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# NON-EXISTENCE OF PRESCRIBABLE CONFORMALLY EQUIVARIANT DILATATION IN SPACE

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Abstract. In this paper, we study the prescribable conformally equivariant dilatations for orientation preserving quasiconformal homeomorphisms. The complex dilatation is a prescribable conformally equivariant dilatation in $\mathbb{R}^2$. A Schottky set is a subset of the unit sphere $\mathbb{S}^n$ whose complement is the union of at least three disjoint open balls. By using the result of Bonk, Kleiner, and Merenkov that there are rigid Schottky sets of positive measure in each dimension at least 3, we prove that it is not possible to have a prescribable conformally equivariant dilatation in $\mathbb{R}^n$, where $n \geq 3$.

## 1. Introduction

Let $G$ be a domain in $\mathbb{R}^n$, where $n \geq 2$. A homeomorphism $f$ of $G$ onto a domain in $\mathbb{R}^n$ is called quasiconformal if its circular dilatation

$$H(x,f) = \limsup_{r\to 0} \frac{\max\{|f(x)-f(y)| : |y-x| = r\}}{\min\{|f(x)-f(y)| : |y-x| = r\}} \geq 1$$

satisfies $\sup\{H(x,f) : x \in G\} < +\infty$, and $K-$quasiconformal, where $K \geq 1$, if $H(x,f) \leq K$ for almost every $x \in G$ with respect to the Lebesgue measure of $\mathbb{R}^n$. For the basic properties of quasiconformal mappings we refer to [3], [1], [5].

The measurable Riemann mapping theorem states that in dimension 2, one can prescribe for quasiconformal mappings a quantity, the complex dilatation, with certain equivariance properties with respect to conformal mappings. Let $f$ be an orientation preserving homeomorphism between two plane domains. The complex dilatation of $f$ at a point $z$ where $f$ is differentiable with $f_z(z) \neq 0$ is defined as

$$\mu_f(z) = \frac{f_{\overline{z}}(z)}{f_z(z)}.$$

By the measurable Riemann mapping theorem, for any given complex-valued function $\mu$ defined on $G$ with $\|\mu\|_\infty = \operatorname{ess\,sup}_{z\in G} |\mu(z)| < 1$, there exists an orientation preserving quasiconformal mapping $f$ on $G$ with the complex dilatation $\mu_f = \mu$ almost everywhere. It is still an open problem whether there exists a similar quantity, which we also call a dilatation, that can be prescribed for a quasiconformal mapping in $\mathbb{R}^n$, where $n \geq 3$. In this paper we show that no such dilatation can be prescribed if the dilatation is to have sufficiently many equivariance properties,

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with respect to conformal mappings that mimic those of the complex dilatation in the plane, by using the rigidity of certain Schottky sets.

We denote the transpose of a matrix $A$ by $A^T$ and the Jacobian determinant of $f$ at $z$ by $J(z, f)$. If $x \in \mathbb{R}^n$, we denote the Euclidean norm of $x$ by $|x|$. We recall some properties of the complex dilatation $\mu_f$ of an orientation preserving quasiconformal homeomorphism $f$:

(i) if also $\mu_{f_1} = \mu = \mu_f$ almost everywhere in the domain $G \subset \mathbb{R}^2$, then $f_1 = g \circ f$, where $g$ is a conformal mapping;

(ii) if $g$ is a conformal mapping, then $\mu_{f \circ g}(z) = \mu_f(g(z))\overline{g'(z)}/g'(z)$ and hence $|\mu_{f \circ g}(z)| = |\mu_f(g(z))|$ for a.e. $z$;

(iii) $\mu_f(z) = 0$ for a.e. $z$ if, and only if, $f$ is a conformal mapping;

(iv) if $||\mu_f||_\infty = k < 1$, then $f$ is $K-$quasiconformal, where $K = (1+k)/(1-k)$;

(v) if $f'(z)$ is the derivative matrix of $f$ at a point $z$ where $f$ is differentiable and $f_z(z) \neq 0$, then $\mu_f(z)$ depends only on the positive definite matrix $B(z) = f'(z)^T f'(z)$.

If $f$ is an orientation reversing homeomorphism of a plane domain, then its complex conjugate $\overline{f}$ is orientation preserving and $\mu_{\overline{f}}$ has the above properties.

In higher dimensions, we now define a conformally equivariant dilatation to be any reasonable quantity, in a sense that we make precise below, that depends only on the matrix $f'(z)^T f'(z)$ and has properties comparable to those listed above with respect to conformal mappings. There are many quantities that would qualify as conformally equivariant dilatations in this sense. We say that such a dilatation is prescribable if it can be prescribed in the same way as the complex dilatation can be prescribed in dimension two by the measurable Riemann mapping theorem. Thus we formulate the following definitions.

Fix an integer $n \geq 2$. Let us denote the set of all positive definite $n \times n-$matrices $B$ with real entries by $P_n$. Note that if $A$ is a non-singular $n \times n-$matrix with real entries, then $B = A^T A \in P_n$. Using the matrix elements of $B \in P_n$, we may consider $P_n$ to be a subset of $\mathbb{R}^{n^2}$.

**Definition 1.1.** Let $m$ be a positive integer and let $\nu : P_n \to \mathbb{R}^m$ be a function. We say that $\nu$ is a **prescribable conformally equivariant dilatation** if the following conditions are satisfied:

(1) The function $\nu$ is measurable and its values form a measurable subset $E$ of the unit ball of $\mathbb{R}^m$.

