

EXCURSION AND RETURN TIMES OF A GEODESIC TO A SUBSET OF A HYPERBOLIC RIEMANN SURFACE

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ABSTRACT. We calculate the asymptotic average rate at which a generic geodesic on a finite area hyperbolic 2-orbifold returns to a subsurface with geodesic boundary. As a consequence we get the average time a generic geodesic spends in such a subsurface. Related results are obtained for excursions into a collar neighborhood of a simple closed geodesic and the associated distribution of excursion depths.

1. INTRODUCTION

The geodesic flow on the unit tangent bundle of a finite area hyperbolic Riemann surface is ergodic [15]. One important, well known consequence is that on average, the time spent by a generic geodesic in a subset of the surface is equal to the relative area of the set. At first glance it does not appear that this approach tells us anything directly about other specific aspects of the behavior of the geodesic relative to the set: for example, about the average rate at which the geodesic returns to the set; the average length of each visit to the set, called the excursion time; or the average time between visits—all related quantities. The reason for this is that these values are determined by certain aspects of the geometry of the sets beyond just the relative area. Nevertheless, for a reasonably large and interesting selection of sets, once the geometry is properly accounted for, it is the classical Ergodic Theorem for flows from which one can infer the existence of a limit and compute its value.

In the cases considered here, where the subset is a subsurface with geodesic boundary or a collar neighborhood of a simple closed geodesic, it is possible to provide precise descriptions of these types of behavior for a generic geodesic relative to the set. In the case of subsurfaces the return time depends only on the length of the boundary and the area of the surface, whereas excursion times are determined by the area of the surface along with the length of the boundary. In the second case the return time depends on the geodesic's length, the width of its collar and the area of the surface, while excursion times depend only on the width of the collar. This last fact is at first surprising, yet it mirrors the result from [11], where it was shown that the average excursion time into an embedded cusp neighborhood is π , independent of the area of the neighborhood and, furthermore, independent of the surface. In the latter case it is also possible to describe the distributions of the maximal depths of geodesic excursions—again something that depends only on the geometry of the collar. This is reminiscent of results from [5] and [10].

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The return time to a cusp relative to the depth of the excursion was studied by Sullivan in [17], leading in various directions to generalizations and tangents. One of these tangents was followed by Nakada in [14], where he found the average return time to a cusp neighborhood of a finite volume 3-manifold as one step in his study of approximation properties of rationals in an imaginary quadratic number field. His approach was taken up by Stratmann [16], who used it to get estimates for the average return time of a generic geodesic in a hyperbolic manifold to a cusp neighborhood, where generic means with respect to the Liouville-Patterson measure on the unit tangent bundle. We employed their methods in [11] to determine average return and excursion times to a cusp neighborhood on a finite area hyperbolic surface, and, like Nakada and Stratmann, we used these values to establish metrical results for approximation by the cusps of a Fuchsian group.

Generic properties of geodesics returning to neighborhoods of a closed geodesic in a negatively curved manifold were studied by Hersonsky and Paulin in [13]. Among other things, they looked at geometric versions of one of Khintchin's theorems and related generalizations of Sullivan's logarithm law for geodesics.

2. BACKGROUND AND MAIN RESULTS

2.1. Geodesic boundary. A finite area hyperbolic 2-orbifold S is the quotient of the Poincaré upper half-plane \mathbb{H} by a discrete group $H \subset \mathrm{PSL}_2(\mathbb{R})$.

Let $\gamma_1, \dots, \gamma_n$ be a collection of disjoint simple closed geodesics on S , which together bound a subsurface M of S . We write $\Gamma = \bigcup \gamma_i$ for the boundary of M . The length of a closed geodesic β is written $l(\beta)$. The total length of the boundary of M is then $l(\Gamma) = \sum l(\gamma_i)$.

Each vector v in the unit tangent bundle $T_1 S$ uniquely determines a geodesic ray $\alpha_v : [0, \infty) \rightarrow S$ which is the projection to S of the forward orbit $\{G_t(v) \mid t \in [0, \infty)\}$ of the geodesic flow on the unit tangent bundle. If $\alpha_v := \alpha_v([0, \infty))$ intersects the boundary Γ of M infinitely often, then there is a sequence of pairs of parameter values $\{(t_i, s_i)\}_{i=1}^\infty$ with $t_i < s_i < t_{i+1}$ so that

$$(2.1) \quad \bigcup_{i=1}^\infty \alpha_v[(t_i, s_i)] = \alpha_v \cap M.$$

In other words, α_v meets the set M in precisely the arcs $\alpha_v[(t_i, s_i)]$ of α_v . We shall refer to these arcs as the excursions of α_v into M and to the sequence of parameters as the excursion parameters of α_v . Let $\#X$ denote the cardinality of the set X ; $\mathrm{area}(T)$ shall denote the hyperbolic area of $T \subset S$. Area is measured in the hyperbolic metric. Henceforth we shall assume that the area of S is finite.

Then the asymptotic average rate at which a geodesic returns to the set M is given as follows.

Theorem 1. *For almost all $v \in T_1 S$, the excursion parameters of α_v satisfy*

$$(2.2) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \#\{n \mid t_n < t\} = \frac{l(\Gamma)}{\pi \mathrm{area}(S)}.$$

Since geodesics are parameterized by arc length, the length of the arc $\alpha[(t_i, s_i)]$ is $s_i - t_i$. One consequence of the theorem is the value for the asymptotic average length of an excursion.

