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ON HARMONIC NON-COMMUTATIVE L^p-OPERATORS ON LOCALLY COMPACT QUANTUM GROUPS

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Abstract. For a locally compact quantum group $\mathbb{G}$ with tracial Haar weight φ and a quantum measure μ on $\mathbb{G}$, we study the space $\mathcal{H}^p_\mu(\mathbb{G})$ of μ-harmonic operators in the non-commutative L^p-space $\mathcal{L}^p(\mathbb{G})$ associated to the Haar weight φ. The main result states that if μ is non-degenerate, then $\mathcal{H}^p_\mu(\mathbb{G})$ is trivial for all $1 \leq p < \infty$.

1. Introduction and preliminaries

Non-commutative Poisson boundaries of (discrete) quantum groups $\mathbb{G}$ was first introduced and studied by Izumi in [6]. Motivated by the classical setting, in fact, he defined the Poisson boundary of $\mathbb{G}$ associated to a 'quantum measure' μ as the space of μ-harmonic 'functions', i.e., the fixed point space of the Markov operator associated to μ. For discrete quantum groups, this was further studied by several authors (cf. [7], [14], [15]). Poisson boundaries in the locally compact quantum group setting was studied by Neufang, Ruan and the author in [9]. Quantum versions of several important classical results regarding harmonic functions were proved there. In particular, triviality of special classes of harmonic functions, such as $\mathcal{C}_0$-functions, was proved.

Another important fact regarding classical harmonic functions on locally compact groups is that for $1 \leq p < \infty$, any L^p-harmonic function associated to an adapted probability measure is trivial. The main result of this paper is a quantum version of this result. But, in order to talk about μ-harmonic elements in the non-commutative L^p-spaces, we first need to define the convolution action by μ on such spaces.

In his PhD thesis [4], Cooney studied the non-commutative L^p-spaces associated to the Haar weight φ of a locally compact quantum group $\mathbb{G}$. He mainly considered Haagerup's version and could prove that in the Kac algebra setting, the convolution action of an 'absolutely continuous quantum measure' can be extended to the Haagerup non-commutative L^p-spaces. So, we cannot consider harmonic operators in the general setting of all locally compact quantum groups. Moreover, in the case of non-tracial φ, there are different ways to define the non-commutative L^p-spaces. Although all these spaces are isometrically isomorphic as Banach spaces, the identifications are not necessarily compatible with the quantum group structure, so it is not clear whether the space of μ-harmonic L^p-operators is the same, as a Banach space, for all different definitions of non-commutative L^p-spaces.

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Therefore, in this paper, instead of restricting ourselves to the Kac algebra setting, we consider locally compact quantum groups $\mathbb{G}$ whose Haar weight φ is a trace. In this case, the convolution action is extended to the non-commutative L^p-spaces, and the main result of the paper states that in the case of a non-degenerate quantum measure μ, for $1 \leq p < \infty$, any μ-harmonic element which lies in the non-commutative L^p-space of φ is trivial.

First, let us introduce our terminology and recall some results on locally compact quantum groups which we will be using in this paper. For more details, we refer the reader to [11].

A *locally compact quantum group* $\mathbb{G}$ is a quadruple $(M, \Gamma, \varphi, \psi)$, where M is a von Neumann algebra with a co-associative co-multiplication $\Gamma : M \to M\bar{\otimes}M$, and φ and ψ are (normal faithful semi-finite) left and right Haar weights on M, respectively. We write $\mathfrak{M}_\varphi^+ = \{x \in M^+ : \varphi(x) < \infty\}$ and $\mathfrak{N}_\varphi = \{x \in M^+ : \varphi(x^*x) < \infty\}$, and we denote by Λ_φ the inclusion of $\mathfrak{N}_\varphi$ into the GNS Hilbert space H_φ of φ. For each locally compact quantum group $\mathbb{G}$, there exists a *left fundamental unitary operator* W on $H_\varphi \otimes H_\varphi$ which satisfies the pentagonal relation and such that the co-multiplication Γ on M can be expressed as

$$\Gamma(x) = W^*(1 \otimes x)W \quad (x \in M).$$

There exists an anti-automorphism R on M, called the *unitary antipode*, such that $R^2 = \iota$, and

$$\Gamma \circ R = \chi(R \otimes R) \circ \Gamma,$$

where $\chi(x \otimes y) = (y \otimes x)$ is the flip map. It can be easily seen that if φ is a left Haar weight, then φR defines a right Haar weight on M.

