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A NON-UNITAL *-ALGEBRA HAS UC^*NP IF AND ONLY IF ITS UNITIZATION HAS UC^*NP

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Abstract. The result stated in the title is proved, thereby disproving the result shown in a 1983 paper by B. A. Barnes (Theorem 4.1).

1. Introduction

Let A be a non-unital algebra such that $a \in A$ and $aA = \{0\}$ or $Aa = \{0\}$ implies $a = 0$. Let $A_e = \{a + \lambda e : a \in A, \lambda \in \mathbb{C}\}$ be the unitization of A with the unit element denoted by e. For an (algebra) norm $\| \cdot \|$ on A, define the algebra norms on A_e as

$$\| a + \lambda e \|_{op} := \sup\{\| ab + \lambda b \| : b \in A, \| b \| \leq 1\} \quad \text{and} \quad \| a + \lambda e \|_1 := \| a \| + |\lambda|$$

for all $a + \lambda e \in A_e$. We must note that throughout this paper, no norm on A is assumed to be complete.

A C^**-norm* is a norm $\| \cdot \|$ on a $*$-algebra A such that $\| a^*a \| = \| a \|^2$ $(a \in A)$. The $*$-algebra A has *unique C^*-norm property (UC^*NP)* if A admits exactly one C^*-norm. The UC^*NP and the twin property of $*$-regularity were discovered by Barnes [2]. They are of significance in harmonic analysis [6, Section 10.5] and have inspired the study of unique uniform norm property (UUNP) in Banach algebras [5, Section 4.6]. For a non-unital, commutative Banach $*$-algebra A, Dabhi and Dedania have proved [4, Corollary 2.3(ii)] that A has UC^*NP iff A_e has UC^*NP. Here we prove the same for any non-unital $*$-algebra, not necessarily commutative. As a result, Theorem 4.1 in [2] is false.

2. Results

Suppose that $\| \cdot \|$ is a unital norm on the algebra A_e (i.e., $\| e \| = 1$). Then $\| a + \lambda e \|_{op} \leq \| a + \lambda e \| \leq \| a + \lambda e \|_1$ $(a + \lambda e \in A_e)$. Theorem 2.1 below implies that the norm $\| \cdot \|$ is equivalent to either $\| \cdot \|_{op}$ or $\| \cdot \|_1$. This theorem is inspired by [1].

Theorem 2.1. *Let $\| \cdot \|$ be a unital norm on A_e.*

(1) *If A is closed in $(A_e, \| \cdot \|)$, then $\| a + \lambda e \|_1 \leq 3 \| a + \lambda e \|$ $(a + \lambda e \in A_e)$.*
(2) *If A is dense in $(A_e, \| \cdot \|)$, then $\| a + \lambda e \|_{op} = \| a + \lambda e \|$ $(a + \lambda e \in A_e)$.*

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Proof. (1) Let $a+\lambda e \in A_e$. We may assume that $\| a+\lambda e \|_1 = 1$. First suppose that $|\lambda| \leq 1/3$. Then $\| a \| = 1-|\lambda| \geq 2/3$. So $\| a+\lambda e \| \geq \| a \| - |\lambda| \geq 2/3-1/3 = 1/3$. Hence $\| a+\lambda e \|_1 \leq 3 \| a+\lambda e \|$. Secondly, suppose that $|\lambda| > 1/3$. Since A is closed in $(A_e, \| \cdot \|)$, the multiplicative linear functional $\varphi_\infty(a+\lambda e) := \lambda \ (a+\lambda e \in A_e)$ is $\| \cdot \|$-continuous. Therefore $1 = |\varphi_\infty(e)| = |\varphi_\infty(b-e)| \leq \| b-e \| \ \ (b \in A)$. In particular, $1/3 \leq |\lambda| \leq |\lambda| \ \| -\frac{a}{\lambda} - e \| = \| a+\lambda e \|$. Thus $\| a+\lambda e \|_1 \leq 3 \| a+\lambda e \|$.

