DISCRETE GABOR FRAMES IN $\ell^2(\mathbb{Z}^d)$

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ABSTRACT. The theory of Gabor frames for the infinite dimensional signal/function space $L^2(\mathbb{R}^d)$ and for the finite dimensional signal space $\mathbb{R}^d$ (or $\mathbb{C}^d$) has been extensively investigated in the last twenty years. However, very little has been done for the Gabor theory in the infinite dimensional discrete signal space $\ell^2(\mathbb{Z}^d)$, especially when $d > 1$. In this paper we investigate the general theory for discrete Gabor frames in $\ell^2(\mathbb{Z}^d)$. We focus on a few fundamental aspects of the theory such as the density/incompleteness theorem for frames and super-frames, the characterizations for dual frame pairs and orthogonal (strongly disjoint) frames, and the existence theorem for the tight dual frame of the Gabor type. The existence result for Gabor frames (resp. super-frames) requires a generalization of a standard result on common subgroup coset representatives.

1. INTRODUCTION

In 1946, D. Gabor proposed the short-time frequency analysis to expand a signal in $L^2(\mathbb{R})$ with the building blocks $\{g_{m,n}\}$, where

$$g_{m,n}(x) = e^{2\pi imbx} g(x - na), \quad m, n \in \mathbb{Z},$$

for a pair of fixed parameters $a, b \in \mathbb{R}$ and $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is called a *Gabor family*. In many applications, such as signal processing, we require this Gabor family to be either an orthonormal basis or a frame for $L^2(\mathbb{R})$ in order to provide for stable decomposition and reconstruction of signals. Although in the last twenty years most of the research in this area has been focused on the function space $L^2(\mathbb{R})$, there are practical reasons for studying the discrete version of Gabor analysis on $\mathbb{R}^n$ or $\ell^2(\mathbb{Z}^d)$. While much work has been done on the Gabor theory in the finite dimensional signal spaces (e.g., $\mathbb{R}^n$ or $\mathbb{C}^d$) (cf. [6, 20, 21, 22]), very little has been done on the higher dimensional signal space $\ell^2(\mathbb{Z}^d)$ [2, 5, 17, 18]. Like the extension of the Gabor theory from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}^d)$ (cf. [15]), there are some technical difficulties generalizing the $\ell^2(\mathbb{Z})$ -space Gabor theory to $\ell^2(\mathbb{Z}^d)$ due to the complexity of the involved high dimensional lattices in $\mathbb{Z}^d$. The purpose of this paper is to examine the analogues of a few fundamental theorems about Gabor frames in $\ell^2(\mathbb{Z}^d)$ that are well known in $L^2(\mathbb{R}^d)$. These theorems include the well known density/incompleteness theorem for frames, the characterizations for dual frame pairs and orthogonal (strongly disjoint) frames, and the existence theorem for the tight dual frame of the Gabor type. In

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particular, with the help of a generalization of an existence theorem for common subgroup coset representatives (which is related to the well known Hall's/marriage theorem), we extend the density/incompleteness theorem to Gabor super-frames.

In order to discuss the main results, we first recall and introduce some definitions and notation about frames for Hilbert spaces. A *frame* for a separable (real or complex) Hilbert space H is a sequence $\{x_j\}_{j \in \mathcal{J}}$ of H such that there exist two positive constants $A, B > 0$ with the property that

$$A\|x\|^2 \leq \sum_{j \in \mathcal{J}} |\langle x, x_j \rangle|^2 \leq B\|x\|^2$$

holds for every $x \in H$. The optimal constants (maximal for A and minimal for B) are called *frame bounds*. A λ -*tight frame* is a frame with equal frame bounds $A = B = \lambda$. It is called a *Parseval frame* if $A = B = 1$. A *uniform frame* is a frame when all the elements in the frame sequence have the same norm. If we require only that the right inequality holds, then we call the sequence a *Bessel sequence*.

For a frame $\{x_j\}_{j \in \mathcal{J}}$ of H , the associated *analysis operator* is the linear mapping $\Theta : H \rightarrow \ell^2(\mathcal{J})$ defined by

$$\Theta(x) = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle e_j,$$

where $\{e_j\}$ is the standard orthonormal basis for $\ell^2(\mathcal{J})$. The adjoint operator Θ^* of Θ is given by

$$\Theta^*\left(\sum_{j \in \mathcal{J}} c_j e_j\right) = \sum_{j \in \mathcal{J}} c_j x_j.$$

If we let $S = \Theta^* \Theta$, then we have

$$Sx = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle x_j, \quad x \in H.$$

Thus S is a positive invertible bounded linear operator on H , which is called the *frame operator* for $\{x_j\}_{j \in \mathcal{J}}$. A direct calculation yields

$$\begin{aligned} x &= \sum_{j \in \mathcal{J}} \langle x, S^{-1/2} x_j \rangle S^{-1/2} x_j \\ &= \sum_{j \in \mathcal{J}} \langle x, S^{-1} x_j \rangle x_j \\ &= \sum_{j \in \mathcal{J}} \langle x, x_j \rangle S^{-1} x_j \quad (x \in H). \end{aligned}$$

This tells us that $\{S^{-1/2} x_j\}_{j \in \mathcal{J}}$ is a Parseval frame, and $\{S^{-1} x_j\}_{j \in \mathcal{J}}$ is also a frame for H . The frame $\{S^{-1} x_j\}_{j \in \mathcal{J}}$ is called the *canonical (or standard) dual* of $\{x_j\}_{j \in \mathcal{J}}$.