Corollary 1. *For almost all $v \in T_1S$,*

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (s_i - t_i) = \frac{\pi \operatorname{area}(M)}{l(\Gamma)}.$$

Remark 1. The average distance between the starting points of the first $n + 1$ excursions is computed by the sum

$$\frac{1}{n} \sum_{i=1}^n (t_{i+1} - t_i) = \frac{t_{n+1} - t_1}{n}.$$

Since n is essentially $\#\{i \mid t_i \leq t_n\}$, by taking the limit as $n \rightarrow \infty$, we see that the average distance between consecutive excursions is generically equal to the inverse of the value for the limit in (2.2), $\frac{\pi \operatorname{area}(S)}{l(\Gamma)}$.

2.2. Collar neighborhoods. Let γ be a simple closed geodesic on S . Let $\mathbf{C}_r(\gamma) = \mathbf{C}_r$ be the collar neighborhood of γ . This is the set of points within a fixed distance r of γ . If r is not too large ($r < R_{l(\gamma)} = \log \coth \frac{l(\gamma)}{4}$; see Section 2.5), the r -collar about γ is divided by γ into disjoint open cylinders denoted A_r and B_r , which we will call half-collars. If a subsurface M is specified, then we shall suppose that $A_r \subset M$. Let $\lambda_{A_r} = \partial \mathbf{C}_r \cap \partial A_r$ and $\lambda_{B_r} = \partial \mathbf{C}_r \cap \partial B_r$, where ∂D denotes the boundary of D in S . The values r , $l(\gamma)$ and $\operatorname{area}(\mathbf{C}_r)$ are not independent, but rather, any two of the values determine the third. This will be made precise in Section 4.1, Proposition 2.

We shall suppose the geodesic ray α_v , determined by the direction v in T_1S , intersects \mathbf{C}_r infinitely often. Let $P_r = \{(t_i, s_i) \mid i \in \mathbb{Z}^+\}$ and $Q_r = \{(\rho_i, \eta_i) \mid i \in \mathbb{Z}^+\}$ be sequences of pairs of parameter values with $t_i < s_i < t_{i+1}$, and $\rho_i < \eta_i < \rho_{i+1}$ so that $\alpha_v(t_i) \in \lambda_{A_r}$, $\alpha_v(\rho_i) \in \lambda_{B_r}$, $\{\alpha_v(s_i), \alpha_v(\eta_i)\} \subset \{\lambda_{A_r}, \lambda_{B_r}\}$ and

$$(2.4) \quad \bigcup_{P_r} \alpha_v[(t_i, s_i)] \cup \bigcup_{Q_r} \alpha_v[(\rho_i, \eta_i)] = \alpha_v \cap \mathbf{C}_r.$$

In other words, these parameters determine the segments of α_v in \mathbf{C}_r with both endpoints on its boundary. We will focus on those that enter \mathbf{C}_r at λ_{A_r} . P_r is called the set of excursion parameters. Let $P'_r = \{(t'_i, s'_i)\}$ be the subset of excursion parameters, reindexed by consecutive numbers, where both values $\alpha_v(t'_i)$ and $\alpha_v(s'_i)$ lie in λ_{A_r} .

Theorem 2. *For $r < R_{l(\gamma)}$ and almost all $v \in T_1S$,*

$$(2.5) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \#\{n \mid t_n < t\} = \frac{l(\gamma) \cosh r}{\pi \operatorname{area}(S)} = \frac{\sqrt{l(\gamma)^2 + (\frac{1}{2} \operatorname{area}(\mathbf{C}_r))^2}}{\pi \operatorname{area}(S)},$$

$$(2.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \#\{n \mid t'_n < t\} = \frac{l(\gamma)(\cosh r - 1)}{\pi \operatorname{area}(S)} = \frac{\sqrt{l(\gamma)^2 + (\frac{1}{2} \operatorname{area}(\mathbf{C}_r))^2} - l(\gamma)}{\pi \operatorname{area}(S)}.$$

In this case the analogue of Corollary 1, the asymptotic average length of an excursion entering at λ_{A_r} into a collar neighborhood, is given by the following.

Corollary 2. *For $r < R$ and almost all $v \in T_1S$,*

$$(2.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (s_i - t_i) = \pi \tanh r = \frac{\pi \operatorname{area}(\mathbf{C}_r)}{\sqrt{l(\gamma)^2 + \operatorname{area}(\mathbf{C}_r)^2}}.$$

2.3. The distribution of collar excursions. In [10] the number theoretic distributions from the Doeblin-Lenstra Conjecture [5],[8] and the related Theorem of Bosma [4] were reinterpreted and expanded in a geometric setting. The analogous notions can be defined in terms of the distribution of the depths of maximal penetration of geodesic excursions into a collar neighborhood of a simple closed geodesic. Given a geodesic α_v with excursion parameters P_r , define $D_v(i) = \inf\{\text{dist}(\alpha_v(t), \gamma) \mid t_i < t < s_i\}$, where $\text{dist}(c, \gamma)$ is the distance between the point c and the geodesic γ . This is the depth of the i^{th} excursion. Using Theorem 2 we show in the next corollary that the depths are nicely distributed in a fashion that is independent of $l(\gamma)$. Choose $R_0 < R_{l(\gamma)}$.