(2) Let $a+\lambda e \in A_e$. Since A is dense in $(A_e, \| \cdot \|)$, there exists a sequence $\{c_n\}$ in A such that $\| c_n \| = 1$ $(n \in \mathbb{N})$ and $c_n \longrightarrow e$ in $\| \cdot \|$ as $n \longrightarrow \infty$. Then

$$\begin{aligned} \| a+\lambda e \|_{op} &\leq \| a+\lambda e \| = \| (a+\lambda e)e \| = \lim_{n\to\infty} \| (a+\lambda e)c_n \| \\ &\leq \sup\{\| (a+\lambda e)b \| : b \in A; \| b \| \leq 1\} = \| a+\lambda e \|_{op} . \end{aligned}$$

Thus $\| a+\lambda e \|_{op} = \| a+\lambda e \|$. □

Corollary 2.2. *Let A be a non-unital algebra.*

(1) *Any norm $\| \cdot \|$ on A_e is equivalent to either $\| \cdot \|_{op}$ or $\| \cdot \|_1$.*
(2) [1, Corollary 2] *Let $\| \cdot \|$ be a complete norm on A such that $\| a \|_{op} = \| a \|$ $(a \in A)$. Then $\| a+\lambda e \|_1 \leq 3 \| a+\lambda e \|_{op}$ $(a+\lambda e \in A_e)$.*

Proof. (1) Without loss of generality, we may assume that $\| \cdot \|$ is unital. Now this is immediate from Theorem 2.1.

(2) Let $| \cdot | = \| \cdot \|_{op}$ on A_e. Then $| \cdot |$ is a unital norm on A_e. Since $\| \cdot \|$ is a complete norm on A, $| \cdot | = \| \cdot \|_{op}$ is complete on A_e, so A is closed in $(A_e, | \cdot |)$. So, by Theorem 2.1(1), $|a+\lambda e|_1 \leq 3|a+\lambda e| (a+\lambda e \in A_e)$. Hence

$$\begin{aligned} \| a+\lambda e \|_1 &= \| a \| + |\lambda| = \| a \|_{op} + |\lambda| \ \text{(by hypothesis)} \\ &= |a| + |\lambda| = |a+\lambda|_1 \leq 3|a+\lambda e| = 3 \| a+\lambda e \|_{op} . \end{aligned}$$

This proves (2). □

Let A be a non-unital $*$-algebra with UC^*NP. In Lemma 2.3, we show that A_e cannot have more than two C^*-norms. Then in Theorem 2.6, we prove that, in fact, A_e must have UC^*NP.

Lemma 2.3. *Let A be a non-unital $*$-algebra with UC^*NP. Then A_e has at most two C^*-norms.*

Proof. Assume that A has UC^*NP. Let $\| \cdot \|$ be the unique C^*-norm on A. Then $\| \cdot \|_{op}$ is a C^*-norm on A_e due to [3, Proposition 2.2(b)]. First we *claim* that $\| \cdot \|_{op}$ is the minimum C^*-norm on A_e. If $| \cdot |$ is a C^*-norm on A_e, then it is also a C^*-norm on A. So, by the hypothesis, $| \cdot | = \| \cdot \|$ on A. Hence, $\| \cdot \|_{op} = | \cdot |_{op} \leq | \cdot |$ on A_e. This proves our claim.

Now let $||| \cdot |||$ be a C^*-norm on A_e other than $\| \cdot \|_{op}$. Because any two equivalent C^*-norms are identical, it is enough to show that $||| \cdot ||| \cong \| \cdot \|_1$ on A_e. Since A has UC^*NP, $||| \cdot ||| = \| \cdot \|$ on A. Hence $||| \cdot |||_{op} = \| \cdot \|_{op}$ on A_e. Now A must be closed in $(A_e, ||| \cdot |||)$. Otherwise, by Theorem 2.1(2), $||| \cdot ||| = ||| \cdot |||_{op}$ on A_e, and so $||| \cdot ||| = ||| \cdot |||_{op} = \| \cdot \|_{op}$ on A_e, which is a contradiction. Then, by Theorem 2.1(1),

$$\begin{aligned} |||a+\lambda e||| &\leq |||a||| + |\lambda| = \| a \| + |\lambda| = \| a+\lambda e \|_1 = \| a \| + |\lambda| \\ &= |||a||| + |\lambda| = |||a+\lambda e|||_1 \leq 3|||a+\lambda e|||. \end{aligned}$$