(2) $\nu(B) = 0$ if, and only if, $B$ is of the form $B = rI_n$, where $r > 0$ and $I_n$ is the identity matrix.

(3) $\nu(rB) = \nu(B)$ whenever $B \in P_n$ and $r > 0$.

(4) $|\nu(P^T BP)| = |\nu(B)|$ whenever $B \in P_n$ and $P$ is an $n \times n$ orthogonal matrix.

(5) If $P$ is an $n \times n$ orthogonal matrix (not necessarily with positive determinant) and $B_1, B_2 \in P_n$ with $\nu(B_1) = \nu(B_2)$, then $\nu(P^T B_1 P) = \nu(P^T B_2 P)$.

(6) If $f$ is a $K-$quasiconformal mapping in $\mathbb{R}^n$, then for a number $k \in [0, 1)$ depending only on $K$ and the function $\nu$, but not on $f$ or $x$, we have $|\nu(f'(x)^T f'(x))| \leq k$ for almost every $x \in \mathbb{R}^n$.

(7) Suppose that $\kappa : \mathbb{R}^n \to E$ is a measurable function with $\|\kappa\|_\infty < 1$. Then there is a quasiconformal (orientation preserving, homeomorphic) mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu(f'(x)^T f'(x)) = \kappa(x)$ for almost every $x \in \mathbb{R}^n$, and if $g : \mathbb{R}^n \to \mathbb{R}^n$ is a quasiconformal mapping such that $\nu(g'(x)^T g'(x)) = \kappa(x)$ for

almost every $x \in \mathbb{R}^n$, then $g$ is of the form $g = M \circ f$, where $M$ is a Möbius transformation.

*Remarks.* 1. We have not specified $m$. The number $m$ can be anything as long as all the conditions above are satisfied. Presumably it is not possible to satisfy the second condition, that is, the uniqueness condition of property (7) if $m < n$ since then too little is being prescribed. However, the point is that we do not need to know this in order to discuss the above definition and its properties.

2. We could conceivably have $m > n$. This does not necessarily mean that too much is being prescribed since the set $E$ might be small enough (for example, the intersection of an $n-$dimensional manifold with the unit ball of $\mathbb{R}^m$) to effectively limit the number of quantities to be prescribed. By not specifying $E$, we are allowing for more possibilities.

3. If $y \in E$ and $z \in \mathbb{R}^m$ with $|z| = |y|$, we need not have $z \in E$ as far as the above assumptions are concerned. Condition (4) could at least theoretically be satisfied without $E$ being so large as to contain the whole unit ball.

4. In Definition 1.1, property (7) corresponds to the measurable Riemann mapping theorem, and the uniqueness part of property (7) corresponds to property (i) of the complex dilatation. At the same time, property (7) indicates an invariance property of the dilatation with respect to the composition of mappings when the outer function is a conformal mapping. Such a property, when the inner function is a conformal mapping, is given by (4), which corresponds to the second part of property (ii) for the complex dilatation. Properties (2), (5), and (6) in Definition 1.1 correspond to properties (iii), the first half of (ii), and (iv), respectively, for the complex dilatation. Property (3) in Definition 1.1 corresponds to the special case of property (i) for the complex dilatation where the conformal mapping $g$ is given by $g(z) = rz$. The very fact that $\nu$ is to depend only on $f'(x)^T f'(x)$ (which is property (v) for the complex dilatation) and not on properties of $f'(x)$ that are lost when passing to $f'(x)^T f'(x)$, together with property (3), corresponds to property (i) of the complex dilatation.

**Example.** In dimension 2, identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, write $f = u + iv$, $z = x + iy$, and

$$f'(z)^T f'(z) = B(z) = B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix},$$

using the fact that $B(z)$ is symmetric. For any $B \in P_2$, write

$$\alpha = b_{11} - b_{22} + 2ib_{12}, \quad \beta = b_{11} + b_{22} + 2\sqrt{\det B}. \tag{1.1}$$

Define $\mu(B) = \alpha/\beta \in \mathbb{C}$ when $B \in P_2$. Consider a homeomorphism $f$ of a plane domain at a point $z$ where $f$ is differentiable with a non-zero Jacobian determinant. If $f$ is orientation preserving, then $\mu(B(z)) = \mu_f(z)$. If $f$ is orientation reversing, then $\mu(B(z)) = \mu_{\overline{f}}(z)$. One can now verify that $\mu$ is a prescribable conformally equivariant dilatation in $\mathbb{R}^2$. Property (7) follows from the measurable Riemann mapping theorem. The other properties follow from some calculations, with all but properties (4) and (5) being rather obvious. The calculation for (5), even though routine, is lengthy.

We shall prove the following result.

**Theorem 1.2.** *If $n \geq 3$, then there is no prescribable conformally equivariant dilatation in $\mathbb{R}^n$.*

This means that if a dilatation quantity is found that can be prescribed for quasiconformal mappings in $\mathbb{R}^n$, where $n \geq 3$, then at least one of the conditions in Definition 1.1 must fail.

It turns out that for all practical purposes, in dimension two, the usual complex dilatation is the only prescribable conformally equivariant dilatation with values in $\mathbb{R}^2$, in the sense that any other is a re-parametrization of the usual complex dilatation. We write $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for the unit disk in the complex plane.