Let M_* be the predual of M. Then the pre-adjoint of Γ induces on M_* an associative completely contractive multiplication

$$\star : M_* \hat{\otimes} M_* \ni f_1 \otimes f_2 \longmapsto f_1 \star f_2 = (f_1 \otimes f_2) \circ \Gamma \in M_*.$$

The *left regular representation* $\lambda : M_* \to \mathcal{B}(H_\varphi)$ is defined by

$$\lambda : M_* \ni f \longmapsto \lambda(f) = (f \otimes \iota)(W) \in \mathcal{B}(H_\varphi),$$

which is an injective and completely contractive algebra homomorphism from M_* into $\mathcal{B}(H_\varphi)$. Then $\hat{M} = \{\lambda(f) : f \in M_*\}''$ is the von Neumann algebra associated with the dual quantum group $\hat{\mathbb{G}}$. It follows that $W \in M\bar{\otimes}\hat{M}$. We also define the completely contractive injection

$$\hat{\lambda} : \hat{M}_* \ni \hat{f} \longmapsto \hat{\lambda}(\hat{f}) = (\iota \otimes \hat{f})(W) \in M.$$

The *reduced quantum group C^*-algebra*

$$\mathcal{C}_0(\mathbb{G}) = \overline{\hat{\lambda}(L_1(\hat{\mathbb{G}}))}^{\|\cdot\|}$$

is a weak* dense C^*-subalgebra of M. Let $\mathcal{M}(\mathbb{G})$ denote the operator dual $\mathcal{C}_0(\mathbb{G})^*$. There exists a completely contractive multiplication on $\mathcal{M}(\mathbb{G})$ given by the convolution

$$\star : \mathcal{M}(\mathbb{G})\widehat{\otimes}\mathcal{M}(\mathbb{G}) \ni \mu \otimes \nu \longmapsto \mu \star \nu = \mu(\iota \otimes \nu)\Gamma = \nu(\mu \otimes \iota)\Gamma \in \mathcal{M}(\mathbb{G})$$

such that $\mathcal{M}(\mathbb{G})$ contains M_* as a norm closed two-sided ideal. Therefore, for each $\mu \in \mathcal{M}(\mathbb{G})$, we obtain a pair of completely bounded maps

$$f \longmapsto \mu \star f \quad \text{and} \quad f \longmapsto f \star \mu$$

on M_* through the left and right convolution products of $\mathcal{M}(\mathbb{G})$. The adjoint maps give the convolution actions $x \mapsto \mu \star x$ and $x \mapsto x \star \mu$ that are normal completely bounded maps on M.

We denote by $\mathcal{P}(\mathbb{G})$ the set of all states on $\mathcal{C}_0(\mathbb{G})$ (i.e., 'the quantum probability measures'). For any such element the convolution action is a *Markov operator*, i.e., a unital normal completely positive map, on M.

Now assume that the left Haar weight φ on $\mathbb{G}$ is a trace, and let $\psi = \varphi R$ be the right Haar weight. We denote by $\mathcal{L}^p(\mathbb{G})$ and $\tilde{\mathcal{L}}^p(\mathbb{G})$ the non-commutative L^p-spaces associated to φ and ψ, respectively. These spaces are obtained by taking the closure of $\mathfrak{M}_\varphi$ and $\mathfrak{M}_\psi$ under the norms $\|x\| = \varphi(|x|^p)^{\frac{1}{p}}$ and $\|x\| = \psi(|x|^p)^{\frac{1}{p}}$, respectively (see [12] for details). We denote by $\mathcal{L}^\infty(\mathbb{G})$ the von Neumann algebra M. Similarly to the classical case, one can also construct the non-commutative L^p-spaces using the complex interpolation method (cf. [5], [10], [13]). The map

$$\mathfrak{M}_\varphi \ni x \longmapsto \varphi \cdot x \in M_* \tag{1.1}$$

extends to an isometric isomorphism between $\mathcal{L}^1(\mathbb{G})$ and M_*, where $\langle \varphi \cdot x\,,\, y\rangle = \varphi(xy)$.