Corollary 3. *For $r \leq R_0$ and almost all $v \in T_1S$ the limit*

$$(2.8) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \#\{i \mid 1 \leq i \leq n, D_v(i) \leq r\}$$

is defined and takes the value

$$(2.9) \quad \delta_v(r) = \frac{\cosh r}{\cosh R_0}.$$

2.4. The invariant measure. The unit tangent bundle of the hyperbolic plane \mathbb{H} can be given as a cartesian product $T_1\mathbb{H} = \mathbb{H} \times S^1$. In these coordinates the natural invariant measure for the geodesic flow \tilde{G} has the form $\tilde{m} = dAd\phi$. There is another set of very useful coordinates in which $T_1\mathbb{H}$ is described, up to measure zero, as the set of triples $(x, y, t) \in \mathbb{R}^3$ with $x \neq y$. Then (x, y, t) corresponds to the vector $v = \dot{\alpha}(t) \in T_1\mathbb{H}$, where α is the geodesic in \mathbb{H} oriented from the endpoint $\alpha_- = x$ to $\alpha_+ = y$ and parameterized so that $\alpha(0)$ is the Euclidean midpoint of the semicircle $\alpha(\mathbb{R})$. The geodesic flow on $T_1\mathbb{H}$ satisfies $\tilde{G}_s(x, y, t) = (x, y, t + s)$. Furthermore, the geodesic flow on T_1S has the invariant probability measure μ , whose lift to $T_1\mathbb{H}$ is equal to [15]:

$$(2.10) \quad \tilde{\mu} = \frac{1}{2\pi \text{area}(S)} \tilde{m} = \frac{1}{\pi \text{area}(S)} (x - y)^{-2} dx dy dt.$$

Let $\mathcal{E} \subset T_1S$ be the set of vectors v for which the orbit $G_t(v)$ is dense in T_1S . As a consequence of the Ergodic Theorem and the Poincaré Recurrence Theorem, [15], \mathcal{E} has full measure in T_1S .

2.5. The normalization of H . Given a simple closed geodesic γ on S we may suppose that the Fuchsian group H has been normalized so that the imaginary axis \mathcal{I} in \mathbb{H} covers γ . Then the stabilizer of \mathcal{I} in H is generated by the transformation $g(z) = \zeta z$ with $\log \zeta = l(\gamma)$. The arc $\tilde{\gamma} = \{it \mid 1 \leq t < \zeta\} \subset \mathcal{I}$ is mapped injectively onto γ by the covering projection.

$\mathbf{C}_r(\gamma)$ is the image of the r -neighborhood $\mathbf{C}_r(\mathcal{I})$ of \mathcal{I} under the covering projection. It follows from the Collar Lemma (see [1] and [7] for the many references) that there is a value $R_{l(\gamma)} = \log \coth \frac{l(\gamma)}{4}$ so that for $r < R_{l(\gamma)}$, the collar $\mathbf{C}_r(\mathcal{I})$ is mapped disjointly from itself by any $h \in H$ which is not a power of g . Consequently, $\mathbf{C}_r(\gamma)$ is a cylinder embedded in S , as we have been assuming. Henceforth, we shall suppose that $r < R_{l(\gamma)}$.

The curves λ_{A_r} and λ_{B_r} lift to straight lines $\tilde{\lambda}_{A_r}$ and $\tilde{\lambda}_{B_r}$ bounding $\mathbf{C}_r(\mathcal{I})$. They emanate from the origin and make an angle ϕ with \mathcal{I} . We shall further stipulate that the lift of the half-collar A_r in $\mathbf{C}_r(\mathcal{I})$ lies in the left half-plane.

3. PROOFS FOR REGIONS WITH GEODESIC BOUNDARY

3.1. The single geodesic. Let $\mathcal{L}^*(\gamma) = \mathcal{L}^*(\gamma, A_r)$ be the subset of the unit tangent bundle over γ where $v \in \mathcal{L}^*(\gamma)$ if there exists τ_0 so that for $0 < t < \tau_0$ the points of $\alpha_v(t)$ lie in A_r . This is the subset of vectors based on γ that point into A_r . Note that the set does not depend on the choice of $r < R_{l(\gamma)}$. Then $\mathcal{L}(\gamma) = \mathcal{L}^*(\gamma) \cap \mathcal{E}$ is a cross-section for the geodesic flow on T_1S , [3]. In other words, for almost all $v \in T_1S$ there exists an increasing sequence of values τ_i so that $G_{\tau_i}(v) \in \mathcal{L}(\gamma)$.

Given $\varepsilon < R_{l(\gamma)} - r$, define the ε -thickened section $\mathcal{L}_\varepsilon(\gamma) = \{G_t(v) \mid t \in [0, \varepsilon], v \in \mathcal{L}(\gamma)\}$. Analysis of the thickened section is the main tool in the proof of Theorem 1.