**Theorem 1.3.** *Suppose that $n = 2$. If $\nu$ is a prescribable conformally equivariant dilatation with values in $\mathbb{R}^2$, then there is a measurable one-to-one function $\varphi$ of $\mathbb{D}$ onto a measurable subset $E = \varphi(\mathbb{D})$ of $\mathbb{D}$ such that for all $B \in P_2$, we have*

$$\nu(B) = \varphi(\mu(B)), \tag{1.2}$$

*where $\mu(B) = \alpha/\beta$, given by (1.1), is the usual complex dilatation, and where $\varphi$ further satisfies the following conditions:*

*(A) $|\varphi(z)| = h(|z|)$ for some function $h : [0, 1) \to [0, 1)$ (that is, $|\varphi(z)|$ depends only on $|z|$);*

*(B) we have $h(k) = 0$ if, and only if, $k = 0$;*

*(C) for each $k \in (0, 1)$, we have $\sup\{h(r) : 0 \leq r \leq k\} < 1$;*

*(D) for each $k \in (0, 1)$, we have $\sup\{r \in [0, 1) : 0 \leq h(r) \leq k\} < 1$;*

*(E) the function $\varphi$ has the property that whenever $\kappa : \mathbb{R}^2 \to E = \varphi(\mathbb{D})$ is measurable with $||\kappa||_\infty < 1$, the function $\varphi^{-1} \circ \kappa$ is also measurable.*

*Furthermore, if $\varphi$ has the above properties, then $\nu$ given by (1.2) is a prescribable conformally equivariant dilatation on $P_2$ with values in $\mathbb{R}^2$.*

Since $\varphi$ is one-to-one, we could consider the function $\varphi(\mu(B))$ to be a re-parametrization of the usual complex dilatation $\mu(B)$. We note that $\varphi(\mathbb{D})$ need not be all of $\mathbb{D}$. The function $h$ must be measurable, taking values arbitrarily close to 1, but we need not have $h([0, 1)) = [0, 1)$.

If $\varphi^{-1}$ is Borel measurable, then condition (E) is certainly satisfied.

When prescribing a function only almost everywhere, it would amount to no loss of generality to assume that the function is Borel measurable rather than measurable, since every measurable function agrees almost everywhere with some Borel measurable function. However, instead of assuming that all functions involved are Borel measurable, we prefer to use the naturally arising condition (E).

**Example.** The function $\mu \mapsto 2\mu/(1 + |\mu|^2)$ is a homeomorphism of $\mathbb{D}$ onto itself. If $\mu(B) = \alpha/\beta$ is as above, we have

$$\frac{2\mu(B)}{1 + |\mu(B)|^2} = \frac{b_{11} - b_{22} + 2ib_{12}}{b_{11} + b_{22}}, \tag{1.3}$$

which gives a re-parametrization of the complex dilatation by means of a formula that depends on the matrix elements of $B$ in a simpler manner than $\mu(B)$.

## 2. Preliminary results

We recall some definitions and results from the paper [2] by Bonk, Kleiner, and Merenkov.

**Definition 2.1.** A Schottky set is a subset of $\mathbb{S}^n$ whose complement is the union of at least three disjoint open balls.

We can write a Schottky set in the form

$$S = \mathbb{S}^n \setminus \bigcup_{i\in I} B_i, \tag{2.1}$$

where the sets $B_i$, $i \in I$, are pairwise disjoint open balls in $\mathbb{S}^n$. Here $I$ is an index set, obviously countable. For each $i \in I$, let $R_i : \mathbb{S}^n \to \mathbb{S}^n$ be the reflection in the peripheral sphere $\partial B_i$.

**Definition 2.2.** The subgroup of the group of all Möbius transformations on $\mathbb{S}^n$ generated by the reflections $R_i$, $i \in I$, is called the Schottky group associated with $S$ and is denoted by $\Gamma_S$.

The Schottky group consists of all Möbius transformations $U$ of the form

$$U = R_{i_1} \circ R_{i_2} \circ \cdots \circ R_{i_k},$$

where $k \in \mathbb{N}$ and $i_1, \cdots, i_k \in I$.

**Definition 2.3.** Let $f : X \to Y$ be a homeomorphism between two metric spaces $(X, d_X)$ and $(Y, d_Y)$. The map $f$ is called $\eta$-quasisymmetric, where $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism, if for all distinct $x, y, z \in X$, we have

$$\frac{d_Y\left(f(x), f(y)\right)}{d_Y\left(f(x), f(z)\right)} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right). \tag{2.2}$$

Every Möbius transformation on $\mathbb{S}^n$ is a quasisymmetric map and sends Schottky sets to Schottky sets.

**Definition 2.4.** A Schottky set $S \subset \mathbb{S}^n$ is rigid if every quasisymmetric map of $S$ onto any other Schottky set $S' \subset \mathbb{S}^n$ is the restriction of a Möbius transformation.

**Lemma 2.5.** *Suppose that $U$ is an open subset in $\mathbb{R}^n$ with $0 \in U$, and $f : U \to \mathbb{R}^n$ is a mapping that is differentiable at $0$. If there exists a set $S \subset U$ that has a Lebesgue density point at $0$ such that $f|S = id_S$, then $Df(0) = id_{\mathbb{R}^n}$.*

Lemma 2.5 above is Lemma 7.3 in [2]. We will make essential use of the following result of Bonk, Kleiner, and Merenkov ([2], Theorem 1.3).