2. μ-HARMONIC OPERATORS

We assume that $\mu \in \mathcal{P}(\mathbb{G})$ throughout this section. By invariance of the left Haar weight φ, we can easily see that $\mathcal{L}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G})$ is invariant under the left convolution action by μ. Since $\varphi = \psi R$ is a trace, by [11, Proposition 5.20] we have

$$R\big((\iota \otimes \varphi)\Gamma(a)(1 \otimes b)\big) = (\iota \otimes \varphi)\big((1 \otimes a)\Gamma(b)\big)$$

for all $a, b \in \mathfrak{N}_\varphi$. Therefore we obtain

(2.1)
$$\langle\, \mu \star (\varphi \cdot a)\,,\, b\,\rangle = \langle\, \mu\,,\, (\iota \otimes \varphi)\big((1 \otimes a)\Gamma(b)\big)\,\rangle = \langle\, \mu R\,,\, (\iota \otimes \varphi)\big(\Gamma(a)(1 \otimes b)\big)\,\rangle$$

$$= \langle\, (\mu R) \star (b \cdot \varphi)\,,\, a\,\rangle = \langle\, (b \cdot \varphi)\,,\, (\mu R) \star a\,\rangle = \langle\, \varphi \cdot ((\mu R) \star a)\,,\, b\,\rangle\,. \tag{2.2}$$

Since the map (1.1) is an isometry, this shows that the convolution action

$$\mathfrak{M}_\varphi \ni x \longmapsto \varphi \cdot x \longmapsto (\mu R) \star (\varphi \cdot x) = \varphi \cdot (\mu \star x) \longmapsto \mu \star x \in \mathfrak{M}_\varphi$$

extends to an operator on $\mathcal{L}^1(\mathbb{G})$ with the same norm as the convolution operator by μ on $\mathcal{L}^\infty(\mathbb{G})$. Now, interpolating between $\mathcal{L}^1(\mathbb{G})$ and $\mathcal{L}^\infty(\mathbb{G})$, we can extend the convolution action

$$\mathcal{L}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G}) \ni x \longmapsto \mu \star x \in \mathcal{L}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G})$$

to $\mathcal{L}^p(\mathbb{G})$. An operator $x \in \mathcal{L}^p(\mathbb{G})$ is called μ-harmonic if $\mu \star x = x$ and $\mathcal{H}^p_\mu(\mathbb{G}) = \{x \in \mathcal{L}^p(\mathbb{G}) : \mu \star x = x\}$ is the space of μ-harmonic operators. It is easy to see that $\mathcal{H}^p_\mu(\mathbb{G})$ is a weak* closed subspace of $\mathcal{L}^p(\mathbb{G})$ for all $1 < p \leq \infty$.

Similarly to the case $p = \infty$, we have a projection $E^p_\mu : \mathcal{L}^p(\mathbb{G}) \to \mathcal{H}^p_\mu(\mathbb{G})$ constructed as follows. Let $\mathcal{U}$ be a free ultra-filter on $\mathbb{N}$, and define $E^p_\mu : \mathcal{L}^p(\mathbb{G}) \to \mathcal{L}^p(\mathbb{G})$ by the weak* limit

$$E^p_\mu(x) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^{n} \mu^k \star x.$$

Then it is easy to see that $E^p_\mu \circ E^p_\mu = E^p_\mu$ and that $\mathcal{H}^p_\mu(\mathbb{G}) = E^p_\mu(\mathcal{L}^p(\mathbb{G}))$. Moreover, by considering the convolution action on $\mathcal{L}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G})$ and passing to limits, we can see that E^p_μ is also positive.