Proposition 1.

$$(3.1) \quad \mu(\mathcal{L}_\varepsilon(\gamma)) = \frac{\varepsilon l(\gamma)}{\pi \text{area}(S)}.$$

Proof. The cross-section $\mathcal{L}(\gamma)$ and the thickened section $\mathcal{L}_\varepsilon(\gamma)$ lift to the subsets $\mathcal{L}(\tilde{\gamma})$ of $T_1\mathbb{H}$ over $\tilde{\gamma}$ and its thickened section $\mathcal{L}_\varepsilon(\tilde{\gamma})$, respectively. Furthermore, the projection from $\mathcal{L}_\varepsilon(\tilde{\gamma})$ to $\mathcal{L}_\varepsilon(\gamma)$ is injective. In this way we reduce the computation in Proposition 1 to a calculation in (x, y, t) -coordinates in \mathbb{R}^3 of $\tilde{\mu}(\mathcal{L}_\varepsilon(\tilde{\gamma})) = \mu(\mathcal{L}_\varepsilon(\gamma))$.

For a given $x \in \mathbb{R}^+$ and $t > 0$, the point y at which the geodesic \overline{xy} with endpoints x and y meets the point $it \in \mathcal{I}$ is the solution to the equation

$$(3.2) \quad |it - \frac{x+y}{2}|^2 = (\frac{x-y}{2})^2.$$

This is $y = -\frac{t^2}{x}$. Therefore for a given $x > 0$, $I_x = [-\frac{\zeta^2}{x}, -\frac{1}{x})$ is the interval of values $y \in \mathbb{R}$ for which \overline{xy} intersects the interval $\tilde{\gamma}$.

For $x \in \mathbb{R}$ and $y \in I_x$, let t_{xy} denote the parameter value for which the unit tangent vector $(x, y, t_{xy}) \in \mathcal{L}(\tilde{\gamma})$. Then

$$(3.3) \quad \mathcal{L}_\varepsilon(\tilde{\gamma}) = \{(x, y, t) \mid x \in \mathbb{R}, y \in I_x \text{ and } t_{xy} \leq t \leq t_{xy} + \varepsilon\}.$$

Consequently,

$$\begin{aligned} (3.4) \quad \tilde{\mu}(\mathcal{L}_\varepsilon(\tilde{\gamma})) &= \frac{1}{\pi \text{area}(S)} \int_{\mathbb{R}^+} \int_{I_x} \int_{t_{xy}}^{t_{xy}+\varepsilon} (x-y)^{-2} dt dy dx \\ &= \frac{\varepsilon}{\pi \text{area}(S)} \int_0^\infty \int_{-\frac{\zeta^2}{x}}^{-\frac{1}{x}} (x-y)^{-2} dy dx. \\ &= \frac{\varepsilon}{\pi \text{area}(S)} \log \zeta = \frac{\varepsilon}{\pi \text{area}(S)} l(\gamma). \end{aligned}$$

□

3.2. Many geodesics. We shall prove a theorem that is slightly more general than Theorem 1. Let $\Gamma = \{\gamma_i\}_{i=1}^n$ be a finite sequence of mutually disjoint simple closed geodesics, except that we allow geodesics to appear twice in the list. To each γ_i in the sequence, associate a distinct half-collar A_i . With Γ as before and $\mathcal{A} = \bigcup A_i$, define $\mathcal{L}(\Gamma, \mathcal{A}) = \bigcup \mathcal{L}(\gamma_i, A_i)$ and $\mathcal{L}_\varepsilon(\Gamma, \mathcal{A}) = \bigcup \mathcal{L}_\varepsilon(\gamma_i, A_i)$. In this case there is not necessarily a subsurface M with which the half-collars are associated.

Suppose $v \in \mathcal{E}$. Then $G_t(v)$ will meet $\mathcal{L}(\Gamma, \mathcal{A})$ in a sequence of points $G_{\tau_j}(v)$. In other words, $\alpha_v(\tau_j)$ will lie on one of the γ_i with a tangent pointing into A_i . Let $\mathcal{N}_v(t) = \#\{i \mid \tau_i < t\}$ denote the number of times the orbit lies in $\mathcal{L}(\Gamma, \mathcal{A})$ or

the number of times α_v crosses one of the geodesics in the direction stipulated by a half-collar.

Theorem 3. *For almost all $v \in T_1 S$,*

$$(3.5) \quad \lim_{t \rightarrow \infty} \frac{\mathcal{N}_v(t)}{t} = \frac{l(\Gamma)}{\pi \operatorname{area}(S)}.$$

Proof. The characteristic function of a set Y is written χ_Y . For ε small we have the inequalities

$$(3.6) \quad \int_0^t \chi_{\mathcal{L}_\varepsilon(\Gamma, \mathcal{A})}(G_\tau(v)) d\tau - 2\varepsilon \leq \varepsilon \mathcal{N}_v(t) \leq \int_0^t \chi_{\mathcal{L}_\varepsilon(\Gamma, \mathcal{A})}(G_\tau(v)) d\tau + 2\varepsilon.$$