**Theorem 2.6.** *For each $n \geq 3$ there exists a Schottky set in $\mathbb{S}^n$ that has positive measure and is rigid.*

## 3. Proof of Theorem 1.2

*Proof.* To get a contradiction, suppose that there exists a prescribable conformally equivariant dilatation $\nu$ in $\mathbb{R}^n$, for some $n \geq 3$. We will be referring to the properties given in Definition 1.1. By Theorem 2.6, there is a rigid Schottky set $S = \mathbb{S}^n \setminus \bigcup_{i\in I} B_i$ in $\mathbb{R}^n$ that has positive measure. Let $G = \Gamma_S$ be the Schottky group associated to $S$.

We write $H = \bigcup_{g\in G} g\left(\bigcup_{i\in I} \partial B_i\right)$ for the $G$−invariant set of zero Lebesgue measure that consists of all images under elements of $G$ of the peripheral spheres of $S$.

Choose a non-singular $n \times n$–matrix $A_0$ with positive determinant such that $A_0$ cannot be written as $rP$, where $r > 0$ and $P$ is an orthogonal matrix. Set $B_0 = A_0^T A_0$. Write $\nu_0 = \nu(B_0)$. Since $A_0 \neq rP$, where $r > 0$ and $P$ is an orthogonal matrix, we cannot write $B_0$ in the form $kI_n$ for some $k > 0$. Thus $\nu_0 = \nu(B_0) \neq 0$ by property (2). Since $x \mapsto A_0x$ is an orientation preserving quasiconformal mapping of $\mathbb{R}^n$, it follows from property (6) that $|\nu_0| < 1$ and $\nu_0 \in E$, where $E$ is a certain measurable subset of the unit ball of $\mathbb{R}^m$.

We define a function $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ as follows, using the set $S$. If $x \in S \setminus H$, we set $\kappa(x) = \nu_0$. Suppose that $x \in \mathbb{R}^n \setminus S$ is such that for all $g \in G$, we have $g(x) \notin S$. Then we set $\kappa(x) = 0$. If $x \in H$, we set $\kappa(x) = 0$. Otherwise, $x \in \mathbb{R}^n \setminus S$ and there is $g \in G \setminus \{id\}$ such that $g(x) \in S \setminus H$. Here $id$ denotes the identity mapping. Then the element $g \in G$ and the point $y = g(x) \in S \setminus H$ are unique by [2, comment after Lemma 5.1, p. 421]. We set $\kappa(x) = \nu(g'(x)^T B_0 g'(x))$. We may write $g'(x) = rP$, where $r > 0$ and $P$ is an orthogonal matrix, both depending on $g$ and $x$. By properties (3) and (4), we have $|\kappa(x)| = |\nu(B_0)| = |\nu_0|$.

Now we have defined $\kappa$ in $\mathbb{R}^n$: it is a measurable function with $\|\kappa\|_\infty < 1$, and the values of $\kappa$ lie in $E$. In fact, $0 < \|\kappa\|_\infty = |\nu(B_0)| < 1$. By property (7), there is an orientation preserving quasiconformal homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu(f'(x)^T f'(x)) = \kappa(x)$ for almost every $x \in \mathbb{R}^n$, and if $F$ is another such mapping, then $F$ is of the form $F = M \circ f$, where $M$ is a Möbius transformation. Let $C$ be a set of measure zero such that $\nu(f'(x)^T f'(x)) = \kappa(x)$ for all $x \in \mathbb{R}^n \setminus C$. Making $C$ larger, if necessary, while retaining the property that $C$ is of zero measure, we may replace $C$ by $\bigcup_{g\in G} g(C)$ without changing notation and assume that if $x \in C$ and $g \in G$, then $g(x) \in C$.

Pick $g \in G \setminus \{id\}$ and set $h = f \circ g$. Write $k(x) = h'(x)^T h'(x)$. We wish to determine the function $\nu(k(x))$, but we only need to do so for almost every $x$. Hence we may assume that $x \notin C$. Furthermore, we may assume that $x \notin H$.

Suppose first that there is $\gamma_1 \in G$ such that $y = \gamma_1(x) \in S \setminus H$. Since $x \notin H$, the element $\gamma_1$ of $G$ is unique. Then $\gamma_2 = \gamma_1 \circ g^{-1} \in G$. For $z = g(x)$, we have $\gamma_2(z) = (\gamma_1 \circ g^{-1})(z) = \gamma_1(g^{-1}(z)) = \gamma_1(x) = y \in S$. Let $Q_1 = g'(x)$, $Q_2 = \gamma_1'(x)$, and $Q_3 = \gamma_2'(z)$. Since $\gamma_1 = \gamma_2 \circ g$, we obtain $\gamma_1'(x) = \gamma_2'(g(x))g'(x) = \gamma_2'(z)g'(x)$. Therefore, we have $Q_2 = Q_3Q_1$, which implies that $Q_3 = Q_2Q_1^{-1}$. Note that since $g, \gamma_1, \gamma_2 \in G$, there are $r_j > 0$ and orthogonal matrices $S_j$ such that $Q_j = r_jS_j$ for $j = 1, 2, 3$. Thus we have $S_3 = S_2S_1^{-1}$ and $S_2 = S_3S_1$.

