

TRAVELING WAVES AND WEAK SOLUTIONS FOR AN EQUATION WITH DEGENERATE DISPERSION

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ABSTRACT. We consider the following family of equations:

$$u_t = 2uu_{xxx} - u_x u_{xx} + 2kuu_x.$$

Here $k \neq 0$ is a constant and $x \in [-L_0, L_0]$. We demonstrate that for these equations there are compactly supported traveling wave solutions (which are in H^2), and the Cauchy problem (with H^2 initial data) possesses a weak solution which exists locally in time. These are the first degenerate dispersive evolution PDE where both of these features are known to hold simultaneously. Moreover, if $k < 0$ or L_0 is not too large, the solution exists globally in time.

1. INTRODUCTION

Nonlinear dispersive partial differential equations, such as the nonlinear Schrödinger (NLS) equation or Korteweg-deVries (KdV) equations, have been the subject of intense mathematical scrutiny. The importance of *linear* dispersive effects in understanding the dynamics of these equations cannot be understated, particularly in establishing the existence of special solutions such as solitons and also in more general studies of the associated Cauchy problem. However, there are a number of dispersive equations in which the mechanism which generates the dispersive effects is itself nonlinear. For instance, consider the following class of quasilinear equations of KdV type studied in [6]:

$$(1) \quad u_t = g_3 u_{xxx} + g_2 u_{xx} + g_1 u_x + g_0 u,$$

where $g_j = g_j(\partial_x^j u, \dots, u, x, t)$. Any dispersive character of these equations is due to the leading “Airy”-like term $g_3 u_{xxx}$. In [6], the authors show that if the dispersive effects are, roughly speaking, uniform, then the equations are well-posed in Sobolev space and moreover exhibit the Kato-smoothing effects common to semilinear dispersive equations. Specifically they require g_3 to be uniformly bounded away from zero; that is, $|g_3| \geq c > 0$. They also require $g_2 \geq 0$ so that $g_2 u_{xx}$ does not act like a backwards diffusion operator. Likewise, in the study of the well-posedness of quasilinear NLS equations in [10], there is an assumption that dispersive effects are uniform.

Nevertheless there are problems in which the dispersive effects may vanish. We call such equations *degenerate dispersive equations*, or DDE for short. The most famous example is the $K(m, n)$ family of equations developed in [18]:

$$u_t = (u^n)_x + (u^m)_{xxx}.$$

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Observe that these equations are in fact of the form (1) but do not satisfy (if $m \geq 2$) the hypotheses of [6]. Other commonly studied DDE are degenerate variations of NLS equations [23], Klein-Gordon equations [19], model equations for granular media [11, 15], the Camassa-Holm equation¹ [3], models for magma dynamics [22] and equations which arise in the numerical analysis of the KdV equation [9]. The most striking effect of degenerate dispersion is the existence of nonsmooth coherent structures with compact support. While there are numerous careful numerical studies ([7, 12, 20, 21]) of compact traveling waves (i.e. compactons) and other types of compact structures, there are very few rigorous results concerning the existence and behavior of general solutions to the Cauchy problem ([2, 7, 8]).

In fact, recent results in [1] demonstrate that the degenerate Airy equation

$$(2) \quad u_t = 2uu_{xxx}$$

is in fact *ill-posed* in the sense that solutions do not depend continuously on the initial data. Equation (2), like $K(m, n)$, is of the form (1) but does not meet the “uniformly dispersive” hypothesis for well-posedness in [6]; it is perhaps the simplest equation which could be said to be degenerate and dispersive. The article [1] does not address the issue of existence and uniqueness of solutions to (2). It is natural, then, to ask if there are any DDE of the form of (1) for which some components of well-posedness are true for data in Sobolev spaces.

This paper answers this question in the affirmative. Specifically, we do so for the following family of equations:

$$(3) \quad u_t = 2uu_{xxx} - u_x u_{xx} + 2kuu_x.$$

Here $k \neq 0$ is a constant and $t > 0$. We take $x \in X := [-L_0, L_0]$ and assume u satisfies periodic boundary conditions. We will be considering the Cauchy problem:

$$(4) \quad u(x, 0) = \phi(x) \in H^2.$$

By rescaling the time and space variables we can set $k = \pm 1$. Note this has the additional effect of changing the width of X , but since we view L_0 as a parameter in the problem, this change is not substantial. Note that (3) is a modification of the $K(2, 2)$ equation: $v_t = (v^2)_{xxx} + (v^2)_x = 2vv_{xxx} + 6v_x v_{xx} + 2vv_x$.

