SUB-RIEMANNIAN BALLS IN CR SASAKIAN MANIFOLDS

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ABSTRACT. By using sub-Riemannian Li-Yau type heat kernel estimates we prove global estimates for the sub-Riemannian distance of CR Sasakian manifolds with nonnegative horizontal Webster-Tanaka Ricci curvature. In particular, in this setting, large sub-Riemannian balls are comparable to Riemannian balls.

1. INTRODUCTION

Let $\mathbb{M}$ be a complete strictly pseudo-convex CR Sasakian manifold with real dimension $2n + 1$. Let θ be a pseudo-hermitian form on $\mathbb{M}$ with respect to which the Levi form L_θ is positive definite. The kernel of θ determines a horizontal bundle $\mathcal{H}$. Now denote by T the Reeb vector field on $\mathbb{M}$, i.e., the characteristic direction of θ . We denote by ∇ the Tanaka-Webster connection of $\mathbb{M}$.

We recall that the CR manifold $(\mathbb{M}, \theta)$ is called Sasakian if the pseudo-hermitian torsion of ∇ vanishes, in the sense that $\mathbf{T}(T, X) = 0$, for every $X \in \mathcal{H}$. For instance the standard CR structures on the Heisenberg group $\mathbb{H}_{2n+1}$ and the sphere $\mathbb{S}^{2n+1}$ are Sasakian. In every Sasakian manifold the Reeb vector field T is a sub-Riemannian Killing vector field (see for instance Theorem 1.5 on p. 42 and Lemma 1.5 on p. 43 in [3]).

We consider the family of scaled Riemannian metrics g_τ , $\tau > 0$, such that for $X, Y \in \mathcal{H}$,

$$(1.1) \quad g_\tau(X, Y) = d\theta(X, JY), \quad g_\tau(X, T) = 0, \quad g_\tau(T, T) = \frac{1}{\tau^2},$$

where J is the complex structure on $\mathbb{M}$. We denote by d_τ the distance corresponding to the Riemannian structure g_τ and by d the sub-Riemannian distance on $\mathbb{M}$. It is well known that $d_\tau(x, y) \rightarrow d(x, y)$ when $\tau \rightarrow 0$. Our goal is to prove the following theorem:

Theorem 1.1. *Let $\mathcal{R}$ be the Ricci curvature of the Webster-Tanaka connection ∇ . If for every $X \in \mathcal{H}$,*

$$\mathcal{R}(X, X) \geq 0,$$

then for every $x, y \in \mathbb{M}$,

$$d_\tau(x, y) \leq d(x, y) \leq A_n d_\tau(x, y) + B_n \sqrt{\tau} d_\tau(x, y)^{1/2},$$

where A_n and B_n are two positive universal constants depending only on n .

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To put things in perspective, estimates between the sub-Riemannian distance and Riemannian ones have been extensively studied in the literature (see for instance [4], [5], [6], [7], [8]). But in these cited works, such estimates are local in nature. To the knowledge of the authors, Theorem 1.1 is the first result that gives global and uniform estimates for a large class of sub-Riemannian metrics. It is consistent with the well known Nagel-Stein-Wainger estimate [7], which implies that, at small scales, $d(x, y) \leq Cd_\tau(x, y)^{\frac{1}{2}}$, and shows that, at large scales, due to curvature effects we have $d(x, y) \simeq d_\tau(x, y)$.

2. LI-YAU AND HARNACK ESTIMATES FOR THE HEAT KERNEL ON SASAKIAN MANIFOLDS

2.1. Curvature dimension inequalities and heat kernel bounds. We first recall some results that will be needed in the sequel and that can be found in [1] and [2]. We denote by Δ the sub-Laplacian on $\mathbb{M}$ and by $\nabla^{\mathcal{H}}$ the horizontal gradient. For smooth functions $f : \mathbb{M} \rightarrow \mathbb{R}$, set

$$(2.1) \quad \Gamma_2(f) = \frac{1}{2} [\Delta \|\nabla^{\mathcal{H}} f\|^2 - 2 \langle \nabla^{\mathcal{H}} f, \nabla^{\mathcal{H}} \Delta f \rangle]$$

and

$$(2.2) \quad \Gamma_2^T(f) = \frac{1}{2} [\Delta (Tf)^2 - 2(Tf)(T\Delta f)].$$

The following result was obtained in [2] by means of a Bochner type formula.

