obviously satisfied since we have stretched exponential tail estimates for their towers.) More precisely, in order to deduce property A.1 we combine this existence of a Gibbs-Markov structure with [Y2, Theorem 4]; in order to deduce property A.2 we combine this existence of a Gibbs-Markov structure with [G1, Theorem 1.2]; in order to deduce property A.3 we combine this existence of a Gibbs-Markov structure with [G1, Theorem 1.3]; and in order to deduce property A.4 we combine this existence of a Gibbs-Markov structure with [MN1, Theorem 2.9]. (See also [AFLV, Corollaries B1–B4]. Observe that the assumption therein that the acip for F_1 and F_2, respectively, has a density in L^p, for some $p > 1$, and decay of correlation at least as fast as $n^{-\alpha-1}$ for some $\alpha > 2$ is only used to deduce, by [AFLV, Theorem C], the existence of a Gibbs-Markov structure with tail estimates decaying at least as fast as $n^{-\alpha+\epsilon}$ for any $\epsilon > 0$.)

A.1. **Central Limit Theorem.** Let $\varphi \in \mathcal{H}_\gamma$ be such that $\int \varphi d\mu = 0$. Then

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} \varphi \circ F^i \right)^2 d\mu \geq 0 \tag{11}$$

is well defined. We say the *Central Limit Theorem* holds for φ if for all $a \in \mathbb{R}$

$$\mu\left(\left\{ x : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ F^i(x) \leq a \right\} \right) \to \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} dx, \text{ as } n \to \infty,$$

whenever $\sigma^2 > 0$. Additionally, $\sigma^2 = 0$ if and only if φ is a *coboundary* ($\varphi \neq \psi \circ F - \psi$ for any $\psi \in L^2$).

A.2. **Local Limit Theorem.** A function $\varphi : M \to \mathbb{R}$ is said to be *periodic* if there exist $\rho \in \mathbb{R}$, a measurable function $\psi : M \to \mathbb{R}$, $\lambda > 0$, and $q : M \to \mathbb{Z}$ such that

$$\varphi = \rho + \psi - \psi \circ F + \lambda q$$

almost everywhere. Otherwise, it is said to be *aperiodic*.

Let $\varphi \in \mathcal{H}_\gamma$ be such that $\int \varphi d\mu = 0$ and σ^2 be as in (11). Assume that φ is aperiodic (which implies that $\sigma^2 > 0$). We say that the *Local Limit Theorem* holds for φ if for any bounded interval $J \subset \mathbb{R}$, for any real sequence $\{k_n\}_{n\in\mathbb{N}}$ with $k_n/n \to \kappa \in \mathbb{R}$, for any $u \in \mathcal{H}_\gamma$, and for any measurable $v : M \to \mathcal{R}_{\epsilon,\delta}$ we have

$$\sqrt{n}\mu\left(\left\{ x \in M : \sum_{i=0}^{n-1} \varphi \circ F^i(x) \in J + k_n + u(x) + v(F^n x) \right\} \right) \to m(J) \frac{\mathrm{e}^{-\frac{\kappa^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}.$$

A.3. **Berry-Esseen Inequality.** If F admits a Gibbs-Markov induced map of base Δ_0 and return time function R, then for any $\varphi : M \to \mathbb{R}$ define $\varphi_{\Delta_0} : \Delta_0 \to \mathbb{R}$ by

$$\varphi_{\Delta_0}(x) = \sum_{i=0}^{R(x)-1} \varphi(F^i x).$$

Let $\varphi \in \mathcal{H}_\gamma$ be such that $\int \varphi d\mu = 0$ and σ^2 be as in (11). Assume that $\sigma^2 > 0$ and that there exists $0 < \delta \leq 1$ such that $\int |\varphi_{\Delta_0}|^2 \chi_{|\varphi_{\Delta_0}| > z} d\mu \leq \text{const}\, z^{-\delta}$, for large z. If $\delta = 1$, assume also that $\int |\varphi_{\Delta_0}|^3 \chi_{|\varphi_{\Delta_0}| \leq z} d\mu$ is bounded. We say that the *Berry-Esseen Inequality* holds for φ if there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $a \in \mathbb{R}$ we have

$$\left| \mu\left(\left\{ x : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ F^i(x) \leq a \right\} \right) - \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} dx \right| \leq \frac{C}{n^{\delta/2}}.$$