Divide through by $t\varepsilon$ and let $t \rightarrow \infty$. By the Ergodic Theorem and Proposition 1, for almost all $v \in T_1 S$, the left- and right-hand limits converge to

$$\begin{aligned} (1/\varepsilon) \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi_{\mathcal{L}_\varepsilon(\Gamma, \mathcal{A})}(G_\tau(v)) d\tau &= (1/\varepsilon) \sum_{i=1}^n \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \chi_{\mathcal{L}_\varepsilon(\gamma_i, A_i)}(G_\tau(v)) d\tau \\ &= (1/\varepsilon) \sum_{i=1}^n \mu(\mathcal{L}_\varepsilon(\gamma_i)) = \frac{\sum_{i=1}^n l(\gamma_i)}{\pi \operatorname{area}(S)} = \frac{l(\Gamma)}{\pi \operatorname{area}(S)}. \end{aligned}$$

□

Proof of Theorem 1. The theorem follows from Theorem 3 by choosing Γ to be the boundary of M and taking the half-collar A_i associated to γ_i to be the one inside M . □

3.3. Excursion times.

Proof of Corollary 1. With the possible exception of the very first, the sum of the first n excursion lengths is

$$\sum_{i=1}^n (s_i - t_i) = \int_0^{s_n} \chi_{T_1 M}(G_\tau(v)) d\tau.$$

The average can then be written in the form

$$\frac{1}{s_n} \int_0^{s_n} \chi_{T_1 M}(G_\tau(v)) d\tau \times \frac{s_n}{n}.$$

The value n on the right is $\#\{i \mid t_i < s_n\}$, from Theorem 1. By that theorem, the limit as $n \rightarrow \infty$ of s_n/n exists and is the inverse of the right-hand side of equation (2.2). By the Ergodic Theorem, for almost all v the limit of the first factor also exists and is the μ -measure of $T_1 M$; this is the area of M divided by the area of S . Note that in both instances convergence is almost everywhere. Taken together, this completes the proof. □

4. PROOFS FOR COLLARS

4.1. Preliminaries. For the remainder of the paper we shall suppose that the lift of the half-collar A_r lies in the right half-plane. Recall that ϕ is the angle made by the ray $\tilde{\lambda}_{A_r}$ and the imaginary axis. Write $\sin \phi + i \cos \phi = a + ib = p_\phi$. The relationship between $\operatorname{area}(\mathbf{C}_r)$, r and ϕ is given in the following proposition.

Proposition 2. $\text{area}(\mathbf{C}_r(\gamma)) = 2l(\gamma) \tan \phi = 2l(\gamma) \sinh r$. In particular, $\cosh r = \frac{1}{\cos \phi} = \frac{1}{b}$.

Proof. In order to see how r and ϕ are related, we look at the geodesic segment that lies on the unit circle centered at 0 running between the imaginary axis and the line $\tilde{\lambda}_{A_r}$. Using polar coordinates with the hyperbolic metric in \mathbb{H} we see that

$$r = \int_{\frac{\pi}{2}-\phi}^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta} = \log(\sec \phi + \tan \phi).$$

Therefore, $e^r = \sec \phi + \tan \phi = \sqrt{1 + \tan^2 \phi} + \tan \phi$. Solving for $\tan \phi$ gives $\sinh r$.

The area of $\mathbf{C}_r(\gamma)$ is the area of the region in $\mathbf{C}_r(\tilde{\gamma})$ between the two circles centered at the origin with radii 1 and ζ respectively. Again, computing with the hyperbolic metric in polar coordinates we have

$$\text{area}(\mathbf{C}_r(\gamma)) = 2 \int_1^\zeta \frac{1}{r} dr \int_{\frac{\pi}{2}-\phi}^{\frac{\pi}{2}} \frac{1}{\sin^2 \theta} d\theta = 2 \log \zeta \tan \phi = 2l(\gamma) \tan \phi.$$

□

The following simple, geometric lemma will be very useful.

Lemma 1.

- (1) Given $x > 0$, if the geodesic \overline{xy} in \mathbb{H} passes through x and the point tp_ϕ , $t > 0$, then

$$(4.1) \quad y = \frac{axt - t^2}{x - at}.$$

- (2) The geodesic \overline{xy} in \mathbb{H} , $x > y$, tangent to $\tilde{\lambda}_A$ at the point tp_ϕ has endpoints

$$(4.2) \quad x = t\left(\frac{1+b}{a}\right) \text{ and } y = t\left(\frac{1-b}{a}\right).$$

Consequently, given x , \overline{xy} is tangent to $\tilde{\lambda}_A$ when

$$(4.3) \quad y = \left(\frac{1-b}{1+b}\right)x.$$

Proof. As in formula (3.2) the geodesic \overline{xy} contains tp_ϕ if

$$\left|\frac{x+y}{2} - tp_\phi\right|^2 = \left(\frac{x-y}{2}\right)^2.$$

With $p_\phi = a + ib$, we square both sides and simplify to get $xy - tax - tay + t^2 = 0$. Solving for y gives (4.1).

Suppose that \overline{xy} is tangent to $\tilde{\lambda}_A$ at tp_ϕ . Let w be the midpoint on the real axis between y and x . Then the line from w to tp_ϕ is orthogonal to $\tilde{\lambda}_{A_r}$. Equating slopes we have $\frac{tb}{ta-w} = -\frac{a}{b}$ or $w = \frac{t}{a}$. Then the radius of the semicircle \overline{xy} is $|tp_\phi - \frac{t}{a}| = \frac{b}{a}t$. With center and radius in hand it is easy to write x and y , which gives (4.2). Formula (4.3) follows. □

4.2. The thickened section. In order to prove Theorem 2 we shall recycle the approach taken in the proof of Theorem 1, only this time the thickened section will be defined with respect to the boundary λ_{A_r} of the collar neighborhood of γ . The situation is somewhat more involved.