This completes the proof. □

Lemma 2.4. *Let A be a non-unital $*$-algebra. Let $\|\cdot\|$ be a C^*-norm on A. Let $(C^*(A), \|\cdot\|^\sim)$ be the completion of $(A, \|\cdot\|)$. If $(C^*(A), \|\cdot\|^\sim)$ contains the identity, then $\|\cdot\|_{op} = \|\cdot\|^\sim$ on A_e.*

Proof. Clearly, $A_e \subset C^*(A)$. Let $a + \lambda e \in A_e \subset C^*(A)$ be non-zero. Then there exists a sequence (a_n) in A such that $a_n \longrightarrow a + \lambda e$ in $\|\cdot\|^\sim$. Let $b \in A$ be such that $\| b \| \leq 1$. Then $a_n b \longrightarrow ab + \lambda b$ in $\|\cdot\|^\sim = \|\cdot\|$. So

$$\| ab + \lambda b \| = \lim_{n\to\infty} \| a_n b \| \leq \lim_{n\to\infty} \| a_n \| = \lim_{n\to\infty} \| a_n \|^\sim = \| a + \lambda e \|^\sim .$$

Hence $\| a + \lambda e \|_{op} \leq \| a + \lambda e \|^\sim$. For the reverse inequality, consider a sequence (c_n) in A such that $\| c_n \| \leq 1$ and $c_n \longrightarrow \frac{(a+\lambda e)^*}{\|a+\lambda e\|}$ in $\|\cdot\|^\sim$. Then

$$\begin{aligned} \| a + \lambda e \|_{op} &\geq \sup_n \| (a+\lambda e) c_n \| \geq \lim_{n\to\infty} \| (a+\lambda e) c_n \| = \lim_{n\to\infty} \| (a+\lambda e) c_n \|^\sim \\ &= \frac{\| (a+\lambda e)(a+\lambda e)^* \|^\sim}{\| a + \lambda e \|^\sim} = \| a + \lambda e \|^\sim . \end{aligned}$$

Thus $\| a + \lambda e \|_{op} \geq \| a + \lambda e \|^\sim$. $\square$

Lemma 2.5. *Let A be a non-unital $*$-algebra. Let $\|\cdot\|$ be a C^*-norm on A. Let $(C^*(A), \|\cdot\|^\sim)$ be the completion of $(A, \|\cdot\|)$. If $(C^*(A), \|\cdot\|^\sim)$ does not contain the identity, then $\|\cdot\|_{op} = \|\cdot\|^\sim_{op}$ on A_e.*

Proof. Since $A \subset C^*(A)$, we have $A_e \subset C^*(A)_e$. Let $a + \lambda e \in A_e$. Then

$$\begin{aligned} \| a + \lambda e \|_{op} &= \sup\{\| (a+\lambda e) b \| : b \in A; \| b \| \leq 1\} \\ &= \sup\{\| (a+\lambda e) b \|^\sim : b \in A; \| b \|^\sim \leq 1\} \\ &\leq \sup\{\| (a+\lambda e) b \|^\sim : b \in C^*(A); \| b \|^\sim \leq 1\} \\ &= \| a + \lambda e \|^\sim_{op} . \end{aligned}$$