Theorem 2.1. *Assume that for every $X \in \mathcal{H}$,*

$$\mathcal{R}(X, X) \geq 0.$$

Then for every $f \in C^\infty(\mathbb{M})$ and any $\nu > 0$,

$$\Gamma_2(f) + \nu \Gamma_2^T(f) \geq \frac{1}{2n} (\Delta f)^2 - \frac{1}{\nu} \|\nabla^{\mathcal{H}} f\|^2 + \frac{n}{2} (Tf)^2.$$

We denote by $p(t, x, y)$ the heat kernel of $\mathbb{M}$, that is, the fundamental solution of the heat equation $\frac{\partial f}{\partial t} = \Delta f$. The following global lower and upper bounds were proved in [1].

Theorem 2.2. *Assume that for every $X \in \mathcal{H}$,*

$$\mathcal{R}(X, X) \geq 0.$$

For any $0 < \varepsilon \leq 1$ there exists a constant $C(\varepsilon) = C(n, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \rightarrow 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has

$$\begin{aligned} & \frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))} \exp \left(- \left(1 + \frac{3}{n} \right) \frac{d(x, y)^2}{(4 - \varepsilon)t} \right) \\ & \leq p(t, x, y) \leq \frac{C(\varepsilon)}{\mu(B(x, \sqrt{t}))} \exp \left(- \frac{d(x, y)^2}{(4 + \varepsilon)t} \right). \end{aligned}$$

In the above theorem, d is the sub-Riemannian distance, $B(x, \sqrt{t})$ is the sub-Riemannian ball with center x and radius $\sqrt{t}$, and μ is the volume corresponding to the volume form $\theta \wedge (d\theta)^n$.

2.2. Harnack type estimates. From now on and in the sequel we assume that for every $X \in \mathcal{H}$, $\mathcal{R}(X, X) \geq 0$. We first have the following Li-Yau type estimate for the heat kernel.

Proposition 2.3. *For $t > 0$,*

$$\|\nabla^{\mathcal{H}} \ln p_t\|^2 + \frac{n}{3} t (T \ln p_t)^2 \leq \left(1 + \frac{3}{n}\right) \frac{\Delta p_t}{p_t} + \frac{n \left(1 + \frac{3}{n}\right)^2}{t}.$$

Proof. The result is essentially proved in [2], but due to the simplicity of the argument, we reproduce, without the details, the proof by sake of completeness. Fix $T > 0$ and consider the functional

$$\Phi(t) = \frac{3}{n} (T-t)^2 P_t \left(\frac{\|\nabla^{\mathcal{H}} p_{T-t}\|^2}{p_{T-t}} \right) + (T-t)^3 P_t \left(\frac{(T p_{T-t})^2}{p_{T-t}} \right),$$

where P_t is the heat semigroup associated with Δ . Since T is a Killing vector field, for any smooth function f we have

$$\langle \nabla^{\mathcal{H}} f, \nabla^{\mathcal{H}} (Tf)^2 \rangle = (Tf)(T\|\nabla^{\mathcal{H}} f\|^2).$$

Differentiating Φ and using the above yield

$$\begin{aligned} \Phi'(t) &= \frac{6}{n} (T-t)^2 P_t (p_{T-t} \Gamma_2(\ln p_{T-t})) + 2(T-t)^3 P_t (p_{T-t} \Gamma_2^T(\ln p_{T-t})) \\ &\quad - \frac{6}{n} (T-t) P_t \left(\frac{\|\nabla^{\mathcal{H}} p_{T-t}\|^2}{p_{T-t}} \right) - 3(T-t)^2 P_t \left(\frac{(T p_{T-t})^2}{p_{T-t}} \right). \end{aligned}$$

From Theorem 2.1, we have

$$\begin{aligned} &\frac{6}{n} (T-t)^2 p_{T-t} \Gamma_2(\ln p_{T-t}) + 2(T-t)^3 p_{T-t} \Gamma_2^T(\ln p_{T-t}) \\ &\geq \frac{3}{n^2} (T-t)^2 p_{T-t} (\Delta \ln p_{T-t})^2 - \frac{18}{n^2} (T-t) p_{T-t} \|\nabla^{\mathcal{H}} \ln p_{T-t}\|^2 \\ &\quad + 3(T-t)^2 p_{T-t} (T \ln p_{T-t})^2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \Phi'(t) &\geq \frac{3}{n^2} (T-t)^2 P_t (p_{T-t} (\Delta \ln p_{T-t})^2) \\ &\quad - \left(\frac{18}{n^2} + \frac{6}{n} \right) (T-t) P_t (p_{T-t} \|\nabla^{\mathcal{H}} \ln p_{T-t}\|^2). \end{aligned}$$

Now, for every $\gamma(t)$, we have

$$\begin{aligned} (\Delta(\ln p_{T-t}))^2 &\geq 2\gamma(t) \Delta \ln p_{T-t} - \gamma(t)^2 \\ &\geq 2\gamma(t) \left(\frac{\Delta p_{T-t}}{p_{T-t}} - \|\nabla^{\mathcal{H}} \ln p_{T-t}\|^2 \right) - \gamma(t)^2. \end{aligned}$$