We demonstrate **(a)** that (3) has compactly supported traveling wave solutions and **(b)** the Cauchy problem (with H^2 initial data) for (3) possesses a weak solution which exists locally in time. Equation (3) is the first degenerate dispersive evolution PDE where **(a)** and **(b)** are known to hold simultaneously. Since many studies of degenerate dispersive equations focus on interactions between compactly supported traveling waves, it is of fundamental importance that the Cauchy problem for initial data *in the same class as the traveling waves* be understood. In our case, our traveling wave solutions are in fact H^2 , as is the case for the $K(m, n)$ equations.

In Section 2 we first prove some *a priori* estimates for smooth solutions of (3). Namely, we show that the H^2 norm is bounded uniformly in terms of the initial data, at least for finite times. In certain situations, the H^2 norm is bounded for all $t > 0$. Subsequently we use those estimates to prove:

¹The Camassa-Holm equation is both nonlocal and completely integrable. As such, it is not obvious how to generalize results which hold for it to other systems.

Theorem 1. For $k = \pm 1$, for all $L_0 > 0$ and all $\phi \in H^2(X)$, there exists $T^* = T^*(k, L_0, \phi) \in (0, \infty]$ so that for all $0 < T < T^*$ there is a function

$$u \in L^2(0, T; H^{7/4})$$

which is a weak solution of (3). Specifically u satisfies (6) below. Additionally, if $k = -1$ or $L_0 < \pi/\sqrt{2}$, then $T^* = +\infty$.

Remark 1. The regularity we use above, $H^{7/4}$, is not sharp. Our result could be shown to hold in the Sobolev space $H^{2-\epsilon}$, where ϵ is any small and positive number. We choose $\epsilon = 1/4$ primarily for concreteness.

Surprisingly, we find that for each wave speed $c > 0$, there are multiple solitary wave solutions to (3) with a continuum of different amplitudes. As is the case for the $K(2, 2)$ compactons, our traveling waves are not smooth, having a jump in the second derivative. There are several substantial differences between the $k = 1$ and $k = -1$ cases. If $k = 1$, there are solitary waves of arbitrarily large amplitude, though their width is bounded by a universal constant, independent of the domain size. On the other hand, if $k = -1$, the solitary waves have a maximal amplitude but can be as wide as the whole of X . In the case where $X = \mathbf{R}$, we find front solutions whose support is a half-line. In Section 3 we derive explicit formulae for the traveling wave solutions.

2. EXISTENCE OF WEAK SOLUTIONS

2.1. Energy estimates. In this section we prove several *a priori* estimates for (3). Let

$$E[f] := \int_X \left(\frac{1}{2} f^2(x) - \frac{5k}{4} f_x^2(x) + \frac{1}{2} f_{xx}^2(x) \right) dx.$$

Proposition 1. If u is a sufficiently smooth solution of (3) with initial condition (4), we have for all t

$$E[u(t)] = E[\phi].$$

Proof. Let

$$\begin{aligned} E_0(t) &:= \frac{1}{2} \int_X u^2(x, t) dx, & E_1(t) &:= -\frac{5k}{4} \int_X u_x^2(x, t) dx, \\ E_2(t) &:= \frac{1}{2} \int_X u_{xx}^2(x, t) dx. \end{aligned}$$

Differentiating E_0 with respect to time, using the form of (3) given by (12) below, and then integrating by parts gives

$$\begin{aligned} \dot{E}_0 &= \int_X uu_t dx \\ &= \int_X u(2uu_{xx} - \frac{3}{2}u_x^2 + ku^2)_x dx \\ &= - \int_X u_x(2uu_{xx} - \frac{3}{2}u_x^2 + ku^2) dx. \end{aligned}$$