Given a frame $\{x_j\}_{j \in \mathcal{J}}$, if it is not a Riesz basis, then, besides the canonical dual, there exist many (in fact, infinitely many) other frames $\{y_j\}_{j \in \mathcal{J}}$ for H that yield a *reconstruction formula* for H :

$$x = \sum_{j \in \mathcal{J}} \langle x, x_j \rangle y_j, \quad x \in H.$$

A frame $\{y_j\}_{j \in \mathcal{J}}$ satisfying the above reconstruction formula is called an *alternate dual frame* or just simply a *dual frame* for $\{x_j\}_{j \in \mathcal{J}}$. The connection between the canonical dual and the alternate duals is given by the following:

Proposition 1.1. Let $\{x_j\}_{j \in \mathcal{J}}$ be a frame for H and S be its frame operator. Then a Bessel sequence $\{y_j\}_{j \in \mathcal{J}}$ is an alternate dual for $\{x_j\}_{j \in \mathcal{J}}$ if and only if there exists a Bessel sequence $\{h_j\}_{j \in \mathcal{J}}$ such that $y_j = S^{-1}x_j + h_j$ for every $j \in \mathcal{J}$ and

$$\sum_{j \in \mathcal{J}} \langle x, x_j \rangle h_j = 0 \quad (\forall x \in H).$$

Two Bessel sequences $\{x_j\}_{j \in \mathcal{J}}$ and $\{h_j\}_{j \in \mathcal{J}}$ satisfying the above equation in Proposition 1.1 are said to be *orthogonal* or *strongly disjoint*. Assume that $\{\phi_j^{(\ell)}\}_{j \in \mathcal{J}}$ are Parseval frames for Hilbert spaces H_ℓ , $\ell = 1, \dots, k$. Then we say that $(\{\phi_j^{(1)}\}, \{\phi_j^{(2)}\}, \dots, \{\phi_j^{(k)}\})$ is a *disjoint k -tuple* if $\{\phi_j^{(1)} \oplus \dots \oplus \phi_j^{(k)} : j \in \mathcal{J}\}$ is a frame for the orthogonal direct sum space $H_1 \oplus \dots \oplus H_k$, and is a *strongly disjoint k -tuple* if it is a Parseval frame for the direct sum space. A strongly disjoint k -tuple is also called a *super-frame* of length k [1, 10, 14].

Lemma 1.2 ([14]). A k -tuple $(\{\phi_j^{(1)}\}, \{\phi_j^{(2)}\}, \dots, \{\phi_j^{(k)}\})$ is a super-frame for $H \oplus \dots \oplus H$ if and only if both of the following hold:

- (i) For each ℓ , $\{\phi_j^{(\ell)}\}$ is a Parseval frame for H .
- (ii) $\{\phi_j^{(m)}\}$ and $\{\phi_j^{(n)}\}$ are orthogonal when $m \neq n$.

The theory of Gabor frames for $L^2(\mathbb{R}^d)$ focuses on the frame properties of the Gabor family $\{g_{m,n} : m, n \in \mathbb{Z}\}$. Here we consider another special case: Gabor frames for $\ell^2(\mathbb{Z}^d)$. In this case the Hilbert space is taken as $H = \ell^2(\mathbb{Z}^d)$ with the natural inner product

$$\langle f, g \rangle = \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) \overline{g(\mathbf{n})}.$$

Let G be a subgroup of $\mathbb{Z}^d$. We say that a set $\mathcal{D}$ *tiles* $\mathbb{Z}^d$ by G if $\{G + \mathbf{m} : \mathbf{m} \in \mathcal{D}\}$ is a disjoint partition of $\mathbb{Z}^d$ (in this case $\mathcal{D}$ is also called a *complete digit set* for $\mathbb{Z}^d/G$). If we only have the disjointness $(G + \mathbf{m}) \cap (G + \mathbf{n}) = \emptyset$ for $\mathbf{m}, \mathbf{n} \in \mathcal{D}$ and $\mathbf{m} \neq \mathbf{n}$, then we say that $\mathcal{D}$ *packs* $\mathbb{Z}^d$ by G .

Given two invertible integer matrices $A, B \in M_{d \times d}(\mathbb{Z})$, let Ω be a complete digit set of $B^*\mathbb{Z}^d$ in $\mathbb{Z}^d$, where B^* is the transpose of B . An element $g \in \ell^2(\mathbb{Z}^d)$ generates a family of elements $G(A, B, \Omega, g) := \{g_{\mathbf{k}, \mathbf{m}} : \mathbf{k} \in \Omega, \mathbf{m} \in \mathbb{Z}^d\}$ via time translations and frequency modulations given by

$$g_{\mathbf{k}, \mathbf{m}}(\mathbf{n}) = e^{2\pi i \langle \mathbf{k}, B^{-1}\mathbf{n} \rangle} g(\mathbf{n} - A\mathbf{m}), \quad \mathbf{n} \in \mathbb{Z}^d.$$

If $\{g_{\mathbf{k}, \mathbf{m}}\}$ is a frame for H , then it is called a *Gabor* (or *Weyl-Heisenberg*) *frame*. The element g is referred to as the *Gabor atom* or *Gabor (mother) wavelet*. It can be checked that for any Gabor frame $\{g_{\mathbf{k}, \mathbf{m}}\}$, there exists another Gabor frame $\{h_{\mathbf{k}, \mathbf{m}}\}$ such that for any $f \in \ell^2(\mathbb{Z}^d)$, there is a Gabor expansion

$$f = \sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} c_{\mathbf{k}, \mathbf{m}} g_{\mathbf{k}, \mathbf{m}},$$

where $c_{\mathbf{k}, \mathbf{m}} = \langle f, h_{\mathbf{k}, \mathbf{m}} \rangle$ are called *Gabor coefficients*. An easy way to obtain this is to use the frame operator, which is given by

$$Sf = \sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle f, g_{\mathbf{k}, \mathbf{m}} \rangle g_{\mathbf{k}, \mathbf{m}}.$$

A direct calculation shows that S commutes with both the time translation and frequency modulation operators, and so we have that the canonical dual of $\{g_{\mathbf{k},\mathbf{m}}\}$ is $\{(S^{-1}g)_{\mathbf{k},\mathbf{m}}\}$. Thus $h = S^{-1}g$ is a Gabor atom which produces a dual of the Gabor type for $\{g_{\mathbf{k},\mathbf{m}}\}$. In general, if two Gabor atoms, g and h , generate dual frames $\{g_{\mathbf{k},\mathbf{m}}\}$ and $\{h_{\mathbf{k},\mathbf{m}}\}$, we will say that (g, h) is a *dual frame pair* or simply a *dual pair*.