Therefore we get

$$\begin{aligned} \frac{3}{n^2} (T-t)^2 P_t (p_{T-t} (\Delta \ln p_{T-t})^2) &\geq \frac{6}{n^2} (T-t)^2 \gamma(t) (\Delta p_{T-t} - P_t (p_{T-t} \|\nabla^{\mathcal{H}} \ln p_{T-t}\|^2)) \\ &\quad - \frac{3}{n^2} (T-t)^2 \gamma(t)^2 p_{T-t}. \end{aligned}$$

This implies

$$\begin{aligned} \Phi'(t) &\geq \frac{6}{n^2}(T-t)^2\gamma(t)\Delta p_T - \frac{3}{n^2}(T-t)^2\gamma(t)^2 p_T \\ &\quad - \left(\frac{18}{n^2} + \frac{6}{n} + \frac{6}{n^2}(T-t)\gamma(t) \right) (T-t)P_t(p_{T-t}\|\nabla^{\mathcal{H}} \ln p_{T-t}\|^2). \end{aligned}$$

Choosing $\gamma(t) = -\frac{n+3}{T-t}$ then leads to

$$\Phi'(t) \geq -\frac{6(n+3)}{n^2}(T-t)\Delta p_T - \frac{3}{n^2}(n+3)^2.$$

By integrating the last inequality from 0 to T , we obtain

$$-\Phi(0) \geq -\frac{3(n+3)}{n^2}T^2\Delta p_T - \frac{3}{n^2}(n+3)^2T,$$

which is the required inequality. $\square$

We can deduce from the previous Li-Yau type inequality the following Harnack inequality.

Theorem 2.4. *For $x, y, z \in \mathbb{M}$, $s < t$,*

$$p(s, x, y) \leq p(t, x, z) \left(\frac{t}{s} \right)^{n+3} \exp \left(\left(1 + \frac{3}{n} \right) \left(\frac{1}{4(t-s)} + \frac{\frac{3}{n}\tau^2 \ln \frac{t}{s}}{4(t-s)^2} \right) d_\tau(x, y)^2 \right),$$

$s < t,$

where d_τ denotes the Riemannian metric introduced in (1.1).

Proof. From Proposition 2.3,

$$\|\nabla^{\mathcal{H}} \ln p_t\|^2 \leq \left(1 + \frac{3}{n} \right) \frac{\Delta p_t}{p_t} + \frac{n \left(1 + \frac{3}{n} \right)^2}{t}$$

and

$$\frac{n}{3}t(T \ln p_t)^2 \leq \left(1 + \frac{3}{n} \right) \frac{\Delta p_t}{p_t} + \frac{n \left(1 + \frac{3}{n} \right)^2}{t}.$$

Therefore we have that for every $\tau > 0$,

(2.3)

$$\|\nabla^{\mathcal{H}} \ln p_t\|^2 + \tau^2(T \ln p_t)^2 \leq \left(1 + \frac{3\tau^2}{nt} \right) \left(1 + \frac{3}{n} \right) \frac{\Delta p_t}{p_t} + \frac{n \left(1 + \frac{3}{n} \right)^2}{t} \left(1 + \frac{3\tau^2}{nt} \right).$$

Now let $x, y, z \in \mathbb{M}$ and let $\gamma : [s, t] \rightarrow \mathbb{M}$, $s < t$, be an absolutely continuous path such that $\gamma(s) = y, \gamma(t) = z$. We first write (2.3) in the form

$$(2.4) \quad g_\tau(\nabla^\tau \ln p_u, \nabla^\tau \ln p_u) \leq a(u) \frac{\Delta p_u}{p_u} + b(u),$$

where ∇^τ denotes the Riemannian gradient of the metric g^τ ; that is,

$$g_\tau(\nabla^\tau \ln p_u, \nabla^\tau \ln p_u) = \|\nabla^{\mathcal{H}} \ln p_t\|^2 + \tau^2(T \ln p_t)^2$$

and

$$\begin{aligned} a(u) &= \left(1 + \frac{3\tau^2}{nu} \right) \left(1 + \frac{3}{n} \right), \\ b(u) &= \frac{n \left(1 + \frac{3}{n} \right)^2}{u} \left(1 + \frac{3\tau^2}{nu} \right). \end{aligned}$$

Let us now consider

$$\phi(u) = \ln p_u(x, \gamma(t)).$$

We compute

$$\phi'(u) = (\partial_u \ln p_u(x, \gamma(u)) + g_\tau(\nabla^\tau \ln p_u(x, \gamma(u)), \gamma'(u)).$$