For the reverse inequality, let $b \in C^*(A)$ be such that $\| b \|^\sim \leq 1$. Then there exists a sequence (b_n) in A such that $\| b_n \| \leq 1$ and $b_n \longrightarrow b$ in $\|\cdot\|^\sim$. So

$$\begin{aligned} \| (a+\lambda e) b \|^\sim &= \lim_{n\to\infty} \| (a+\lambda e) b_n \|^\sim = \lim_{n\to\infty} \| (a+\lambda e) b_n \| \\ &\leq \sup_n \| (a+\lambda e) b_n \| \leq \| a + \lambda e \|_{op} . \end{aligned}$$

Thus $\| a + \lambda e \|^\sim_{op} \leq \| a + \lambda e \|_{op}$. $\square$

The next result disproves [2, Theorem 4.1]. The gap in that proof lies in the first line. It is claimed that $C^*(A)$ can be identified with a closed maximal ideal of $C^*(A_e)$ of codimension one. But this is not true. In [2, Example 4.4], we have $C^*(A_e) = C^*(A)$.

Theorem 2.6. *A non-unital $*$-algebra A has UC^*NP iff A_e has UC^*NP.*

Proof. Let A have UC^*NP. Then, by Lemma 2.3, A_e has at most two C^*-norms. Also, by the first paragraph in the proof of Lemma 2.3, A_e has a minimum C^*-norm. Let $\|\cdot\|$ and $|||\cdot|||$ be these two C^*-norms on A_e with $\|\cdot\| \leq |||\cdot|||$. Note that $\|\cdot\|_{op} = \|\cdot\|$ on A_e. Since A has UC^*NP, $\|\cdot\| = |||\cdot|||$ on A. Let $(C^*(A), \|\cdot\|^\sim)$ and $(C^*(A_e), |||\cdot|||^\sim)$ be the completions of $(A, \|\cdot\|)$ and $(A_e, |||\cdot|||)$, respectively.

We embed $C^*(A)$ into $C^*(A_e)$ as a $*$-subalgebra. Let $a \in C^*(A)$. Then there exists a sequence (a_n) in A converging to a in $\|\cdot\|^\sim$. Hence (a_n) is a Cauchy sequence in $\|\cdot\|=\|\cdot\|^{\sim}$. So (a_n) is a Cauchy sequence in $|||\cdot|||$ because we have $\|\cdot\|=|||\cdot|||$ on A. Thus it is a Cauchy sequence in $(A_e,|||\cdot|||)$ and so it converges to some b in $C^*(A_e)$. Hence the map $j:C^*(A)\to C^*(A_e)$ defined by $j(a)=b$ is a well defined, one-to-one $*$-homomorphism. Thus we have $C^*(A)\hookrightarrow C^*(A_e)$.

Assume that $C^*(A)$ has identity. Then $A_e\hookrightarrow C^*(A)\hookrightarrow C^*(A_e)$. Since every C^*-algebra has UC^*NP, we have $\|\cdot\|^{\sim}=|||\cdot|||^{\sim}$ on $C^*(A)$. So, by Lemma 2.4, we have $\|\cdot\|=\|\cdot\|_{op}=\|\cdot\|^{\sim}=|||\cdot|||^{\sim}=|||\cdot|||$ on A_e. Now, assume that $C^*(A)$ does not have the identity. Then $C^*(A)\hookrightarrow C^*(A)_e\hookrightarrow C^*(A_e)$. Since $C^*(A)_e$ is also a C^*-algebra with $\|\cdot\|^{\sim}_{op}$, we have $\|\cdot\|^{\sim}_{op}=|||\cdot|||^{\sim}$ on $C^*(A)_e$ and hence on A_e because $A_e\hookrightarrow C^*(A)_e$. By Lemma 2.5, $\|\cdot\|^{\sim}_{op}=\|\cdot\|_{op}$ on A_e. Therefore $\|\cdot\|=\|\cdot\|_{op}=\|\cdot\|^{\sim}_{op}=|||\cdot|||^{\sim}=|||\cdot|||$ on A_e. Thus A_e has UC^*NP.