Recall that $g(z) = \zeta z$ covers the geodesic γ . Let \mathbf{s} denote the segment of $\tilde{\lambda}_{A_r}$ between the points p_ϕ and ζp_ϕ . It is a preimage of λ_{A_r} under the covering projection. Henceforth x will always be a positive number.

We define a notion of intersection between the directed geodesic $\beta = \overline{xy}$ and the arc \mathbf{s} that only counts the first point at which the geodesic intersects $\tilde{\lambda}_{A_r}$. More precisely, if $\beta \cap \tilde{\lambda}_A = \emptyset$, then set $\overline{xy} \hat{\cap} \mathbf{s} = \emptyset$. Also, if β intersects $\tilde{\lambda}_A$ in the points $\beta(t_1)$ and $\beta(t_2)$, $t_1 \leq t_2$, then

$$\overline{xy} \hat{\cap} \mathbf{s} = \begin{cases} \{\beta(t_1)\} & \text{if } \beta(t_1) \in \mathbf{s} \\ \emptyset & \text{otherwise.} \end{cases}$$

We shall define several sections of the unit tangent bundle over λ_{A_r} by specifying sets of triples (x, y, t_{xy}) in the unit tangent bundle over \mathbf{s} . For $i = 0, 1, 2, 3$ let

$$\mathcal{J}^{i*}(\mathbf{s}) = \{(x, y, t_{xy}) | R^i \text{ and } \overline{xy} \hat{\cap} \mathbf{s} \neq \emptyset\},$$

where R^0, \dots, R^3 denote respectively the conditions, $R^0 : x > 0, y \in \mathbb{R}$, $R^1 : 0 < y < x$, $R^2 : 0 < x < y$, $R^3 : y < 0 < x$. Given x, y , if $\overline{xy} \hat{\cap} \mathbf{s} \neq \emptyset$, then there is a unique t_{xy} so that (x, y, t_{xy}) is the vector tangent to the directed geodesic \overline{xy} at the point of intersection with \mathbf{s} .

As in Section 3.1, define $\mathcal{J}^i(\mathbf{s}) = \mathcal{E} \cap \mathcal{J}^{i*}(\mathbf{s})$ and the thickened sections

$$\mathcal{J}_\varepsilon^i(\mathbf{s}) = \{(x, y, t) | (x, y, t_{xy}) \in \mathcal{J}^i(\mathbf{s}) \text{ and } t_{xy} \leq t \leq t_{xy} + \varepsilon\}.$$

$\mathcal{J}_\varepsilon^i(\mathbf{s})$ projects to the thickened section $\mathcal{J}_\varepsilon^i(\lambda_{A_r})$ in the unit tangent bundle of S over the boundary of the collar. We let $\mathcal{L}_\varepsilon(\mathbf{s})$ denote the union of the sections $\mathcal{J}_\varepsilon^i(\mathbf{s})$.

In order to see what these sets represent, note that up to measure zero, $\mathcal{J}^0(\lambda_{A_r})$ is equal to the set of all vectors over λ_{A_r} pointing into the collar. $\mathcal{J}^3(\lambda_{A_r})$ is the subset determining geodesics that cross γ . By elementary hyperbolic geometry such geodesics will then exit the collar at λ_B . Removing $\mathcal{J}^3(\lambda_{A_r})$ from $\mathcal{J}^0(\lambda_{A_r})$ results in two disjoint sets, which are $\mathcal{J}^1(\lambda_{A_r})$ and $\mathcal{J}^2(\lambda_{A_r})$. Geodesics determined by vectors from these subsets enter λ_{A_r} from different sides of $\mathcal{J}^3(\lambda_{A_r})$ and exit the collar at λ_{A_r} . Together these are all the geodesics that enter and exit via λ_{A_r} .

Proposition 3. $\mu(\mathcal{J}_\varepsilon^1(\lambda_{A_r}) \cup \mathcal{J}_\varepsilon^3(\lambda_{A_r})) = \frac{\varepsilon(1 + \cosh r)l(\gamma)}{2\pi \text{ area}(S)}.$

Proof. The computation is done in (x, y, t) coordinates for the sets $\mathcal{J}_\varepsilon^i(\mathbf{s})$. We shall describe the limits of integration by specifying potential x values and then giving the corresponding set of y values using Lemma 1. Three cases are distinguished for the domain of x . In all we have $y < x$. For now we make the additional assumption that $\zeta a < (1 + b)/a$.