A.4. **Almost Sure Invariance Principle.** Given $d \geq 1$ and a Hölder continuous $\varphi\colon M \to \mathbb{R}^d$ with mean zero, we denote

$$S_n = \sum_{i=0}^{n-1} \varphi \circ F^i, \quad \text{for each } n \geq 1.$$

We say that φ satisfies an *Almost Sure Invariance Principle* (ASIP) if there exists $\lambda > 0$ and a probability space supporting a sequence of random variables $\{S_n^*\}_n$ (which can be $\{S_n\}_n$ in the $d = 1$ case) and a d-dimensional Brownian motion $W(t)$ such that

(1) $\{S_n\}_n$ and $\{S_n^*\}_n$ are equally distributed;
(2) $S_n^* = W(n) + O(n^{1/2-\lambda})$, as $n \to \infty$, almost everywhere.

Satisfying an ASIP is a strong statistical property that implies other limiting laws such as the Central Limit Theorem, the Functional Central Limit Theorem and the Law of the Iterated Logarithm.

References

[A] J. F. Alves, *Strong statistical stability of non-uniformly expanding maps*, Nonlinearity 17 (2004), no. 4, 1193–1215. MR2069701 (2005e:37043)

[AA] J.F. Alves and V. Araújo, *Random perturbations of non-uniformly expanding maps*, Astérisque 286 (2003), 25–62. MR2052296 (2005e:37058)

[AFLV] J.F. Alves, J.M. Freitas, S. Luzzatto, and S. Vaienti, *From rates of mixing to recurrence times via large deviations*, Adv. Math. 228 (2011), 1203–1236. MR2822221

[ALP] J.F. Alves, S. Luzzatto, and V. Pinheiro, *Markov structures and decay of correlations for non-uniformly expanding dynamical systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), 817–839. MR2172861 (2006e:37047)

[AV] J.F. Alves and M. Viana, *Statistical stability for robust classes of maps with non-uniform expansion*, Ergodic Theory Dynam. Systems 22 (2002), 1–32. MR1889563 (2003d:37037)

[AS] V. Araújo and J. Solano, *Absolutely continuous invariant measures for non-expanding maps*, arXiv:1111.4540v1

[BST] J. Buzzi, O. Sester, and M. Tsujii, *Weakly expanding skew-products of quadratic maps*, Ergodic Theory Dynam. Systems 23 (2003), 1401–1414. MR2018605 (2004m:37034)

[G1] S. Gouëzel, *Berry-Esseen theorem and local limit theorem for non-uniformly expanding maps*, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005), 997–1024. MR2172207 (2007b:60071)

[G2] S. Gouëzel, *Decay of correlations for non uniformly expanding systems*, Bull. Soc. Math. France 134 (2006), 1–31. MR2233699 (2008f:37066)

[KN] G. Keller and T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps*, Comm. Math. Phys. 149 (1992), 31–69. MR1182410 (93i:58123)

[MN1] I. Melbourne, M. Nicol, *Almost sure invariance principle for non-uniformly hyperbolic systems*, Comm. Math. Phys. 260 (2005), 131–146. MR2175992 (2006h:37047)

[MN2] I. Melbourne, M. Nicol, *Large deviations for non-uniformly hyperbolic systems*, Trans. Amer. Math. Soc. 360 (2008), 6661–6676. MR2434305 (2009m:37086)

[S1] D. Schnellmann, *Non-continuous weakly expanding skew-products of quadratic maps with two positive Lyapunov exponents*, Ergodic Theory Dynam. Systems 28 (2008), 245–266. MR2380309 (2010a:37051)

[S2] D. Schnellmann, *Positive Lyapunov exponents for quadratic skew-products over a Misiurewicz-Thurston map*, Nonlinearity 22 (2009), 2681–2695. MR2550691 (2011a:37094)

[V] M. Viana, *Multidimensional non-hyperbolic attractors*, Publ. Math. Inst. Hautes Études Sci. 85 (1997), 63–96. MR1471866 (98j:58073)

[Y1] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. 147 (1998), 585–650. MR1637655 (99h:58140)

[Y2] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. 110 (1999), 153–188. MR1750438 (2001j:37062)

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