By the definition of $\kappa$, we have

$$\kappa(x) = \nu(\gamma_1'(x)^T B_0 \gamma_1'(x)) = \nu(Q_2^T B_0 Q_2).$$

Since $\gamma_2 = \gamma_1 \circ g^{-1}$ and $\gamma_2(z) = y \in S \setminus H$, we have

$$\kappa(z) = \nu(\gamma_2'(z)^T B_0 \gamma_2'(z)) = \nu(Q_3^T B_0 Q_3).$$

By property (3), we obtain $\nu(Q_3^T B_0 Q_3) = \nu(S_3^T B_0 S_3)$.

On the other hand, since $x \notin C$, we obtain $z = g(x) \notin C$. Thus $\nu(f'(z)^T f'(z)) = \kappa(z) = \nu(S_3^T B_0 S_3)$. From $h'(x) = f'(g(x))g'(x) = f'(z)g'(x)$, we have

$$k(x) = g'(x)^T f'(z)^T f'(z) g'(x). \tag{3.1}$$

By properties (3) and (5) and the fact that $\nu(f'(z)^T f'(z)) = \nu(S_3^T B_0 S_3)$, we obtain

$$\nu(k(x)) = \nu(S_1^T f'(z)^T f'(z) S_1) = \nu(S_1^T S_3^T B_0 S_3 S_1) = \nu(S_2^T B_0 S_2).$$

Since $\kappa(x) = \nu(Q_2^T B_0 Q_2) = \nu(S_2^T B_0 S_2)$, we find that $\nu(k(x)) = \kappa(x)$.

If there is no $\gamma \in G$ such that $\gamma(x) \in S$, then by the definition of $\kappa$, we have $\kappa(x) = 0$. By properties (3) and (4) and the fact that $g(x) \notin C$, we get

$$\begin{aligned} |\nu(k(x))| &= |\nu(g'(x)^T f'(g(x))^T f'(g(x)) g'(x))| \\ &= |\nu(Q_1^T f'(g(x))^T f'(g(x)) Q_1)| \\ &= |\nu(S_1^T f'(g(x))^T f'(g(x)) S_1)| \\ &= |\nu(f'(g(x))^T f'(g(x)))| \\ &= |\kappa(g(x))|. \end{aligned}$$

Since $g(x)$ is not in $S$, by the definition of $\kappa$, we have $\kappa(g(x)) = 0$. Thus we obtain $\nu(k(x)) = 0 = \kappa(x)$.

We have proved that $\nu(k(x)) = \nu(h'(x)^T h'(x)) = \kappa(x)$ for almost every $x \in \mathbb{R}^n$.

Suppose first that $g$, and hence $f \circ g$, is orientation preserving. Then the function $F = h = f \circ g$ is also a solution to the problem $\nu(F'(x)^T F'(x)) = \kappa(x)$ almost everywhere, so that by property (7), there is a Möbius transformation $M$ such that $f \circ g = M \circ f$.

Suppose then that $g$ is orientation reversing. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define $H(x) = (x_1, x_2, \ldots, x_{n-1}, -x_n)$. Thus $H$ is an orientation reversing Möbius transformation. Define $\varphi = H \circ f \circ g$ so that $\varphi$ is an orientation preserving quasiconformal homeomorphism. Now $H'(x)$ is a symmetric (diagonal) constant matrix with $H'(x)^T H'(x) = I_n$, the identity matrix. Hence $\varphi' = H' f' g'$, where the derivative matrices are evaluated at the appropriate points, so that $\varphi'^T \varphi' = h'^T h'$. It follows that $\varphi$ also satisfies $\nu(\varphi'(x)^T \varphi'(x)) = \kappa(x)$ almost everywhere, so that by property (7) there is a Möbius transformation $M_1$ such that $H \circ f \circ g = \varphi = M_1 \circ f$. Now $M = H^{-1} \circ M_1 (= H \circ M_1)$ is a Möbius transformation such that $f \circ g = M \circ f$.

This associates to each $g \in G$ a Möbius transformation $M$ such that $f \circ g = M \circ f$, and it is clear that the functions $M$ form a group and that the map $g \mapsto M$ is a group isomorphism. The Möbius group $G'$ formed by the maps $M$ is seen to be a Schottky group as follows, using the ideas of [2], Section 7.

It suffices to consider the reflections that generate $G$. Suppose that $g \in G$ is a reflection in a sphere $T$, so $g(x) = x$ for all $x \in T$ and not for any other $x$. Then $f(x) = f(g(x)) = M(f(x))$ for all $x \in T$. Therefore $f(x)$ is a fixed point of $M$. Similarly, it is seen that if $y$ is a fixed point of $M$ and if we write $y = f(x)$, as we may since $f$ is a homeomorphism, then $x = g(x)$ so that $x \in T$. The Möbius transformation $M$ must be orientation reversing so that its fixed point set is a sphere $T'$ in $\mathbb{S}^n$. Hence $f$ maps $T$ onto $T'$. Thus $S' = f(S)$ is a Schottky set. The group $G'$ is now clearly the Schottky group associated with $S'$.