Similarly, we can extend the right convolution action

$$\tilde{\mathcal{L}}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G}) \ni x \longmapsto x \star \mu \in \tilde{\mathcal{L}}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G})$$

to $\tilde{\mathcal{L}}^p(\mathbb{G})$. Then $\tilde{\mathcal{H}}^p_\mu(\mathbb{G}) = \{x \in \tilde{\mathcal{L}}^p(\mathbb{G}) : x \star \mu = x\}$ is a weak* closed subspace of $\tilde{\mathcal{L}}^p(\mathbb{G})$ and there is a positive projection $\tilde{E}^p_\mu$ on $\tilde{\mathcal{L}}^p(\mathbb{G})$ such that $\tilde{\mathcal{H}}^p_\mu(\mathbb{G}) = \tilde{E}^p_\mu(\tilde{\mathcal{L}}^p(\mathbb{G}))$.

Proposition 2.1. *The unitary antipode R extends to an isometric isomorphism*

$$R : \mathcal{L}^p(\mathbb{G}) \to \tilde{\mathcal{L}}^p(\mathbb{G})$$

such that $R(\mathcal{H}^p_\mu(\mathbb{G})) = \tilde{\mathcal{H}}^p_{\mu R}$ for all $1 \le p \le \infty$.

Proof. Since R is an anti-automorphism, we have

$$\psi(|R(a)|^p) = \psi(R(|a|^p)) = \varphi(|a|^p)$$

for all $a \in \mathfrak{M}_\varphi$. Therefore R extends to an isometry from $\mathcal{L}^p(\mathbb{G})$ onto $\tilde{\mathcal{L}}^p(\mathbb{G})$. Moreover, we have

$$\begin{aligned} R(\mu \star a) &= R\big((\mu \otimes \iota)\Gamma(a)\big) = R\big((\mu \otimes \iota)\Gamma(R^2(a))\big) \\ &= R\big((\iota \otimes \mu)(R \otimes R)\Gamma(R(a))\big) \\ &= (\iota \otimes \mu R)\Gamma(R(a)) = R(a) \star \mu R, \end{aligned}$$

which implies that $R(\mathcal{H}^p_\mu(\mathbb{G})) = \tilde{\mathcal{H}}^p_{\mu R}$. ☐

Therefore, for $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, we can identify each $\mathcal{L}^p(\mathbb{G})$ and $\tilde{\mathcal{L}}^q(\mathbb{G})$ with the dual space of the other via

$$\langle a\,,\, b\rangle = \varphi(aR(b)) = \psi(R(a)b), \qquad a \in \mathcal{L}^p(\mathbb{G})\,, \quad b \in \tilde{\mathcal{L}}^q(\mathbb{G})\,.$$

Theorem 2.2. *Let $1 < p\,, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we have linear isometric isomorphisms*

$$\mathcal{H}^p_\mu(\mathbb{G})^* \cong \tilde{\mathcal{H}}^q_\mu(\mathbb{G}) \qquad \text{and} \qquad \mathcal{H}^p_\mu(\mathbb{G}) \cong \tilde{\mathcal{H}}^q_\mu(\mathbb{G})^*.$$

Proof. Denote

$$\mathcal{J}^p_\mu(\mathbb{G}) := \{x - \mu \star x \,:\, x \in \mathcal{L}^p(\mathbb{G})\}^- \quad \text{and} \quad \tilde{\mathcal{J}}^q_\mu(\mathbb{G}) := \{y - y \star \mu \,:\, y \in \tilde{\mathcal{L}}^q(\mathbb{G})\}^-.$$

Since

$$\begin{aligned} \langle x\,,\, y \star \mu\rangle &= \psi\big(R(x)(y \star \mu)\big) = \psi\big(R(x)(\iota \otimes \mu)\Gamma(y)\big) = \mu\big((\psi \otimes \iota)(R(x) \otimes 1)\Gamma(y)\big) \\ &= \mu R\big((\psi \otimes \iota)\Gamma(R(x))(y \otimes 1)\big) = \psi\big((\iota \otimes \mu R)\Gamma(R(x))\, y\big) \\ &= \psi\big(R\big((\mu \otimes \iota)\Gamma(x)\big)\, y\big) = \psi(R(\mu \star x)\, y) = \langle \mu \star x\,,\, y\rangle \end{aligned}$$