Here is a simple example of a Parseval frame in $\ell^2(\mathbb{Z})$.

Example 1.3. Let $d = 1$, $A = N$ and $B = K$ be such that $N/K \leq 1$, where N, K are two positive integers. Set $\mathcal{D} = \{0, 1, \dots, N-1\}$. Then $\mathcal{D}$ tiles $\mathbb{Z}$ by $A\mathbb{Z}$ and packs $\mathbb{Z}$ by $B\mathbb{Z}$. Let $\{e_i\}_{i \in \mathbb{Z}}$ be the standard orthonormal basis for $\ell^2(\mathbb{Z})$ and let $g = \frac{1}{\sqrt{K}}(e_0 + \dots + e_{N-1})$. Then $\{g_{k,m} : 0 \leq k \leq K-1, m \in \mathbb{Z}\}$ is a Parseval frame.

Proof. The g vector is of the form

$$(\dots, 0, 0, \underbrace{\frac{1}{\sqrt{K}}, \frac{1}{\sqrt{K}}, \dots, \frac{1}{\sqrt{K}}}_{N \text{ coordinates}}, 0, 0, \dots).$$

First, note that since the vector g is of length N , translations by integer multiples of N do not overlap. That is, $\langle g_{k,m}, g_{k,j} \rangle = 0$ for all $j \neq m$. So consider the spaces

$$M_m = \text{span}\{e_{i+mN}\}_{i=0}^{N-1}.$$

Then $\ell^2(\mathbb{Z}) = \bigoplus_{m \in \mathbb{Z}} M_m$ is the orthogonal direct sum of all the subspaces M_m . There-

fore, in order to show that $\{g_{k,m} : 0 \leq k \leq K-1, m \in \mathbb{Z}\}$ is a Parseval frame for $\ell^2(\mathbb{Z})$, it is enough to check that for any fixed m , $\{g_{k,m}\}_{k=0}^{K-1}$ is a Parseval frame for M_m . So consider M_0 , with $\{g_{k,0}\} = \{e^{2\pi i \frac{k}{K}n} g(n)\}$. This space is isomorphic to $\mathbb{C}^N$, and $\{g_{k,0}\}$ is the Parseval frame generated by the K -th roots of unity (cf. [3, 10]). $\square$

The main results of this paper are summarized in the following three theorems. The first is a characterization for tight Gabor frames, dual pairs and orthogonality of Gabor Bessel sequences.

Theorem 1.4. Let $A, B \in M_{d \times d}(\mathbb{Z})$ be two invertible matrices, and $g, h \in \ell^2(\mathbb{Z}^d)$. Assume that two Gabor families $G(A, B, \Omega, g) = \{g_{\mathbf{k},\mathbf{m}} : \mathbf{k} \in \Omega, \mathbf{m} \in \mathbb{Z}^d\}$ and $G(A, B, \Omega, h) = \{h_{\mathbf{k},\mathbf{m}} : \mathbf{k} \in \Omega, \mathbf{m} \in \mathbb{Z}^d\}$ are Bessel sequences, where $g_{\mathbf{k},\mathbf{m}}(\mathbf{n}) = e^{2\pi i \langle \mathbf{k}, B^{-1}\mathbf{n} \rangle} g(\mathbf{n} - A\mathbf{m})$, $\mathbf{n} \in \mathbb{Z}^d$. Then we have:

(i) (g, h) is a dual pair if and only if

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} g(\mathbf{n} - A\mathbf{m}) \overline{h(\mathbf{n} - A\mathbf{m} - B\mathbf{j})} = \frac{1}{|\Omega|} \delta_{\mathbf{0},\mathbf{j}}$$

for $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{n} \in \mathbb{Z}^d$ (in fact, $\mathbf{n}$ in any $A\mathbb{Z}^d$ -tile is enough).

(ii) $\{g_{\mathbf{k},\mathbf{m}}\}$ and $\{h_{\mathbf{k},\mathbf{m}}\}$ are orthogonal if and only if

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} g(\mathbf{n} - A\mathbf{m}) \overline{h(\mathbf{n} - A\mathbf{m} - B\mathbf{j})} = 0$$

holds for $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{n} \in \mathbb{Z}^d$.

The following is the generalized density theorem for Gabor super-frames.

Theorem 1.5. *Let $A, B \in M_{d \times d}(\mathbb{Z})$ be two invertible matrices. Then the following are equivalent:*

- (i) *There exists a Gabor super-frame of length L for $\ell^2(\mathbb{Z}^d)$; i.e. there exist $g_1, \dots, g_L \in \ell^2(\mathbb{Z}^d)$ such that $\{(g_1)_{\mathbf{k}, \mathbf{m}} \oplus \dots \oplus (g_L)_{\mathbf{k}, \mathbf{m}} \mid \mathbf{k} \in \Omega, \mathbf{m} \in A\mathbb{Z}^d\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d) \oplus \dots \oplus \ell^2(\mathbb{Z}^d)$, where $(g_i)_{\mathbf{k}, \mathbf{m}}(\mathbf{n}) = e^{2\pi i \langle \mathbf{k}, B^{-1}\mathbf{n} \rangle} g_i(\mathbf{n} - A\mathbf{m})$, $\mathbf{n} \in \mathbb{Z}^d$.*
- (ii) $|\det(AB^{-1})| \leq \frac{1}{L}$.

A standard density condition for Parseval frames follows as a corollary with $L = 1$. We remark that the implication from (ii) to (i) is straightforward when $d = 1$. The nontrivial part comes from the higher dimension case due to the complexity of the lattices in $\mathbb{Z}^d$. In this case we need to generalize a standard result on the existence of common coset representatives for subgroups with equal finite index to the case where $[G : H] \geq [G : K]$.