Since $f$ is a quasiconformal mapping on $\mathbb{R}^n$, the mapping $f$ is a quasisymmetric mapping on $S$. Since, by assumption, $S$ is a rigid Schottky set and $f(S)$ is also a Schottky set, it follows that $f|S$ is equal to the restriction of a Möbius transformation to $S$. Since we may replace $f$ by $M \circ f$, where $M$ is a fixed Möbius transformation, we may assume that $f|S$ is the identity mapping of $S$. By Lemma 2.5, it follows that at each Lebesgue density point of $S$ where $f$ is differentiable and has a positive Jacobian determinant, and hence at almost every point $x$ of $S$, we have $f'(x) = I_n$ and therefore $\nu(f'(x)^T f'(x)) = \nu(I_n) = 0$. But at almost every $x$, we have $\nu(f'(x)^T f'(x)) = \kappa(x)$, so that $\kappa(x) = 0$ almost everywhere on $S$. But by construction, at all $x \in S \setminus H$, we have $\kappa(x) = \nu(B_0) \neq 0$, which is a contradiction since $H$ has measure zero. This contradiction proves Theorem 1.2. □

## 4. Proof of Theorem 1.3

*Proof.* Suppose that $n = 2$. Let $\nu : P_2 \to \mathbb{R}^2$ be a prescribable conformally equivariant dilatation. We refer to the properties in Definition 1.1. The set $E = \nu(P_2)$ is measurable by assumption.

We identify the set $P_2$ of positive definite $2 \times 2$–matrices $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with the subset $\mathcal{A} = \{(a, b, c) \in \mathbb{R}^3 : a > 0,\ c > 0,\ ac > b^2\}$ of $\mathbb{R}^3$ and equip $P_2$ with the topological and measure space structure that $\mathcal{A}$ inherits as a subset of $\mathbb{R}^3$. The function $(a, b, c) \mapsto \mathcal{F}(a, b, c) = (a/c, b/c, c)$ is a homeomorphism of $\mathcal{A}$ onto $\mathcal{B} = \{(a, b, c) \in \mathbb{R}^3 : c > 0,\ a > b^2\}$. The subset $\mathcal{C}$ of $\mathcal{A}$ is measurable if, and only if, the subset $\mathcal{F}(\mathcal{C})$ of $\mathcal{B}$ is measurable.

Let $A_1$ and $A_2$ be $2 \times 2$–matrices with a positive determinant so that $f_j(z) = A_j z$ (where the complex number $z = \begin{bmatrix} x \\ y \end{bmatrix}$ is viewed as a column vector) is an affine mapping, for $j = 1, 2$. Write $B_j = A_j^T A_j$ and let $\mu_j = \mu(B_j)$ be the usual complex dilatation of the mapping $f_j$ as given by (1.1). Suppose that $\mu_1 \neq \mu_2$. If $\nu(B_1) = \nu(B_2)$, then let $\kappa : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\kappa(z) \equiv \nu(B_1)$ be a constant function in $\mathbb{R}^2$. Now each of $f = f_1$ and $f = f_2$ is a solution to the equation $\nu(f'(z)^T f'(z)) = \kappa(z)$ in $\mathbb{R}^2$. By property (7), it must be the case that $f_2 = g \circ f_1$ for some Möbius transformation $g$, but this cannot be the case since $\mu_1 \neq \mu_2$. It follows that if $\mu_1 \neq \mu_2$, then $\nu(B_1) \neq \nu(B_2)$.

Suppose next that $\mu_1 = \mu_2$. Then, by property (i) of the complex dilatation, we have $f_2 = g \circ f_1$ for some Möbius transformation $g$, and since $f_1$ and $f_2$ fix the point at infinity, it follows that $g$ is of the form $g(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. Hence $A_2 = rPA_1$, where $P$ is an orthogonal matrix with $\det P = 1$ and $r > 0$. It follows that $B_1 = r^2 B_2$, so that by property (3), we have $\nu(B_1) = \nu(B_2)$.

This implies that we can represent $\nu$ as a function of the complex dilatation: $\nu(B) = \varphi(\mu(B))$, where $\varphi$ is a one-to-one function of $\mathbb{D}$ into (but not necessarily onto) $\mathbb{D}$. We have $\varphi(\mathbb{D}) = \nu(P_2) = E$.

We next show that the function $\varphi$ is measurable. If $(a, b, c), (a', b', c') \in \mathcal{A}$, then by (1.3), the matrices corresponding to $(a, b, c)$ and $(a', b', c')$ have the same complex dilatation if, and only if, $(a-c)/(a+c) = (a'-c')/(a'+c')$ and $b/(a+c) = b'/(a'+c')$. Thus the complex dilatation is constant exactly on each curve of the form

$$\{(a, b, c) \in \mathcal{A} : (a - c)/(a + c) = \alpha_1,\ b/(a + c) = \alpha_2\}$$

for real constants $\alpha_1$ and $\alpha_2$, or equivalently on straight half-lines of the form

$$L(\alpha_1, \alpha_2) = \{(a, b, c) \in \mathcal{A} : a/c = \alpha_1,\ b/c = \alpha_2\} = \{(\alpha_1 c, \alpha_2 c, c) : c > 0\},$$

where now $\alpha_1 > 0$ and $\alpha_2 \in \mathbb{R}$ with $\alpha_2^2 < \alpha_1$ since $b^2/c^2 < (ac)/c^2 = a/c$. These half-lines are disjoint for distinct pairs $(\alpha_1, \alpha_2)$, and $\mathcal{A}$ is the disjoint union of such half-lines. Thus $\mathcal{F}(L(\alpha_1, \alpha_2)) = \{(\alpha_1, \alpha_2, c) : c > 0\} \subset \mathcal{B}$.