for all $x \in \mathfrak{M}_\varphi$ and $y \in \mathfrak{M}_\psi$, it follows that $\mathcal{H}^p_\mu(\mathbb{G}) = \tilde{\mathcal{J}}^q_\mu(\mathbb{G})^\perp$, and therefore

$$\mathcal{H}^p_\mu(\mathbb{G})^* = \frac{\tilde{\mathcal{L}}^q(\mathbb{G})}{\mathcal{H}^p_\mu(\mathbb{G})^\perp} = \frac{\tilde{\mathcal{L}}^q(\mathbb{G})}{\tilde{\mathcal{J}}^q_\mu(\mathbb{G})}\,.$$

In the following we show that the correspondence

$$\frac{\tilde{\mathcal{L}}^q(\mathbb{G})}{\tilde{\mathcal{J}}^q_\mu(\mathbb{G})} \ni y + \tilde{\mathcal{J}}^q_\mu(\mathbb{G}) \longmapsto \tilde{E}^q_\mu(y) \in \tilde{\mathcal{H}}^q_\mu(\mathbb{G})$$

defines a linear isometric isomorphism. First we observe that

$$\tilde{E}^q_\mu(y \star \mu - y) = \lim_{\mathcal{U}} \left((y \star \mu - y) \star \sum_1^n \frac{\mu^k}{n} \right) = 0$$

for all $y \in \tilde{\mathcal{L}}^q(\mathbb{G})$, which implies that the above map is well-defined. It is obviously onto. To check the injectivity, first note that

$$y - y \star \mu^k = (y - y \star \mu) + (y \star \mu - y \star \mu^2) + (y \star \mu^{k-1} - y \star \mu^k) \in \tilde{\mathcal{J}}^q_\mu(\mathbb{G}),$$

for all $k \in \mathbb{N}$. Now suppose that $\tilde{E}^q_\mu(y) = 0$. Then, by the above and by the weak* closeness of $\tilde{\mathcal{J}}^q_\mu(\mathbb{G})$, we have

$$y = y - \tilde{E}^q_\mu(y) = y - \left(\lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n y \star \mu^k \right) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n \left(y - y \star \mu^k \right) \in \tilde{\mathcal{J}}^q_\mu(\mathbb{G}),$$

and therefore the injectivity of the map follows. Moreover, since $\tilde{E}^q_\mu$ is an idempotent, it follows that

$$y + \tilde{\mathcal{J}}^q_\mu(\mathbb{G}) = \tilde{E}^q_\mu(y) + \tilde{\mathcal{J}}^q_\mu(\mathbb{G}).$$

Therefore

$$\|y + \tilde{\mathcal{J}}^q_\mu(\mathbb{G})\| \leq \|\tilde{E}^q_\mu(y)\|.$$

On the other hand, we have

$$\begin{aligned} \|\tilde{E}^q_\mu(y)\| &= \sup \left\{ \left| \langle \tilde{E}^q_\mu(y) \, , \, x \rangle \right| \, : \, x \in \mathcal{L}^p(\mathbb{G}) \, , \ \|x\| \leq 1 \right\} \\ &= \sup \left\{ \left| \langle y \, , \, E^p_\mu(x) \rangle \right| \, : \, x \in \mathcal{L}^p(\mathbb{G}) \, , \ \|x\| \leq 1 \right\} \\ &\leq \|y + \tilde{\mathcal{J}}^q_\mu(\mathbb{G})\| \, . \end{aligned}$$

This shows that the map is isometric and so yields the first identification. The second identification is proved along similar lines. □

Proposition 2.3. *For $1 < p \leq \infty$ the space $\mathcal{H}^p_\mu(\mathbb{G})$ is generated by its positive elements.*