Since $2uu_x u_{xx} = u(u_x^2)_x$ and $u^2 u_x = 1/3(u^3)_x$, we can rewrite the last line above as

$$- \int_X (u(u_x^2)_x - \frac{3}{2}u_x^3 + \frac{k}{3}(u^3)_x) dx.$$

Now, $\int_X (u^3)_x$ vanishes due to the periodic boundary conditions. A final integration by parts then gives

$$\dot{E}_0 = \frac{5}{2} \int_X u_x^3 dx.$$

Similarly for E_1 , we have after after integrating by parts in x one time:

$$\begin{aligned} \dot{E}_1 &= -\frac{5k}{2} \int_X u_x u_{xt} dx \\ &= -\frac{5k}{2} \int_X u_x (2uu_{xx} - \frac{3}{2}u_x^2 + ku^2)_{xx} dx \\ &= \frac{5k}{2} \int_X u_{xx} (2uu_{xx} - \frac{3}{2}u_x^2 + ku^2)_x dx. \end{aligned}$$

Applying the derivative and multiplying out the integrand convert the last line above to

$$\begin{aligned} &\frac{5k}{2} \int_X u_{xx} (-u_x u_{xx} + 2uu_{xxx} + 2kuu_x) dx \\ &= \frac{5k}{2} \int_X (-u_x u_{xx}^2 + 2uu_{xx} u_{xxx} + 2kuu_x u_{xx}) dx. \end{aligned}$$

We then use the fact that $2uu_{xx} u_{xxx} = u(u_{xx}^2)_x$ to get

$$\frac{5k}{2} \int_X (-u_x u_{xx}^2 + u(u_{xx}^2)_x + ku(u_x^2)_x) dx.$$

A final integration by parts in the middle term results in

$$\dot{E}_1 = -5k \int_X u_x u_{xx}^2 dx - \frac{5k^2}{2} \int_X u_x^3 dx.$$

Also, for E_2 , we apply the time derivative, use (12) (see below) and integrate by parts one time:

$$\begin{aligned} \dot{E}_2 &= \int_X u_{xx} u_{xxt} dx \\ &= \int_X u_{xx} (2uu_{xx} - \frac{3}{2}u_x^2 + ku^2)_{xxx} dx \\ &= - \int_X u_{xxx} (2uu_{xx} - \frac{3}{2}u_x^2 + ku^2)_{xx} dx. \end{aligned}$$

If we apply the two x -derivatives to the quantity in parentheses above we arrive at

$$\begin{aligned} &- \int_X u_{xxx} (-u_x u_{xx} + 2uu_{xxx} + 2kuu_x)_x dx \\ &= - \int_X u_{xxx} (-u_{xx}^2 + u_x u_{xxx} + 2uu_{xxxx} + 2kuu_{xx} + 2ku_x^2) dx \\ &= - \int_X (-u_{xxx} u_{xx}^2 + u_x u_{xxx}^2 + 2uu_{xxx} u_{xxxx} + 2kuu_{xx} u_{xxx} + 2ku_x^2 u_{xxx}) dx. \end{aligned}$$

Since $u_{xxx} u_{xx}^2 = 1/3(u_{xx}^3)_x$ and $u_x u_{xxx}^2 + 2uu_{xxx} u_{xxxx} = (uu_{xxx}^2)_x$ we have

$$- \int_X (-\frac{1}{3}(u_{xx}^3)_x + (uu_{xxx}^2)_x + 2kuu_{xx} u_{xxx} + 2ku_x^2 u_{xxx}) dx.$$

The first two terms are perfect derivatives and thus vanish upon integration due to the periodic boundary conditions. Observing that $2uu_{xx}u_{xxx} = u(u_{xx}^2)_x$ and integrating by parts, the term $2ku_x^2u_{xxx}$ gives

$$\begin{aligned} & - \int_X (2ku_{xx}u_{xxx} + 2ku_x^2u_{xxx})dx \\ & = - \int_X (ku(u_{xx}^2)_x - 4ku_xu_{xx}^2)dx. \end{aligned}$$

A final integration by parts gives

$$\dot{E}_2 = 5k \int_X u_x u_{xx}^2 dx.$$

Since $k = \pm 1$, we have $k^2 = 1$, and therefore

$$\dot{E} = \dot{E}_0 + \dot{E}_1 + \dot{E}_2 = 0.$$

□

When $k = -1$, it is clear that $E^{1/2}$ is equivalent to the usual norm on H^2 . On the other hand, when $k = 1$, then E_1 is negative and $E^{1/2}$ is not necessarily equivalent to the H^2 norm. We have