In frame theory, tight frames are the ones that have attracted particular attention due to their simplicity (e.g. the canonical dual is a scalar multiple of the tight frame itself) and due to some other useful features in applications (e.g. tight frames are optimal for erasures, etc.). When a frame itself is not a tight frame, the canonical dual frame cannot be tight. However, it is possible that tight (alternate) dual frames exist even when a given frame is not a tight one. The existence of tight dual frames for nontight frames could be a useful feature to have for either theoretical or practical reasons. In encoding-decoding applications, due to the irregularity of the applied problem the favorite or suitable frame for engineers in encoding may not necessarily be a tight one, and quite often the condition number of the frame operator for the frame (which is equal to the condition number of the frame operator for the canonical dual frame) could be very large. This usually causes very unstable reconstructions (decoding). However, the condition number of the frame operator of a tight canonical dual frame is always one. Our last theorem completely characterizes the cases when a Gabor frame admits a tight dual of the Gabor form.

Theorem 1.6. *Let $A, B \in M_{d \times d}(\mathbb{Z})$ be two invertible matrices. Then the following are equivalent:*

- (i) *for every Gabor frame $\{g_{\mathbf{k}, \mathbf{m}}\}$ with lower frame bound strictly greater than 1, there exists a Parseval Gabor frame $\{h_{\mathbf{k}, \mathbf{m}}\}$ such that (g, h) is a dual pair,*
- (ii) $|\det(AB^{-1})| \leq \frac{1}{2}$.

This is an analogy of one of the main results obtained in [12], where we deal with the Gabor frames for $L^2(\mathbb{R}^d)$. We refer to [12, 13] for more background and results about the existence of tight Gabor duals. The rest of this paper is organized as follows. In section 2, we prove Theorem 1.4 and discuss some consequences of this result. The proofs of Theorem 1.5 and Theorem 1.6 will be given in section 3.

2. CHARACTERIZATION OF TIGHT GABOR FRAMES AND DUAL FRAMES

Before we prove Theorem 1.4, we need some basic lemmas on the nature of modulation in higher dimensions. The first lemma shows that modulation values depend only on the $B^*\mathbb{Z}^d$ -tile, and the second lemma generalizes the behavior of the roots of unity.

Lemma 2.1. *Modulation is well defined for the quotient group $\mathbb{Z}^d/B^*\mathbb{Z}^d$. That is, $e^{2\pi i\langle \mathbf{x}, B^{-1}\mathbf{n} \rangle} = e^{2\pi i\langle \mathbf{y}, B^{-1}\mathbf{n} \rangle}$ if $\mathbf{x}, \mathbf{y} \in \mathbf{k}_i + B^*\mathbb{Z}^d$ for some $\mathbf{k}_i \in \Omega$ and the $B^*\mathbb{Z}^d$ translates of Ω tile $\mathbb{Z}^d$.*

Proof. Let $\mathbf{x} = \mathbf{k}_i + B^*\mathbf{v}$ and $\mathbf{y} = \mathbf{k}_i + B^*\mathbf{w}$ for some $\mathbf{k}_i \in \Omega$. Then

$$\begin{aligned} e^{2\pi i\langle \mathbf{x}, B^{-1}\mathbf{n} \rangle} &= e^{2\pi i\langle \mathbf{k}_i + B^*\mathbf{v}, B^{-1}\mathbf{n} \rangle} \\ &= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle} \cdot e^{2\pi i\langle B^*\mathbf{v}, B^{-1}\mathbf{n} \rangle} \\ &= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle} \cdot e^{2\pi i\langle \mathbf{v}, \mathbf{n} \rangle} \\ &= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle} \cdot (1) \\ &= e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle} \cdot e^{2\pi i\langle \mathbf{w}, \mathbf{n} \rangle} \\ &= e^{2\pi i\langle \mathbf{k}_i + B^*\mathbf{w}, B^{-1}\mathbf{n} \rangle} \\ &= e^{2\pi i\langle \mathbf{y}, B^{-1}\mathbf{n} \rangle}. \end{aligned}$$

□

Lemma 2.2. *If Ω is a $B^*\mathbb{Z}^d$ -tile of $\mathbb{Z}^d$, then*

$$\sum_{\mathbf{k} \in \Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n} \rangle} = \begin{cases} |\Omega| & \text{if } \mathbf{n} \in B\mathbb{Z}^d \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathbf{k}_i \in \Omega$ be arbitrary. Then

$$\begin{aligned} e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle} \left(\sum_{\mathbf{k} \in \Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n} \rangle} \right) &= \sum_{\mathbf{k} \in \Omega} e^{2\pi i\langle \mathbf{k}_i + \mathbf{k}, B^{-1}\mathbf{n} \rangle} \\ &= \sum_{\tilde{\mathbf{k}} \in \Omega} e^{2\pi i\langle \tilde{\mathbf{k}}, B^{-1}\mathbf{n} \rangle}. \end{aligned}$$

So either $\sum_{\mathbf{k} \in \Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n} \rangle} = 0$ or $e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle} = 1$. But $e^{2\pi i\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle} = 1$ if and only if $\langle \mathbf{k}_i, B^{-1}\mathbf{n} \rangle \in \mathbb{Z}$. And since $\mathbf{k}_i$ was arbitrary, we have $B^{-1}\mathbf{n} \in \mathbb{Z}^d$ or, equivalently, $\mathbf{n} \in B\mathbb{Z}^d$. □