Proof. By considering the polar decomposition, we observe that $\mathcal{L}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G})$ is self-adjoint. Let $x \in \mathcal{H}^p_\mu(\mathbb{G})$, and $\mathcal{L}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G}) \ni x_n \to x$ in $\mathcal{L}^p(\mathbb{G})$. Using the continuity of the adjoint on $\mathcal{L}^p(\mathbb{G})$, we obtain

$$\mu \star x^* = \lim_n \mu \star x_n^* = \lim_n (\mu \star x_n)^* = (\lim_n \mu \star x_n)^* = x^*,$$

where the limits are taken in $\mathcal{L}^p(\mathbb{G})$. Therefore, $\mathcal{H}^p_\mu(\mathbb{G})$ is self-adjoint and so is generated by its self-adjoint elements. Now, let x be a self-adjoint element in $\mathcal{L}^p(\mathbb{G})$, and let $x = x_+ - x_-$ where both x_+ and x_- are in $\mathcal{L}^p(\mathbb{G})^+$. Then we have

$$x = E^p_\mu(x) = E^p_\mu(x_+) - E^p_\mu(x_-),$$

which yields the result by positivity of the map E^p_μ. □

Main Theorem: Case $1 < p < \infty$. A state $\mu \in \mathcal{P}(\mathbb{G})$ is called *non-degenerate* on $\mathcal{C}_0(\mathbb{G})$ if for every non-zero element $x \in \mathcal{C}_0(\mathbb{G})^+$ there exists $n \in \mathbb{N}$ such that $\langle x\,,\,\mu^n\rangle \neq 0$.

Theorem 2.4. *Let $\mathbb{G}$ be a non-compact, locally compact quantum group with a tracial (left) Haar weight φ, and let $\mu \in \mathcal{P}(\mathbb{G})$ be non-degenerate. Then for all $1 < p < \infty$ we have*

$$\mathcal{H}^p_\mu(\mathbb{G}) = \{0\}\,.$$

Proof. First let $1 < p \leq 2$, and suppose $0 \leq x \in \mathcal{H}^p_\mu(\mathbb{G})$ with $\|x\|_p = 1$. Define

$$\tilde{\mu} := \sum_{i=n}^{\infty} \frac{\mu^n}{2^n}.$$

Since μ is non-degenerate, $\tilde{\mu}$ is faithful, and $\tilde{\mu} \star x = x$. Now, let $q \geq 2$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Using the duality between $\mathcal{L}^p(\mathbb{G})$ and $\tilde{\mathcal{L}}^q(\mathbb{G})$, we assign to each pair $a \in \mathcal{L}^p(\mathbb{G})$ and $b \in \tilde{\mathcal{L}}^q(\mathbb{G})$ an element $\Omega_{a,b} \in \mathcal{L}^\infty(\mathbb{G})$ defined by

$$\langle f\,,\,\Omega_{a,b}\rangle = \langle f \star a\,,\,b\rangle \qquad (f \in M_*).$$

We clearly have $\|\Omega_{a,b}\| \leq \|a\|_p\|b\|_q$. Now, choose $y \in \tilde{\mathcal{L}}^q(\mathbb{G})$, $\|y\|_q = 1$, such that $\langle x\,,\,y\rangle = 1$. We claim that $\Omega_{x,y} \in \mathcal{C}_0(\mathbb{G})$ (in fact, $\Omega_{a,b} \in \mathcal{C}_0(\mathbb{G})$ for all $a \in \mathcal{L}^p(\mathbb{G})$ and $b \in \tilde{\mathcal{L}}^q(\mathbb{G})$). To see this, assume that

$$x = \int_0^\infty \lambda\, de_\lambda$$

is the spectral decomposition of x, and let

$$x_n = \int_{\frac{1}{n}}^{n} \lambda\, de_\lambda.$$

Then $x_n \in \mathcal{L}^p(\mathbb{G}) \cap \mathcal{L}^\infty(\mathbb{G}) \subseteq \mathfrak{N}_\varphi$, $\|x_n\|_p \leq \|x\|_p$, and $\|x - x_n\|_p \to 0$. Also let $y_n \in \mathfrak{N}_\psi$ be such that

$$\|y_n - y\|_q \to 0.$$

Denote by $\omega_{\eta,\zeta}$ the vector functional associated with $\eta\,,\zeta \in H_\varphi$. Then, for $f \in M_*$ we have