Lemma 2. *If $k = 1$, then $E^{1/2}$ is equivalent to the usual norm on H^2 if and only if $0 < L_0 < \frac{\sqrt{2}\pi}{2}$.*

Proof. Since $X = [-L_0, L_0]$, by Plancherel's theorem we have

$$E(t) = 2L_0 \sum_{n \in \mathbf{Z}} \left(\frac{1}{2} - \frac{5}{4} \left(\frac{n\pi}{L_0} \right)^2 + \frac{1}{2} \left(\frac{n\pi}{L_0} \right)^4 \right) |\hat{u}_n|^2,$$

where $\hat{u}_n = (2L_0)^{-1} \int_{-L_0}^{L_0} u(x, t) e^{-in\pi x/L_0} dx$ are the usual Fourier coefficients.

The polynomial

$$\mu(n) := \frac{1}{2} - \frac{5}{4} \left(\frac{n\pi}{L_0} \right)^2 + \frac{1}{2} \left(\frac{n\pi}{L_0} \right)^4$$

has $\mu(n) > 0$ if and only if $|n\pi/L_0| > \sqrt{2}$ or $|n\pi/L_0| < \sqrt{2}/2$. Therefore, if there are no integers n with $|n|$ in the set $[\sqrt{2}L_0/2\pi, \sqrt{2}L_0/\pi]$ we have $\mu(n) > 0$ for all $n \in \mathbf{Z}$. This in turn implies that there exists $C > 1$ such that for all $n \in \mathbf{Z}$,

$$C^{-1} \left(1 + \left(\frac{n\pi}{L_0} \right)^4 \right) < \mu(n) < C \left(1 + \left(\frac{n\pi}{L_0} \right)^4 \right).$$

This then gives the equivalence of $E^{1/2}$ to the normal H^2 norm.

Thus we only need to determine when there are no integers n with $|n|$ in the set $[\sqrt{2}L_0\pi/2, \sqrt{2}L_0\pi]$. This happens precisely when $L_0 < \sqrt{2}\pi/2$. On the other hand, if $L_0 \geq \sqrt{2}\pi/2$, $\mu(n)$ is not strictly bounded above zero on the integers, and we cannot have the equivalence of the norms. □

The following corollary gives a useful *a priori* estimate in situations where $E^{1/2}$ is not a norm on H^2 . Let

$$N[f] := \frac{1}{2} \|f\|_{H^2}^2 = \frac{1}{2} (\|f\|_{L^2}^2 + \|f_{xx}\|_{L^2}^2).$$

Corollary 3. For $k = \pm 1$ and any $L_0 > 0$, we have for any sufficiently smooth solution of (3):

$$\frac{d}{dt}N[u(t)] \leq CN^{3/2}[u(t)].$$

Proof. Since $N = N[u(t)] = E_0(t) + E_2(t)$, we have

$$\begin{aligned} \dot{N} &= \frac{5}{2} \int_X u_x^3 dx + 5k \int_X u_x u_{xx}^2 dx \\ (5) \quad &\leq C \|u_x\|_{L^\infty} \int_X (u_x^2 + u_{xx}^2) dx \\ &\leq C \|u\|_{H^2}^3 \\ &= CN^{3/2}. \end{aligned}$$

□

2.2. Construction of weak solutions. Let $\psi : X \times [0, \infty) \rightarrow \mathbb{R}$ be a compactly supported test function. Since ψ is compactly supported, there exists $T > 0$ such that $\psi(x, t) = 0$ for all $t \geq T$. If we multiply (3) by ψ , integrate over $X \times [0, \infty)$ and subsequently integrate by parts in each of x and t as appropriate, we arrive at

$$(6) \quad \int_0^T \int_X \left[-(\partial_t \psi)u + (\partial_x^3 \psi + k \partial_x \psi)(u^2) - \frac{7}{2}(\partial_x \psi)(\partial_x u)^2 \right] dx dt = \int_X \psi(x, 0) \phi(x) dx.$$

We say that $u \in L^2(0, T; H^{7/4})$ is a weak solution of (3) provided this relation holds for all test functions ψ . We now prove Theorem 1.

Proof of Theorem 1.