Proof of Theorem 1.4. Let $\xi, \eta \in \ell^2(\mathbb{Z}^d)$ be of finite support; i.e., they have finitely many nonzero coefficients. Then the sum $\sum_{\mathbf{m} \in \mathbb{Z}^d} \sum_{\mathbf{k} \in \Omega} \langle \xi, g_{\mathbf{k}, \mathbf{m}} \rangle \langle h_{\mathbf{k}, \mathbf{m}}, \eta \rangle$ becomes

$$\begin{aligned} &\sum_{\mathbf{m} \in \mathbb{Z}^d} \sum_{\mathbf{k} \in \Omega} \left(\sum_{\mathbf{n} \in \mathbb{Z}^d} \xi(\mathbf{n}) \overline{e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{n} \rangle} g(\mathbf{n} - \mathbf{A}\mathbf{m})} \right) \left(\sum_{\mathbf{j} \in \mathbb{Z}^d} e^{2\pi i\langle \mathbf{k}, B^{-1}\mathbf{j} \rangle} h(\mathbf{j} - \mathbf{A}\mathbf{m}) \overline{\eta(\mathbf{j})} \right) \\ &= \sum_{\mathbf{m} \in \mathbb{Z}^d} \sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{n}, \mathbf{j} \in \mathbb{Z}^d} \xi(\mathbf{n}) \overline{\eta(\mathbf{j})} e^{2\pi i\langle \mathbf{k}, B^{-1}(\mathbf{j} - \mathbf{n}) \rangle} \overline{g(\mathbf{n} - \mathbf{A}\mathbf{m})} h(\mathbf{j} - \mathbf{A}\mathbf{m}) \\ &= \sum_{\mathbf{n}, \mathbf{j} \in \mathbb{Z}^d} \xi(\mathbf{n}) \overline{\eta(\mathbf{j})} \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{k} \in \Omega} e^{2\pi i\langle \mathbf{k}, B^{-1}(\mathbf{j} - \mathbf{n}) \rangle} \right) \overline{g(\mathbf{n} - \mathbf{A}\mathbf{m})} h(\mathbf{j} - \mathbf{A}\mathbf{m}). \end{aligned}$$

So (g, h) is a dual pair if and only if the equalities

$$\begin{aligned} \sum_{\mathbf{n}, \mathbf{j} \in \mathbb{Z}^d} \xi(\mathbf{n}) \overline{\eta(\mathbf{j})} \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{k} \in \Omega} e^{2\pi i \langle \mathbf{k}, B^{-1}(\mathbf{j} - \mathbf{n}) \rangle} \right) \overline{g(\mathbf{n} - A\mathbf{m})} h(\mathbf{j} - A\mathbf{m}) \\ = \langle \xi, \eta \rangle = \sum_{\mathbf{n} \in \mathbb{Z}^d} \xi(\mathbf{n}) \overline{\eta(\mathbf{n})} \end{aligned}$$

hold for all $\xi, \eta \in \ell^2(\mathbb{Z}^d)$ of finite length. Hence we have that (g, h) is a dual pair if and only if

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{k} \in \Omega} e^{2\pi i \langle \mathbf{k}, B^{-1}(\mathbf{j} - \mathbf{n}) \rangle} \right) \overline{g(\mathbf{n} - A\mathbf{m})} h(\mathbf{j} - A\mathbf{m}) = \delta_{\mathbf{j}, \mathbf{n}}.$$

From Lemma 2.2, $\sum_{\mathbf{k} \in \Omega} e^{2\pi i \langle \mathbf{k}, B^{-1}(\mathbf{j} - \mathbf{n}) \rangle} = |\Omega|$ if $\mathbf{j} - \mathbf{n} \in B\mathbb{Z}^d$ and 0 otherwise. So the above equation holds if and only if

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} \overline{g(\mathbf{n} - A\mathbf{m})} h(\mathbf{n} + B\ell - A\mathbf{m}) = \frac{1}{|\Omega|} \delta_{\mathbf{0}, \ell}$$

holds for all $\ell \in \mathbb{Z}^d$. This proves (i) of Theorem 1.4.

Similarly, let $\xi, \eta \in \ell^2(\mathbb{Z}^d)$ be of finite length. Then (g, u) is an orthogonal pair if and only if

$$\sum_{\mathbf{n}, \mathbf{j} \in \mathbb{Z}^d} \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{k} \in \Omega} e^{2\pi i \langle \mathbf{k}, B^{-1}(\mathbf{j} - \mathbf{n}) \rangle} \right) \overline{g(\mathbf{n} - A\mathbf{m})} u(\mathbf{j} - A\mathbf{m}) = 0$$

holds for all $\xi, \eta \in \ell^2(\mathbb{Z}^d)$ of finite length, which in turn is equivalent to the condition that

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} \overline{g(\mathbf{n} - A\mathbf{m})} u(\mathbf{n} + B\ell - A\mathbf{m}) = 0$$

holds for all $\ell \in \mathbb{Z}^d$. This proves (ii) of Theorem 1.4. $\square$

Corollary 2.3. *Let $g \in \ell^2(\mathbb{Z}^d)$. Then $\{g_{\mathbf{k}, \mathbf{m}}\}$ is a λ -tight frame for $\ell^2(\mathbb{Z}^d)$ if and only if*

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} g(\mathbf{n} - A\mathbf{m}) \overline{g(\mathbf{n} - A\mathbf{m} - B\mathbf{j})} = \frac{\lambda}{|\Omega|} \delta_{\mathbf{0}, \mathbf{j}}$$

holds for $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{n} \in \mathbb{Z}^d$.

Proof. This follows from Theorem 1.4 (i) and from the fact that $\{g_{\mathbf{k}, \mathbf{m}}\}$ is a λ -tight frame for $\ell^2(\mathbb{Z}^d)$ if and only if $(g, \frac{1}{\lambda}g)$ is a dual pair. $\square$

The following corollary is one application of this characterization formula.

Corollary 2.4. *Let $\Lambda = \{\mathbf{i}_1, \dots, \mathbf{i}_L\}$ be a subset of $\mathbb{Z}^d$, and $g = \frac{1}{\sqrt{|\Omega|}}(e_{\mathbf{i}_1} + \dots + e_{\mathbf{i}_L})$. Then $\{g_{\mathbf{k}, \mathbf{m}}\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d)$ if and only if Λ tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$ and packs by $B\mathbb{Z}^d$.*

Proof. From Corollary 2.3, $\{g_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d)$ if and only if

$$\begin{aligned} \frac{1}{|\Omega|} \delta_{\mathbf{0},\mathbf{j}} &= \sum_{\mathbf{m} \in \mathbb{Z}^d} g(\mathbf{n} - A\mathbf{m}) \overline{g(\mathbf{n} - A\mathbf{m} - B\mathbf{j})} \\ &= \frac{1}{|\Omega|} \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{i}_s \in \Lambda} e_{\mathbf{0}}(\mathbf{n} - A\mathbf{m} - \mathbf{i}_s) \right) \left(\sum_{\mathbf{i}_s \in \Lambda} e_{\mathbf{0}}(\mathbf{n} - A\mathbf{m} - \mathbf{i}_s - B\mathbf{j}) \right). \end{aligned}$$

For $j = \mathbf{0}$, we have

$$\begin{aligned} \frac{1}{|\Omega|} &= \frac{1}{|\Omega|} \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{i}_s \in \Lambda} e_{\mathbf{0}}(\mathbf{n} - A\mathbf{m} - \mathbf{i}_s) \right)^2 \\ 1 &= \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{i}_s \in \Lambda} e_{\mathbf{0}}(\mathbf{n} - A\mathbf{m} - \mathbf{i}_s) \right)^2, \end{aligned}$$

which holds if and only if Λ tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$.