Conversely, let A_e have UC^*NP. Since A is a $*$-ideal in A_e, A has UC^*NP due to [2, Theorem 2.2(1)]. □

A norm $\|\cdot\|$ on an algebra A is *spectral* if $r_A(a)\leq\|a\|\quad(a\in A)$, where $r_A(\cdot)$ is the spectral radius on A. The algebra A has *Spectral Extension Property* (SEP) if every norm on A is spectral. This property arose in the investigation of incomplete algebra norms on Banach algebras [5, Section 4.5]. The next result is in the direction of a non-commutative analogue of [3, Corollary 3.2].

Theorem 2.7. *Let A be a semisimple, non-unital algebra.*

(1) If A_e has SEP, then A has SEP and A is closed in every norm on A_e.

(2) If A has SEP and if A is closed in every norm on A_e, then A_e has SEP.

Proof. (1) Let $\|\cdot\|$ be any norm on A. Then, by the hypothesis, $\|\cdot\|_1$ is a spectral norm on A_e so that $r_{A_e}(a+\lambda e)\leq\|a+\lambda e\|_1\quad(a+\lambda e\in A_e)$, and hence $r_A(a)=r_{A_e}(a)\leq\|a\|\quad(a\in A)$. Thus A has SEP. Now let $\|\cdot\|$ be any norm on A_e. Since A_e has SEP, $\|\cdot\|$ is a spectral norm on A_e. Define $\varphi_\infty(a+\lambda e):=\lambda\ (a+\lambda e\in A_e)$. Then φ_∞ is a multiplicative linear functional on A_e and $\ker\varphi_\infty=A$. Since $\|\cdot\|$ is a spectral norm on A_e, φ_∞ is $\|\cdot\|$-continuous. Hence $A=\ker\varphi_\infty$ is closed in $(A_e,\|\cdot\|)$.

(2) Let A have SEP. Let $|\cdot|$ be any norm on A_e. It is enough to show that

$$r_{A_e}(a+\lambda e)\leq|a+\lambda e|_{op}\quad(a+\lambda e\in A_e).$$

Set $\|\cdot\|=|\cdot|_{op}$ on A_e. Then $\|\cdot\|$ is a unital norm on A_e. By the hypothesis, A is closed in $(A_e,\|\cdot\|)$. Hence, by Theorem 2.1(1), we have

$$\|(a+\lambda e)\|_1\leq 3\|a+\lambda e\|\quad(a+\lambda e\in A_e).$$

Therefore,

$$r_{A_e}(a+\lambda e)\leq r_{A_e}(a)+|\lambda|=r_A(a)+|\lambda|\leq\|a\|+|\lambda|=\|a+\lambda e\|_1\leq 3\|a+\lambda e\|.$$

Since $r_{A_e}((a+\lambda e)^n)=r_{A_e}(a+\lambda e)^n$ and $\|(a+\lambda e)^n\|\leq\|a+\lambda e\|^n$ for all $n\in\mathbb{N}$, we get $r_{A_e}(a+\lambda e)\leq\|a+\lambda e\|=|a+\lambda e|_{op}\quad(a+\lambda e\in A_e)$. □

Remark 2.8. (1) Even if A is a non-unital $*$-algebra with UC^*NP, the C^*-algebra $C^*(A)$ may have the identity. For example, let $D:=\{(x,y,0)\in\mathbb{R}^3:x^2+y^2\leq 1\}$, let $R:=\{(x,0,t)\in\mathbb{R}^3:-1\leq x\leq 1,\ 0\leq t\leq 1\}$, and let $\Omega:=D\cup R$. Let A be the algebra of all continuous complex-valued functions f on Ω such that f is analytic on the interior of D and $f(0,1,0)=f(0,-1,0)=0$. For $f\in A$, define $f^*(x,y,t)=\overline{f(x,-y,t)}\ ((x,y,t)\in\mathbb{R}^3)$. Then A is a non-unital, commutative Banach $*$-algebra with UC^*NP. As in [2, Example 4.4], the $C^*(A)$ has identity.

(2) We do not know whether the assumption that "A is closed in every norm on A_e" in Theorem 2.7(2) can be omitted.

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