Step 1. Existence of approximate solutions. We begin by introducing a regularized version of equation (3). Let $\varepsilon > 0$ be given, and let \mathcal{J}_ε be a family of Friedrichs mollifiers (for instance, convolution with an appropriate smooth family of functions [14]). Then consider

$$(7) \quad u_t = 2\mathcal{J}_\varepsilon ((\mathcal{J}_\varepsilon u)(\partial_x^3 \mathcal{J}_\varepsilon u)) - \mathcal{J}_\varepsilon ((\partial_x \mathcal{J}_\varepsilon u)(\partial_x^2 \mathcal{J}_\varepsilon u)) + 2k\mathcal{J}_\varepsilon ((\mathcal{J}_\varepsilon u)(\partial_x \mathcal{J}_\varepsilon u)).$$

We use the same initial data as above, (4). It is sometimes useful to rewrite this as

$$(8) \quad \partial_t u = \partial_x^3 \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon u)^2 - \partial_x \mathcal{J}_\varepsilon \left(\frac{7}{2} (\partial_x \mathcal{J}_\varepsilon u)^2 \right) + k \partial_x \mathcal{J}_\varepsilon (\mathcal{J}_\varepsilon u)^2.$$

With the abundance of mollifiers above, it is clear that the right hand side of (7) is a bounded and continuous map from H^2 into itself. Thus the Picard theorem for ODEs on a Banach space applies: there is a solution of (7), denoted $u_\varepsilon(x, t)$, in $C^1(0, T_\varepsilon; H^2)$ for some $T_\varepsilon > 0$.

Step 2. Uniform time of existence. That the time of existence for u_ε depends upon ε is a problem, as we would like to take $\varepsilon \rightarrow 0$. Here we prove the following:

Lemma 4. For $k = \pm 1$ and $L_0 > 0$ there exists $T^* = T^*(k, L_0, \phi) \in (0, \infty]$ such that for all $T \in (0, T^*)$ and $\varepsilon > 0$, the solutions u_ε of (7) satisfy

$$u_\varepsilon \in C^1(0, T; H^2) \subset L^2(0, T; H^2)$$

and

$$\partial_t u_\varepsilon \in C(0, T; H^{-1}) \subset L^2(0, T; H^{-1}).$$

These functions are bounded uniformly in ε in these spaces. Finally, if $k = -1$ or $L_0 < \pi/\sqrt{2}$, then $T^* = \infty$ for all ϕ .

Proof. This lemma follows from the energy estimates derived in the previous section. The placement of mollifiers in (7) is done in such a way that these energy estimates carry over to the regularized equation. Specifically, we have for all $t \in [0, T_\varepsilon)$,

$$(9) \quad E[u_\varepsilon(t)] = E[\phi]$$

and

$$(10) \quad \frac{d}{dt} N[u_\varepsilon(t)] \leq CN^{3/2}[u_\varepsilon(t)].$$

Of course, the verification of (9) and (10) follows along lines similar to the proofs of Proposition 1 and Corollary 3, and we will suppress most of the details. The key point is to show that the mollifiers do not affect the structure of the energy argument.

If we set $F_0(t) := \frac{1}{2} \int_X u_\varepsilon^2 dx$ and $v_\varepsilon = \mathcal{J}_\varepsilon u_\varepsilon$, then

$$\begin{aligned} \dot{F}_0 &= \int_X u_\varepsilon u_{\varepsilon,t} dx \\ &= \int_X u_\varepsilon \mathcal{J}_\varepsilon (2v_\varepsilon v_{\varepsilon,xx} - \frac{3}{2} v_{\varepsilon,x}^2 + kv_\varepsilon^2)_x dx \\ &= \int_X (\mathcal{J}_\varepsilon u_\varepsilon) (2v_\varepsilon v_{\varepsilon,xx} - \frac{3}{2} v_{\varepsilon,x}^2 + kv_\varepsilon^2)_x dx \\ &= \int_X v_\varepsilon (2v_\varepsilon v_{\varepsilon,xx} - \frac{3}{2} v_{\varepsilon,x}^2 + kv_\varepsilon^2)_x dx. \end{aligned}$$