For $j \neq \mathbf{0}$,

$$0 = \frac{1}{|\Omega|} \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\mathbf{i}_s \in \Lambda} e_{\mathbf{0}}(\mathbf{n} - A\mathbf{m} - \mathbf{i}_s) \right) \left(\sum_{\mathbf{i}_s \in \Lambda} e_{\mathbf{0}}(\mathbf{n} - A\mathbf{m} - \mathbf{i}_s - B\mathbf{j}) \right),$$

which holds if and only if Λ packs $\mathbb{Z}^d$ by $B\mathbb{Z}^d$. □

Corollary 2.4 gives an alternate proof for Example 1.3.

3. PROOFS OF THEOREMS 1.4 AND 1.5

We first note that two Bessel sequences are orthogonal if the range spaces of their respective analysis operators are orthogonal, i.e., $\Theta_2^* \Theta_1 = 0$, where Θ_1 and Θ_2 are the analysis operators of the two Bessel sequences. If two Gabor atoms h and v generate two dual frames for a Gabor frame atom g , then $u = h - v$ generates a Gabor sequence which is orthogonal (strongly disjoint) with $\{g_{\mathbf{k},\mathbf{m}}\}$. As a simple consequence of Theorem 1.4 we have

Corollary 3.1. *If $\{g_{\mathbf{k},\mathbf{m}}\}$ and $\{u_{\mathbf{k},\mathbf{m}}\}$ are orthogonal Bessel sequences, then $\langle g, u \rangle = 0$.*

Proof. Since

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} \overline{g(\mathbf{n} - A\mathbf{m})} u(\mathbf{n} + B\ell - A\mathbf{m}) = 0$$

holds for all $\ell, \mathbf{n} \in \mathbb{Z}^d$. Letting $\ell = 0$ and summing over $\mathbf{n}$ in a complete digit set of $A\mathbb{Z}^d$, we then get $\langle g, u \rangle = 0$. □

We remark that in general $\langle g, u \rangle = 0$ does not imply that (g, u) is an orthogonal pair.

Example 3.2. For $d = 1$, let $A = K$ and $B = N$. Consider the standard orthonormal basis vectors e_0 and e_K for $\ell^2(\mathbb{Z})$. Clearly $\langle e_0, e_K \rangle = 0$, but for $n = 0$, $j = -1$,

$$\sum_{m \in \mathbb{Z}} e_0(n - mN) \overline{e_K(n - mN - jK)} = \sum_{m \in \mathbb{Z}} e_0(-mN) \overline{e_K(-mN + K)} = 1,$$

since the term in the sum equals 1 when $m = 0$ and 0 otherwise. Therefore the condition in (ii) of Theorem 1.4 is not satisfied, and so (e_0, e_K) do not form an orthogonal pair.

Another corollary, which will be useful in proving Theorem 1.5, relates an orthogonal pair of Gabor sequences generated by characteristic functions to the tiling properties of their index sets.

Corollary 3.3. *Let $g = \chi_{\Lambda_1}$ and $h = \chi_{\Lambda_2}$ with $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$. Then $\{g_{\mathbf{k}, \mathbf{m}}\}$ and $\{h_{\mathbf{k}, \mathbf{m}}\}$ are orthogonal if and only if Λ_1 and Λ_2 are $B\mathbb{Z}^d$ -translation disjoint.*

Proof. Applying (ii) of Theorem 1.4, we have that (g, h) are an orthogonal pair if and only if

$$\sum_{\mathbf{m} \in \mathbb{Z}^d} \chi_{\Lambda_1}(\mathbf{n} - A\mathbf{m}) \chi_{\Lambda_2}(\mathbf{n} - A\mathbf{m} - B\mathbf{j}) = 0$$

for $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{n} \in \mathbb{Z}^d$. The left side can only be nonzero if $\mathbf{n} - A\mathbf{m} \in \Lambda_1$ and $\mathbf{n} - A\mathbf{m} - B\mathbf{j} \in \Lambda_2$, simultaneously. Hence we get that $\{g_{\mathbf{k}, \mathbf{m}}\}$ and $\{h_{\mathbf{k}, \mathbf{m}}\}$ are orthogonal if and only if Λ_1 and Λ_2 are $B\mathbb{Z}^d$ -translation disjoint. $\square$

We now turn to Theorem 1.5, the density condition for Gabor super-frames in $\ell^2(\mathbb{Z}^d)$. This theorem provides for the existence of Gabor super-frames and Parseval frames based on the determinants of the integer matrices A and B . As mentioned before, the implication from (ii) to (i) is straightforward for $d = 1$ since in this case the tiling of $\mathbb{Z}$ by $a\mathbb{Z}$ and $b\mathbb{Z}$ is not very complicated. In higher dimensions, however, additional work is required and we need to generalize some results concerning common representatives for cosets (Theorem 3.7). We will only discuss the abelian group case, although we can prove that Theorem 3.7 still remains true for arbitrary countable groups (however, the proof is quite different and much more complicated than the abelian group case). It is well known that if an abelian group G has two subgroups of finite index, H, K , then they have a common set of representatives for their cosets if and only if $|G/H| = |G/K|$ (see, for example, [19]). We require something more general for the case when $|G/H| \geq |G/K|$.