Now, for every $\lambda > 0$, we have

$$g_\tau(\nabla^\tau \ln p_u(x, \gamma(u)), \gamma'(u)) \geq -\frac{1}{2\lambda^2} g_\tau(\nabla^\tau \ln p_u, \nabla^\tau \ln p_u) - \frac{\lambda^2}{2} g_\tau(\gamma'(u), \gamma'(u)).$$

Choosing $\lambda = \sqrt{\frac{a(u)}{2}}$ and then using (2.4) yield

$$\phi'(u) \geq -\frac{b(u)}{a(u)} - \frac{1}{4} a(u) g_\tau(\gamma'(u), \gamma'(u)).$$

By integrating this inequality from s to t we get as a result:

$$\ln p(t, x, y) - \ln p(s, x, z) \geq -\int_s^t \frac{b(u)}{a(u)} du - \frac{1}{4} \int_s^t a(u) g_\tau(\gamma'(u), \gamma'(u)) du.$$

We now minimize the quantity $\int_s^t a(u) g_\tau(\gamma'(u), \gamma'(u)) du$ over the set of absolutely continuous paths such that $\gamma(s) = y, \gamma(t) = z$. By using reparametrization of paths, it is seen that

$$\int_s^t a(u) g_\tau(\gamma'(u), \gamma'(u)) du \geq \frac{d_\tau^2(x, y)}{\int_s^t \frac{dv}{a(v)}},$$

with equality achieved for $\gamma(u) = \sigma\left(\frac{\int_s^u \frac{dv}{a(v)}}{\int_s^t \frac{dv}{a(v)}}\right)$, where $\sigma : [0, 1] \rightarrow \mathbb{M}$ is a unit geodesic for the distance d_τ that joins y and z . As a conclusion we get

$$p(s, x, y) \leq \exp\left(\int_s^t \frac{b(u)}{a(u)} du + \frac{d_\tau^2(y, z)}{4 \int_s^t \frac{dv}{a(v)}}\right) p(t, x, z).$$

Finally, from Cauchy-Schwarz inequality, we have

$$\int_s^t \frac{dv}{a(v)} \geq \frac{(t-s)^2}{\int_s^t a(v) dv},$$

and thus

$$p(s, x, y) \leq \exp\left(\int_s^t \frac{b(u)}{a(u)} du + \frac{d_\tau^2(y, z) \int_s^t a(v) dv}{4(t-s)^2}\right) p(t, x, z).$$

□

3. UNIFORM DISTANCE ESTIMATES

We are now ready to prove Theorem 1.1:

Proof. The inequality $d_\tau(x, y) \leq d(x, y)$ is straightforward. We now prove the second inequality. From Theorem 2.4 and Theorem 2.2,

$$\begin{aligned} p(t, x, y) &\geq \frac{1}{2^{n+3}} p(t/2, x, x) \exp\left(-\left(1 + \frac{3}{n}\right) \left(\frac{1}{2t} + \frac{3 \ln 2\tau^2}{nt^2}\right) d_\tau(x, y)^2\right) \\ &\geq \frac{1}{2^{n+3}} \frac{C_0(n)}{\mu(\mathbf{B}(x, \sqrt{t}))} \exp\left(-\left(1 + \frac{3}{n}\right) \left(\frac{1}{2t} + \frac{3 \ln 2\tau^2}{nt^2}\right) d_\tau(x, y)^2\right). \end{aligned}$$

From the Gaussian upper bound of Theorem 2.2 and the previous lower bound, we deduce that for all $t > 0$,

$$\begin{aligned} \ln \frac{2^{n+3}C(\varepsilon)}{C_0(n)} + \left(\frac{1}{2} \left(1 + \frac{3}{n} \right) d_\tau(x, y)^2 - \frac{d(x, y)^2}{4 + \varepsilon} \right) \frac{1}{t} \\ + \left(1 + \frac{3}{n} \right) \left(\frac{3 \ln 2}{n} \right) \tau^2 d_\tau^2(x, y) \frac{1}{t^2} \geq 0. \end{aligned}$$

We now choose $t = \tau d_\tau(x, y)$ and obtain

$$\begin{aligned} d(x, y)^2 \leq (4 + \varepsilon) \left(\ln \frac{2^{n+3}C(\varepsilon)}{C_0(n)} + \left(1 + \frac{3}{n} \right) \left(\frac{3 \ln 2}{n} \right) \right) \tau d_\tau(x, y) \\ + \left(2 + \frac{\varepsilon}{2} \right) \left(1 + \frac{3}{n} \right) d_\tau(x, y)^2. \end{aligned}$$

□

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