If x is larger than $\zeta(1 + b)/a$, then for all $y \in \mathbb{R}$, $\overline{xy} \hat{\cap} \mathbf{s} = \emptyset$. Thus, the first case to consider is when $(1 + b)/a < x < \zeta(1 + b)/a$. It follows from (4.2) that for such an x there is a point y so that \overline{xy} is tangent to $\tilde{\lambda}_A$ and that this point of tangency lies in \mathbf{s} . It also follows from (4.3) that given x , the interval of corresponding y values

will vary between $(\frac{1+b}{1+b})x$, the point at which \overline{xy} is tangent to \mathbf{s} , and $\frac{ax\zeta-\zeta^2}{x-a\zeta}$, where \overline{xy} meets ζp_θ , the upper endpoint of \mathbf{s} . Thus the measure of the set of vectors in $\mathcal{L}_\varepsilon(\mathbf{s})$ determined by these (x, y) values is M_1 , where

$$\begin{aligned} \pi \operatorname{area}(S) M_1 &= \int_{\frac{1+b}{a}}^{\zeta(\frac{1+b}{a})} \int_{\frac{ax\zeta-\zeta^2}{x-a\zeta}}^{(\frac{1+b}{1+b})x} \int_{t_{xy}}^{t_{xy}+\varepsilon} (x-y)^{-2} dt dy dx \\ &= \varepsilon \left[\frac{1+b}{2b} \log \zeta - \log \zeta - \frac{1}{2} \log \left(\frac{(1+b)^2}{a^2} - 2(1+b) + 1 \right) \right. \\ &\quad \left. + \frac{1}{2} \log \left(\frac{(1+b)^2}{a^2} - 2(1+b)\zeta + \zeta^2 \right) \right]. \end{aligned}$$

The second case is when $\zeta a < x < (1+b)/a$. According to the lemma, as y varies between $\frac{ax-1}{x-a}$ and $\frac{ax\zeta-\zeta^2}{x-a\zeta}$, $\overline{xy} \cap \mathbf{s}$ varies between p_ϕ and ζp_ϕ , taking on all values in \mathbf{s} . Writing out the integral as in the previous case, we see that the measure of the corresponding subset of $\mathcal{L}_\varepsilon(\mathbf{s})$ is equal to M_2 , where

$$\begin{aligned} \pi \operatorname{area}(S) M_2 &= \varepsilon \left[\frac{1}{2} \log \left(\frac{(1+b)^2}{a^2} - 2(1+b) + 1 \right) - \frac{1}{2} \log (a^2\zeta^2 - 2a^2\zeta + 1) \right. \\ &\quad \left. + \log \zeta + \log(1-a^2) - \frac{1}{2} \log \left(\frac{(1+b)^2}{a^2} - 2(1+b)\zeta + \zeta^2 \right) \right]. \end{aligned}$$

The final case is where $a < x < \zeta a$. Given x in this interval, \overline{xy} will meet \mathbf{s} at p_ϕ when $y = \frac{ax-1}{x-a}$ and as y goes to $-\infty$, \overline{xy} limits at the vertical line intersecting \mathbf{s} in the point tp_ϕ where $t = \frac{x}{\sin \phi}$. In this case the measure of the associated subset of $\mathcal{L}_\varepsilon(\mathbf{s})$ is M_3 with

$$\pi \operatorname{area}(S) M_3 = \varepsilon \left[\frac{1}{2} \log (a^2\zeta^2 - 2a^2\zeta + 1) - \frac{1}{2} \log(1-a^2) \right].$$

Using Proposition 2 and the facts that $a = \sin \phi$, $b = \cos \phi$ and $\log \zeta = l(\gamma)$, we get

$$M_1 + M_2 + M_3 = \frac{\varepsilon(1+b)l(\gamma)}{2\pi b \operatorname{area}(S)} = \frac{\varepsilon(1+\cosh r)l(\gamma)}{2\pi \operatorname{area}(S)},$$

which is the proposition under the assumption that $\zeta a < (1+b)/a$. If we reverse the inequality a similar computation yields the same result. \square

4.3. Returns to a collar.

Proof of Theorem 2. To begin we show that

$$(4.4) \quad \mu(\mathcal{J}_\varepsilon^3(\lambda_{A_r})) = \mu(\mathcal{L}_\varepsilon(\gamma)) = \frac{\varepsilon l(\gamma)}{\pi \operatorname{area}(S)}.$$

It was observed in Section 3.1 that given $v \in \mathcal{E}$, $G_t(v)$ will meet $\mathcal{L}(\Gamma, \mathcal{A})$ in a sequence of points $G_{\tau_j}(v)$, $j \in \mathbb{Z}^+$. Each time α_v meets the geodesic γ from the A side, it must have either originated in A or else crossed λ_{A_r} first before proceeding to γ . Conversely, each time α_v crosses λ_{A_r} , so that its tangent lies in $\mathcal{J}_\varepsilon^3(\lambda_{A_r})$, it must go on to cross γ . Thus, for each $j \geq 2$ there will be a corresponding value η_j so that $G_{\eta_j}(v) \in \mathcal{J}_\varepsilon^3(\lambda_{A_r})$, and if $G_\eta(v) \in \mathcal{J}_\varepsilon^3(\lambda_{A_r})$, then $\eta = \eta_j$ for some j . In other words, the section $\mathcal{J}_\varepsilon^3(\lambda_{A_r})$ counts the crossing of γ from the A side exactly as does $\mathcal{L}_\varepsilon(\gamma)$.