Let $U$ be an open subset of $\mathbb{D}$. By assumption, the set $\nu^{-1}(U)$ is a measurable subset of $\mathcal{A}$ and hence a measurable subset of $\mathbb{R}^3$. Also, $\nu^{-1}(U)$ is the union of sets of the form $L(\alpha_1, \alpha_2)$.

The homeomorphisms $z \mapsto F_1(z) = (2z)/(1+|z|^2)$ as in (1.3) of $\mathbb{D}$ onto itself and $z = x+iy \mapsto F_2(z) = (x+1+iy)/(1-x)$ of $\mathbb{D}$ onto $\mathcal{D} = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2 : \alpha_2^2 < \alpha_1\} \subset \mathbb{R}^2$ preserve measurable sets. We write $F_3 = F_2 \circ F_1$. With $a/c = \alpha_1$ and $b/c = \alpha_2$, where $\alpha_2^2 < \alpha_1$, we can write $F_2((a-c+2ib)/(a+c)) = (\alpha_1, \alpha_2)$. To prove that $\varphi^{-1}(U)$ is measurable, it suffices to show that $F_3(\varphi^{-1}(U))$ is measurable.

Now $F_3(\varphi^{-1}(U))$ is the set of those pairs $(\alpha_1, \alpha_2)$ such that $L(\alpha_1, \alpha_2) \subset \nu^{-1}(U)$, that is,

$$\begin{aligned} F_3(\varphi^{-1}(U)) &= \big\{(a/c, b/c) : (a,b,c) \in \nu^{-1}(U)\big\} \\ &= \{(a', b') : (a', b', c) \in \mathcal{F}(\nu^{-1}(U))\}. \end{aligned}$$

The set $\mathcal{F}(\nu^{-1}(U))$ is a measurable subset of $\mathcal{B}$. By Tonelli's theorem ([4], p. 309) applied to the characteristic function of $\mathcal{F}(\nu^{-1}(U))$ on $\mathcal{D} \times \mathbb{R}_+$, for almost every fixed $c > 0$, the subset $\{(a', b') : (a', b', c) \in \mathcal{F}(\nu^{-1}(U))\}$ of $\mathbb{R}^2$ is measurable, and since this subset is equal to $F_3(\varphi^{-1}(U))$ for every $c > 0$, we see that $F_3(\varphi^{-1}(U))$, and hence $\varphi^{-1}(U)$, is measurable. This proves that $\varphi$ is a measurable function.

Suppose that $B$ has eigenvalues $\lambda_1$ and $\lambda_2$, where $0 < \lambda_2 \leq \lambda_1$, and write $\lambda = \lambda_1/\lambda_2 \geq 1$ and $k = (\sqrt{\lambda} - 1)/(\sqrt{\lambda} + 1) \in [0,1)$. Then $k = |\mu(B)|$. By properties (3) and (4), we have $|\nu(B)| = |\nu(D)|$, where $D = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$, so we may write $|\nu(B)| = h(k) \geq 0$ for $0 \leq k < 1$. By property (2), we have $h(k) = 0$ if, and only if, $k = 0$, and we have $h(k) < 1$ for all $k < 1$ since the values of $\nu$ lie in $\mathbb{D}$ by assumption. This proves (A) and (B).

Fix $k \in [0,1)$. Suppose that $0 \leq r \leq k$. Let $A$ be a $2 \times 2 - -$matrix with real entries and a positive determinant. Let the affine self-map $f$ of $\mathbb{R}^2$ be given by $f(x) = Ax$, set $B = A^T A$, and suppose that $|\mu(B)| = r$. Then $f$ is a $K-$quasiconformal self-homeomorphism of $\mathbb{R}^2$, where (regardless of $r$) we may take $K = (1+k)/(1-k)$. By property (6) (where we denote by $c$ what is denoted by $k$ in property (6)), it follows that there is a number $c \in [0,1)$ depending only on $K$ (and the function $\nu$ as a whole) such that $|\nu(B)| \leq c$. Now $|\nu(B)| = h(r)$. Thus $\sup\{h(r) : 0 \leq r \leq k\} \leq c < 1$. This proves (C).

Consider property (D). Again fix $k \in [0,1)$. To get a contradiction, suppose that $\sup\{r \in [0,1) : 0 \leq h(r) \leq k\} = 1$. Then there is a sequence $z_j \in \mathbb{D}$ such that $|z_j| \to 1$ as $j \to \infty$, while $|\varphi(z_j)| = h(|z_j|) \leq k$ for all $j$. Define the function $\kappa : \mathbb{R}^2 \to \mathbb{D}$ by $\kappa(z) = \varphi(z_j)$ whenever $j \geq 1$ and $j < |z| \leq j+1$, and $\kappa(z) = 0$ when $|z| \leq 1$. Then $\kappa$ is measurable and $||\kappa||_\infty \leq k$. By property (7), there is a quasiconformal self-homeomorphism $f$ of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^2$ with $|x| > 1$, we have

$$\varphi(\mu(f'(x)^T f'(x))) = \nu(f'(x)^T f'(x)) = \kappa(x) = \varphi(z_j),$$

where $j$ is obtained from the condition $j < |x| \leq j+1$. Since $\varphi$ is one-to-one, this means that $\mu(f'(x)^T f'(x)) = z_j$. But since $|z_j| \to 1$, it follows that $f$ is not quasiconformal in the entire plane, which is a contradiction. This proves (D).