$$\begin{aligned}\langle f\,,\,\Omega_{x_n,y_n}\rangle &= \langle f \star x_n\,,\,y_n\rangle = \langle \lambda(f)\Lambda_\varphi(x_n)\,,\,\Lambda_\varphi(R(y_n))\rangle \\ &= \langle \omega_{\Lambda_\varphi(x_n)\,,\,\Lambda_\varphi(R(y_n))}, \lambda(f)\rangle = \langle f\,,\,\hat{\lambda}(\omega_{\Lambda_\varphi(x_n),\Lambda_\varphi(R(y_n))})\rangle,\end{aligned}$$

which implies that $\Omega_{x_n,y_n} = \hat{\lambda}(\omega_{\Lambda_\varphi(x_n),\Lambda_\varphi(R(y_n))}) \in \mathcal{C}_0(\mathbb{G})$. Moreover, it follows that

$$\begin{aligned}\|\Omega_{x,y} - \Omega_{x_n,y_n}\|_\infty &\leq \|\Omega_{x-x_n,y}\|_\infty + \|\Omega_{x_n,y-y_n}\|_\infty \\ &\leq \|x - x_n\|_p\,\|y\|_q + \|x_n\|_p\,\|y - y_n\|_q \to 0.\end{aligned}$$

This shows that $\Omega_{x,y} \in \mathcal{C}_0(\mathbb{G})$, as claimed. But then we have $\|\Omega_{x,y}\| \leq \|x\|_p\|y\|_q = 1$, and

$$\langle \tilde{\mu}\,,\,\Omega_{x,y}\rangle = \langle \tilde{\mu} \star x\,,\,y\rangle = \langle x\,,\,y\rangle = 1.$$

Since $\tilde{\mu}$ is faithful, it follows that $\Omega_{x,y} = 1$, and therefore $1 \in \mathcal{C}_0(\mathbb{G})$, which contradicts our assumption of $\mathbb{G}$ being non-compact, so $x = 0$. This shows, by Proposition 2.3, that $\mathcal{H}^p_\mu(\mathbb{G}) = \{0\}$ for all $1 < p \leq 2$. Now, a similar argument yields $\tilde{\mathcal{H}}^q_\mu(\mathbb{G}) = 0$ for all $1 < q \leq 2$, which implies by Theorem 2.2 that $\mathcal{H}^p_\mu(\mathbb{G}) = \tilde{\mathcal{H}}^q_\mu(\mathbb{G})^* = \{0\}$ for all $2 \leq p < \infty$. $\square$

Main Theorem: Case $p = 1$**.** Since $\mathcal{L}^1(\mathbb{G})$ is not a dual Banach space, our proof for $1 < p < \infty$ does not work in this case, and so we have to treat this case separately. We do this by first proving a similar result for M_* and then using the identification of the latter with $\mathcal{L}^1(\mathbb{G})$. Note that for the following theorem we do not assume that the Haar weight is a trace.

Theorem 2.5. *Let* $\mathbb{G}$ *be a non-compact, locally compact quantum group, and let* $\mu \in \mathcal{P}(\mathbb{G})$ *be non-degenerate. If* $\omega \in \mathcal{M}(\mathbb{G})$ *is such that* $\mu \star \omega = \omega$*, then* $\omega = 0$*.*

Proof. Assume that $\mu \star \omega = \omega$, and let $\tilde{\mu}$ be as in the proof of Theorem 2.4. So, $\tilde{\mu}$ is faithful, and $\tilde{\mu} \star \omega = \omega$. Therefore we have

$$\lambda(\tilde{\mu})\lambda(\omega)\xi = \lambda(\tilde{\mu} \star \omega)\xi = \lambda(\omega)\xi$$

for all $\xi \in H_\varphi$. Now if $\omega \neq 0$, there exists $\xi \in H_\varphi$ such that $\|\lambda(\omega)\xi\| = 1$. Denote by $\hat{\omega}$ the restriction of $\omega_{\lambda(\omega)\xi}$ to $\hat{M}$. Then $\|\hat{\omega}\| = 1$, and

$$\langle \tilde{\mu} \, , \, \hat{\lambda}(\hat{\omega}) \rangle = \langle \lambda(\tilde{\mu}) \, , \, \hat{\omega} \rangle = \langle \lambda(\tilde{\mu})\lambda(\omega)\xi \, , \, \lambda(\omega)\xi \rangle = \langle \lambda(\omega)\xi \, , \, \lambda(\omega)\xi \rangle = 1.$$