Let G be a countable abelian group and H, K be two subgroups of finite index. Consider the subgroup $K + H$ of G , and let $|G/(K + H)| = N$. Write

$$G = \bigcup_{i=1}^N (d_i + K + H),$$

where $\{d_1, \dots, d_N\}$ is a set of complete coset representatives of $K + H$.

Lemma 3.4. *For any $i, j \leq N$, the number of cosets of K contained in $d_i + K + H$ is the same as the number of cosets of K contained in $d_j + K + H$.*

Proof. Let $\{a_m + K \mid 1 \leq m \leq t\}$ be all of the cosets of K contained in $d_i + K + H$. Then, for every m we have

$$\begin{aligned} (d_j - d_i) + (a_m + K) &\subseteq d_j - d_i + d_i + K + H \\ &= d_j + K + H. \end{aligned}$$

Note that each pair of cosets $(d_j - d_i) + (a_m + K)$ and $(d_j - d_i) + (a_n + K)$ are disjoint for $a_m \neq a_n$, since if $x \in [(d_j - d_i) + (a_m + K)] \cap [(d_j - d_i) + (a_n + K)]$,

$$\begin{aligned} x &= d_i - d_j + a_m + k_1 = d_i - d_j + a_n + k_2, \\ a_m - a_n &= k_2 - k_1, \\ a_m - a_n &\in K. \end{aligned}$$

Thus the number of cosets of K contained in $d_j + K + H$ is greater than or equal to the number of cosets of K contained in $d_i + K + H$. Applying the same argument with d_j and d_i reversed shows that the number of cosets of K contained in $d_i + K + H$ equals the number of cosets of K contained in $d_j + K + H$. $\square$

Similarly we also have

Lemma 3.5. $\forall i, j \leq N$, the number of cosets of H contained in $d_i + K + H$ is the same as the number of cosets of H contained in $d_j + K + H$.

Lemma 3.6. Any coset $g_1 + K$ contained in $d_i + H + K$ has nonempty intersection with any coset $g_2 + H$ contained in $d_i + H + K$.

Proof. For some h_0 , $g_1 + K = d_i + h_0 + K$, and for some k_0 , $g_2 + H = d_i + k_0 + H$. The element $d_i + k_0 + h_0$ is contained in both cosets. $\square$

Theorem 3.7. Let $n = |G/K|$ and $m = |G/H|$. If $n \geq Lm$, then there exist L disjoint sets of elements in G :

$$\begin{aligned} &\{g_{11}, g_{12}, \dots, g_{1m}\}, \\ &\{g_{21}, g_{22}, \dots, g_{2m}\}, \\ &\vdots \\ &\{g_{L1}, g_{L2}, \dots, g_{Lm}\} \end{aligned}$$

such that

- (i) for each $1 \leq i \leq L$, $\{g_{i1}, \dots, g_{im}\}$ tiles G by H for each $1 \leq i \leq L$ and
- (ii) $\{g_{11}, \dots, g_{1m}, g_{21}, \dots, g_{2m}, \dots, g_{L1}, \dots, g_{Lm}\}$ packs by K .

Proof. Since there are n cosets of K , and each $d_i + K + H$ contains the same number of them for $1 \leq i \leq N$, then each $d_i + K + H$ contains $\frac{n}{N}$ cosets of K . Similarly, each $d_i + K + H$ contains $\frac{m}{N}$ cosets of H . Since $n \geq Lm$, $\frac{n}{N} \geq L \frac{m}{N}$. Let $K_{i1}, \dots, K_{i \frac{n}{N}}$ be the cosets of K contained in $d_i + K + H$ and $H_{i1}, \dots, H_{i \frac{m}{N}}$ be the cosets of H contained in $d_i + K + H$. By Lemma 3.6, any coset of K and coset of H contained in $d_i + K + H$ have nonempty intersection. So for all $1 \leq j \leq \frac{m}{N}$ we can choose

$$\begin{aligned} a_{ij}^{(1)} &\in K_{ij} \cap H_{ij}, \\ a_{ij}^{(2)} &\in K_{i,j+\frac{m}{N}} \cap H_{ij}, \\ &\vdots \\ a_{ij}^{(L)} &\in K_{i,j+(L-1)\frac{m}{N}} \cap H_{ij}. \end{aligned}$$

Now, relabel the representatives as

$$\begin{aligned} \{g_{11}, g_{12}, \dots, g_{1m}\} &= \bigcup_{i=1}^N \left\{ a_{i1}^{(1)}, \dots, a_{i\frac{m}{N}}^{(1)} \right\}, \\ \{g_{21}, g_{22}, \dots, g_{2m}\} &= \bigcup_{i=1}^N \left\{ a_{i1}^{(2)}, \dots, a_{i\frac{m}{N}}^{(2)} \right\}, \\ &\vdots \\ \{g_{L1}, g_{L2}, \dots, g_{Lm}\} &= \bigcup_{i=1}^N \left\{ a_{i1}^{(L)}, \dots, a_{i\frac{m}{N}}^{(L)} \right\}. \end{aligned}$$

It remains to show that each $\{g_{i1}, \dots, g_{im}\}$ tiles G by H and $\{g_{ij}\}$ packs by K . Since each $a_{ij}^{(p)}$ represents one of the n different cosets of K , they pack by K (and tile if the equality $n = LM$ holds). In addition, for each fixed $1 \leq p \leq L$, every $a_{ij}^{(p)}$ represents one of the m different cosets of H , and so they tile by H . $\square$

Lemma 3.8. *Let $A, B \in M_{d \times d}(\mathbb{Z})$ be two invertible matrices and $h_i \in \ell^2(\mathbb{Z}^d)$ ($i = 1, 2, \dots, n$). If $\bigcup \{(h_i)_{\mathbf{k}, \mathbf{m}}\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d)$, then*

$$\sum_{i=1}^n \|h_i\|^2 = |\det(AB^{-1})|.$$

Proof. The proof is the same as the proof of Lemma 4.4 in [7]. $\square$

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. For (i) $\implies$ (ii), suppose that $\{(g_1)_{\mathbf{k}, \mathbf{m}} \oplus \dots \oplus (g_L)_{\mathbf{k}, \mathbf{m}} \mid \mathbf{k} \in \Omega, \mathbf{m} \in A\mathbb{Z}^d\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d) \oplus \dots \oplus \ell^2(\mathbb{Z}^d)$. Then

$$\begin{aligned} \|g_1 \oplus \dots \oplus g_L\| &\leq 1, \\ \sum_{i=1}^L \|g_i\| &\leq 1, \\ L \cdot |\det(AB^{-1})| &\leq 1 \end{aligned}$$

since each $\{(g_i)_{\mathbf{k}, \mathbf{m}}\}$ is a Parseval frame for $\ell^2(\mathbb{Z}^d)$. Therefore, $|\det(AB^{-1})| \leq \frac{1}{L}$.