As a consequence of the above, we can replace $\mathcal{L}_\varepsilon(\Gamma, \mathcal{A})$ (with $\Gamma = \gamma$) in formula (3.6) by $\mathcal{J}_\varepsilon^3(\lambda_{A_r})$. It follows that the value in formula (2.2) of Theorem 1 is $(1/\varepsilon)\mu(\mathcal{J}_\varepsilon^3(\lambda_{A_r})) = \frac{l(\gamma)}{\pi \operatorname{area}(S)}$, completing the argument.

The hyperbolic isometry $h(z) = \zeta/\bar{z}$ induces an isometry of $T_1\mathbb{H}$ that interchanges the sets $\mathcal{J}_\varepsilon^1(\mathbf{s})$ and $\mathcal{J}_\varepsilon^2(\mathbf{s})$. Therefore, we have

$$\mu(\mathcal{J}_\varepsilon^1(\lambda_{A_r})) = \mu(\mathcal{J}_\varepsilon^2(\lambda_{A_r})) \text{ or } \mu(\mathcal{J}_\varepsilon^1(\lambda_{A_r}) \cup \mathcal{J}_\varepsilon^2(\lambda_{A_r})) = 2\mu(\mathcal{J}_\varepsilon^1(\lambda_{A_r})).$$

Now making use of the outcomes of formula (4.4) and Proposition 3, we have

$$\mu(\mathcal{J}_\varepsilon^1(\lambda_{A_r})) = \mu(\mathcal{J}_\varepsilon^1(\lambda_{A_r}) \cup \mathcal{J}_\varepsilon^3(\lambda_{A_r})) - \mu(\mathcal{J}_\varepsilon^3(\lambda_{A_r})) = \frac{\varepsilon(\cosh r - 1)l(\gamma)}{2\pi \operatorname{area}(S)}$$

and

$$\mu(\mathcal{J}_\varepsilon^0(\lambda_{A_r})) = 2\mu(\mathcal{J}_\varepsilon^1(\lambda_{A_r})) + \mu(\mathcal{J}_\varepsilon^3(\lambda_{A_r})) = \frac{\varepsilon l(\gamma) \cosh r}{\pi \operatorname{area}(S)}.$$

Let $P = \{(t_i, s_i)\}$ and $P' = \{(t'_i, s'_i)\}$ be the excursion parameters of α_v . For $r > R_{l(\gamma)}$ and $v \in \mathcal{E}$ define the counting functions $N_v^0(r)(t) = \#\{i \mid t_i < t\}$ and $N_v^1(r)(t) = \#\{i \mid t'_i < t\}$. Taking the role of $\mathcal{L}(\gamma)$ from the proof of Theorem 1, the sections $\mathcal{J}^0(\lambda_{A_r})$ and $\hat{\mathcal{J}}(\lambda_{A_r}) = \mathcal{J}^1(\lambda_{A_r}) \cup \mathcal{J}^2(\lambda_{A_r})$ count crossings by α_v of λ_{A_r} that go into A and crossings of λ_{A_r} by α_v into A that exit A through λ_{A_r} , respectively. Thus, rewriting formula (3.6) in the first instance gives, for $\varepsilon > 0$ sufficiently small,

$$\int_0^t \chi_{\mathcal{J}_\varepsilon^0(\lambda_{A_r})}(G_\tau(v)) d\tau - 2\varepsilon \leq \varepsilon N_v^0(r)(t) \leq \int_0^t \chi_{\mathcal{J}_\varepsilon^0(\lambda_{A_r})}(G_\tau(v)) d\tau + 2\varepsilon.$$

Again, dividing by $t\varepsilon$, letting t go to infinity and applying the Ergodic Theorem, we prove the first part of Theorem 2. The proof of the second part of the theorem follows if \mathcal{J}^0 and N^0 are replaced by $\hat{\mathcal{J}}$ and N^1 . \square

Proof of Corollary 2. Let $\mathbf{V}_A \subset T_1\mathbf{C}_r$ denote the set of vectors that are tangent to the geodesic segments in \mathbf{C}_r which begin on λ_{A_r} . Then one has $2\mu(\mathbf{V}_A) = \mu(T_1\mathbf{C}_r)$.

Proceeding as in the proof of Corollary 1, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n (s_i - t_i) &= \lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{s_n} \chi_{\mathbf{V}}(G_\tau(v)) d\tau \\ &= \lim_{n \rightarrow \infty} \frac{1}{s_n} \int_0^{s_n} \chi_{\mathbf{V}}(G_\tau(v)) d\tau \times \frac{s_n}{n}. \end{aligned}$$

By the Ergodic Theorem and Theorem 2, this equals

$$\frac{\operatorname{area}(\mathbf{C}_r)}{2 \operatorname{area}(S)} \times \frac{\pi \operatorname{area}(S)}{l(\gamma) \cosh r} = \pi \tanh r.$$

Here, the latter equality follows from Proposition 2. \square

Proof of Corollary 3. We argue as in [11]. Write $N_v(r)(t) = \#\{j \mid t_j < t\}$ for the function that counts returns to the radius r collar. Using Theorem 2, the distribution $\delta(r)$ can be written as

$$\delta(r) = \lim_{n \rightarrow \infty} \frac{N_v(r)(t)}{N_v(R_0)(t)} = \frac{\cosh r}{\cosh R_0}.$$

This completes the proof of Corollary 2. \square

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