Since $\|\hat{\lambda}(\hat{\omega})\| \leq 1$ and $\tilde{\mu}$ is faithful, it follows that $\hat{\lambda}(\hat{\omega}) = 1$. But this implies that $1 \in \mathcal{C}_0(\mathbb{G})$, which contradicts our assumption of $\mathbb{G}$ being non-compact. Hence, $\omega = 0$. ☐

Theorem 2.6. *Let* $\mathbb{G}$ *be a non-compact, locally compact quantum group with a tracial (left) Haar weight* φ*, and let* $\mu \in \mathcal{P}(\mathbb{G})$ *be non-degenerate. Then*

$$\mathcal{H}^1_\mu(\mathbb{G}) = \{0\} \, .$$

Proof. Let $x \in \mathcal{H}^1_\mu(\mathbb{G})$. We have that $\mu R \in \mathcal{P}(\mathbb{G})$ is non-degenerate, and from equations (2.1) and (2.2) we get

$$\mu R \star (\varphi \cdot x) = \varphi \cdot (\mu \star x) = \varphi \cdot x.$$

Hence, $\varphi \cdot x = 0$ by Theorem 2.5, and therefore $x = 0$. ☐

Remark 2.7. The statements of Theorems 2.4 and 2.6 are not true in general for the case $p = \infty$. Any non-degenerate probability measure on a non-amenable discrete group is a counterexample [8].

Compact Case. We conclude by proving the triviality of μ-harmonic operators in the compact quantum group setting.

Theorem 2.8. *Let* $\mathbb{G}$ *be a compact quantum group with tracial Haar state, and let* $\mu \in \mathcal{P}(\mathbb{G})$ *be non-degenerate. Then* $\mathcal{H}^p_\mu(\mathbb{G}) = \mathbb{C}1$ *for all* $1 \leq p \leq \infty$*.*

Proof. The case $p = \infty$ was proved in the general case in [9]. Let $1 \leq p < \infty$, and assume that $x \in \mathcal{H}^p_\mu(\mathbb{G})$, $x \notin \mathbb{C}1$ and $\|x\|_p = 1$. Then there exists $y \in \tilde{\mathcal{L}}^q(\mathbb{G})$ with $\|y\|_q = 1$ such that $\langle x \, , \, y \rangle = 1$ (we let $q = \infty$ for $p = 1$) and $\langle 1 \, , \, y \rangle = 0$. Then from the proof of Theorem 2.4 (which we can also apply to the case of $p = 1$ and $q = \infty$, since $\mathcal{L}^\infty(\mathbb{G}) \subseteq \mathcal{L}^2(\mathbb{G})$ for a compact quantum group) we have $\Omega_{x,y} = 1$ and

$$\langle \varphi \star x \, , \, y \rangle = \langle \varphi \, , \, 1 \rangle = 1, \tag{2.3}$$

where φ is the Haar state on $\mathbb{G}$. Now, let $x_n \in \mathcal{L}^\infty(\mathbb{G})$ be such that $\|x_n - x\|_p \to 0$. Then

$$\langle \varphi \star x \, , \, y \rangle = \lim_n \langle \varphi \star x_n \, , \, y \rangle = \lim_n \langle \varphi \, , \, x_n \rangle \langle 1 \, , \, y \rangle = 0.$$

But this contradicts (2.3), and therefore $x = 0$. Hence, $\mathcal{H}^p_\mu(\mathbb{G}) = \mathbb{C}1$. ☐

Remark 2.9. All of our results in this paper can be proved, by slight modifications of the arguments, for a state μ on the universal C^*-algebra $C_u(\mathbb{G})$.

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