For (ii) $\implies$ (i), suppose $|\det(AB^{-1})| \leq \frac{1}{L}$. Then $L|\det(A)| \leq |\det(B)|$. Thus $L|\mathbb{Z}^d/A\mathbb{Z}^d| \leq |\mathbb{Z}^d/B\mathbb{Z}^d|$. By Theorem 3.7, there exist L sets of representatives $\{\Lambda_1, \dots, \Lambda_L\}$ with $\Lambda_j = \{\mathbf{i}_{j1}, \dots, \mathbf{i}_{j, |\det A|}\}$, each of which tiles $\mathbb{Z}^d$ by $A\mathbb{Z}^d$ and packs by $B\mathbb{Z}^d$. Therefore, by Corollary 2.4, there exist L Parseval frames for $\ell^2(\mathbb{Z}^d)$, with Gabor atoms $g_i = \frac{1}{\sqrt{|\det B|}} \chi_{\Lambda_i}$. Since Λ_i and Λ_j are $B\mathbb{Z}^d$ -translation disjoint for any i, j , Corollary 3.3 implies that $\{(g_i)_{\mathbf{k}, \mathbf{m}}\}$ and $\{(g_j)_{\mathbf{k}, \mathbf{m}}\}$ are orthogonal. Therefore, by Lemma 1.2, $g_1 \oplus \dots \oplus g_L$ is a Gabor super-frame of length L .

Moreover, $g_1 \oplus \dots \oplus g_L$ is an orthonormal Gabor super-frame only if equality holds. $\square$

Although the density condition for Parseval frames can be proven independently, it follows nicely as a corollary to Theorem 1.5 with $L = 1$.

Corollary 3.9. *The following are equivalent:*

- (i) *There exists a Gabor frame $\{g_{\mathbf{k},\mathbf{m}}\}$ for fixed A, B .*
- (ii) $|\det(AB^{-1})| \leq 1$.

Finally, we sketch a proof of the tight dual theorem and refer the reader to [12].

Proof of Theorem 1.6. For (i) $\implies$ (ii), let $\{g_{\mathbf{k},\mathbf{m}}\}$ be a Gabor frame with frame operator S and lower frame bound $\frac{1}{\|S^{-1}\|} > 1$. By assumption, there is a Parseval frame $\{h_{\mathbf{k},\mathbf{m}}\}$ with (g, h) a dual pair. Let $\phi = h - S^{-1}g$. Then (g, ϕ) form an orthogonal pair. The frame operator for $\{\phi_{\mathbf{k},\mathbf{m}}\}$ is $\Theta_\phi^* \Theta_\phi = I - S^{-1}$, which is invertible, so $\{\phi_{\mathbf{k},\mathbf{m}}\}$ is also a frame. Therefore, there are two orthogonal, Parseval frames, and so $|\det(AB^{-1})| \leq \frac{1}{2}$.

For (ii) $\implies$ (i), let $\{g_{\mathbf{k},\mathbf{m}}\}$ be a Gabor frame with frame operator S and lower frame bound $\frac{1}{\|S^{-1}\|} > 1$. From Lemma 3.7 in [12], there exists a Parseval frame $\{h_{\mathbf{k},\mathbf{m}}\}$ such that (g, h) is an orthogonal pair. Since $\|S^{-1}\| < 1$, $I - S^{-1}$ is a positive operator, and so consider $\phi = S^{-1}g + \sqrt{I - S^{-1}}h$. First, note that $\sqrt{I - S^{-1}}$ commutes with the modulation and translation operators. Also, $(g, \sqrt{I - S^{-1}}h)$ form an orthogonal pair, since

$$\sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle f, (\sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}} \rangle g_{\mathbf{k},\mathbf{m}} = \sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle \sqrt{I - S^{-1}}f, h_{\mathbf{k},\mathbf{m}} \rangle g_{\mathbf{k},\mathbf{m}} = 0$$

Thus (g, ϕ) form a dual pair. It remains to show that $\{\phi_{\mathbf{k},\mathbf{m}}\}$ is a Parseval frame:

$$\begin{aligned} & \sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle f, \phi_{\mathbf{k},\mathbf{m}} \rangle \phi_{\mathbf{k},\mathbf{m}} \\ &= \sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle f, (S^{-1}g + \sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}} \rangle (S^{-1}g + \sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}} \\ &= S^{-1} \left(\sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle f, (S^{-1}g)_{\mathbf{k},\mathbf{m}} \rangle g_{\mathbf{k},\mathbf{m}} \right) \\ &\quad + 0 + 0 + \sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle f, (\sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}} \rangle (\sqrt{I - S^{-1}}h)_{\mathbf{k},\mathbf{m}} \\ &= S^{-1}f + (I - S^{-1}) \left(\sum_{\mathbf{k} \in \Omega} \sum_{\mathbf{m} \in \mathbb{Z}^d} \langle f, h_{\mathbf{k},\mathbf{m}} \rangle h_{\mathbf{k},\mathbf{m}} \right) \\ &= S^{-1}f + f - S^{-1}f \\ &= f. \end{aligned}$$

Therefore, $\{g_{\mathbf{k},\mathbf{m}}\}$ has a Parseval dual frame. □

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