To prove (E), suppose that $\kappa : \mathbb{R}^2 \to E$ is measurable with $||\kappa||_\infty < 1$. By assumption, there is a quasiconformal mapping $f$ of the plane such that $\nu_f(x) = \kappa(x)$ for a.e. $x$, where $\nu_f(x)$ denotes $\nu(f'^T(x) f'(x))$. Since $\nu(B) = \varphi(\mu(B))$, and since $\mu(f'^T(x) f'(x))$ is the usual complex dilatation $\mu_f(x)$ of $f$ at $x$, it follows that $\mu_f(x) = (\varphi^{-1} \circ \nu_f)(x) = (\varphi^{-1} \circ \kappa)(x)$ for a.e. $x$. Since the function $\mu_f$ is measurable, it follows that $\varphi^{-1} \circ \kappa$ is measurable. This proves (E).

We have now shown that if $\nu$ is a prescribable conformally equivariant dilatation, then $\nu$ is of the form claimed in Theorem 1.3.

Conversely, suppose that $\nu$ is of the form stated in Theorem 1.3 and let $E = \nu(P_2) = \varphi(\mathbb{D}) \subset \mathbb{D}$ be measurable. In particular, we are assuming that both $\nu$ and $\varphi$ are measurable and that $\varphi$ is a bijection. We need to prove that $\nu$ has the properties of a prescribable conformally equivariant dilatation.

It is immediate that properties (1), (2), and (3) hold.

If $P$ is an orthogonal matrix, then the complex dilatations of $B$ and $P^TBP$ have equal modulus, so that (A) implies that property (4) holds.

Suppose that $\nu(B_1) = \nu(B_2)$, where $B_1, B_2 \in P_2$. Since $\varphi$ is one-to-one, it follows that $\mu(B_1) = \mu(B_2)$. Let $P$ be an orthogonal $2\times 2$–matrix. If $\det P = 1$, we may write $P$ in the form $P = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for some real $\theta$, and a calculation shows that $\mu(P^TBP) = e^{2i\theta}\mu(B)$. If $\det P = -1$, we may write $P$ in the form $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$, and then $\mu(P^TBP) = \overline{e^{2i\theta}\mu(B)}$. In both cases, it follows from $\mu(B_1) = \mu(B_2)$ that $\mu(P^TB_1P) = \mu(P^TB_2P)$, and hence $\nu(P^TB_1P) = \nu(P^TB_2P)$. So property (5) holds.

Let $f$ be a $K-$quasiconformal mapping in $\mathbb{R}^2$. Set $k = \sup\{h(r) : 0 \leq r \leq (K-1)/(K+1)\} < 1$. Then $k$ depends only on $K$ and on the function $h$ (hence on the function $\nu$). Further we then have $|\nu(f'(z)^Tf'(z))| \leq k$ for almost every $z \in \mathbb{R}^2$. Thus property (6) holds.

Finally, consider property (7). Let $\kappa : \mathbb{R}^2 \to E = \varphi(\mathbb{D})$ be a measurable function with $|\kappa(x)| \leq k_1 < 1$ for almost every $x$. By assumption (D), there is a number $k_2 \in [0,1)$ such that $\sup\{|\mu(B)| : |\nu(B)| = |\varphi(\mu(B))| \leq k_1\} \leq k_2 < 1$.

We define $\tilde{\mu} = \varphi^{-1} \circ \kappa$ so that $\tilde{\mu}$ is measurable by assumption (E). By the above, $||\tilde{\mu}||_\infty \leq k_2 < 1$. By the measurable Riemann mapping theorem, there is a quasiconformal homeomorphism $f$ of $\mathbb{R}^2$ onto itself whose complex dilatation $\mu_f$ satisfies $\mu_f(x) = \tilde{\mu}(x)$ for a.e. $x$. But $\mu_f(x) = \mu(f'^T(x)f'(x))$ so that

$$\nu(f'^T(x)f'(x)) = \varphi(\mu(f'^T(x)f'(x))) = \varphi(\tilde{\mu}(x)) = \kappa(x)$$

for a.e. $x$. Furthermore, if $g$ also is a quasiconformal (orientation preserving, homeomorphic) self-mapping of $\mathbb{R}^2$ such that $\nu(g'(x)^Tg'(x)) = \kappa(x)$ for almost every $x \in \mathbb{R}^2$, it follows from this and from the facts that $\nu(g'(x)^Tg'(x)) = \varphi(\mu(g'(x)^Tg'(x)))$ and

$$\varphi(\mu(g'(x)^Tg'(x))) = \kappa(x) = \varphi(\mu(f'(x)^Tf'(x))),$$

while $\varphi$ is one-to-one, that $\mu(f'(x)^Tf'(x)) = \mu(g'(x)^Tg'(x))$ for almost every $x \in \mathbb{R}^2$. By the uniqueness part of the measurable Riemann mapping theorem, we see that $g = M \circ f$ for some Möbius transformation $M$. This then proves property (7).

This completes the proof of Theorem 1.3. □

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