In going from the second to the third line we have made use of the fact that \mathcal{J}_ε is self-adjoint. Notice that the final line here is the same as that which appears in the expression for \dot{E}_0 in the proof of Proposition 1 except with v_ε appearing instead of u . Thus the same steps used there lead us to

$$\dot{F}_0 = \frac{5}{2} \int_X v_{\varepsilon,x}^3 dx.$$

In exactly the same fashion, if we set

$$F_1(t) := -\frac{5k}{4} \int_X u_{\varepsilon,x}^2 dx \quad \text{and} \quad F_2(t) := \frac{1}{2} \int_X u_{\varepsilon,xx}^2 dx,$$

then

$$\dot{F}_1 = -5k \int_X v_{\varepsilon,x} v_{\varepsilon,xx}^2 dx - \frac{5k^2}{2} \int_X v_{\varepsilon,x}^3 dx, \quad \dot{F}_2 = 5k \int_X v_{\varepsilon,x} v_{\varepsilon,xx}^2 dx.$$

From this, (9) and (10) follow immediately.

If $k = -1$ or $L_0 < \pi/\sqrt{2}$, Lemma 2 tells us that $E^{1/2}[u_\varepsilon(t)]$ is an equivalent norm for H^2 . Thus (9) tells us that the H^2 norm of the solution cannot blow up at any time. Therefore, by the continuation theorem for autonomous ODEs on a Banach space, for any $0 < T < \infty$ the solutions u_ε of (7) exist on the interval $[0, T)$ and are in the space $C^1(0, T; H^2(X))$, bounded uniformly in ε .

On the other hand, if $k = 1$ and $L_0 \geq \pi/\sqrt{2}$, then $E^{1/2}$ is not a norm on H^2 . Instead, we integrate the differential inequality (10) to find that there exists a time $T^* > 0$, which is independent of ε , before which $N[u_\varepsilon(t)] = \frac{1}{2} \|u_\varepsilon\|_{H^2}^2$ must remain

finite. That is, in this case, for any $0 < T < T^*$ the solutions u_ε of (7) exist on the interval $[0, T)$ and are in the space $C^1(0, T; H^2(X))$, bounded uniformly in ε .

That $\partial_t u_\varepsilon$ are bounded uniformly in $C(0, T; H^{-1})$ follows by examining the right hand side of (8). For instance, $u_\varepsilon \in H^2$ implies $u_\varepsilon^2 \in H^2$, which in turn implies $(u_\varepsilon^2)_{xxx} \in H^{-1}$. Thus the first term on the right hand side is in $C(0, T; H^{-1})$. All the other terms are similar. \square

Step 3. Convergence of u_ε to a weak solution. Fix $T < T^*$. The Aubin-Lions lemma states (see [4])

Lemma 5. *Suppose that B_0 , B_1 , and B_{-1} are three separable reflexive Banach spaces with $B_1 \subseteq B_0$ and B_0 continuously embedded in B_{-1} . Suppose that f_m is a bounded sequence in $L^p(0, T; B_1)$ and $\partial_t f_m$ is a bounded sequence in $L^q(0, T; B_{-1})$. Here $1 < p, q < \infty$. Then there is a subsequence $f_{m'}$ which converges in $L^p(0, T; B_0)$.*

X is compact and thus $H^2 \subseteq H^{7/4} \subseteq H^{-1}$. Therefore this lemma together with our Lemma 4 implies that there is a function u such that

$$u_\varepsilon \rightarrow u$$

in $L^2(0, T; H^{7/4})$ as $\varepsilon \rightarrow 0$, along a subsequence. This function u is the weak solution we are looking for, as we are about to see.

By multiplying (8) by ψ , a compactly supported test function, and then integrating in both space and time and integrating by parts, we find that the solutions u_ε satisfy

$$\begin{aligned} (11) \quad \int_0^T \int_X \left[-(\partial_t \psi) u_\varepsilon + (\partial_x^3 \psi + k \partial_x \psi) (\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon u_\varepsilon)^2) - (\partial_x \psi) \left(\frac{7}{2} \mathcal{J}_\varepsilon(\partial_x \mathcal{J}_\varepsilon u_\varepsilon)^2 \right) \right] dx dt \\ = \int_X \psi(x, 0) \phi(x) dx. \end{aligned}$$

Since we have u_ε converging strongly to $u \in L^2(0, T; H^{7/4})$, we have enough regularity to ensure that $\mathcal{J}_\varepsilon(\mathcal{J}_\varepsilon u_\varepsilon)^2$ converges strongly to u^2 and $\mathcal{J}_\varepsilon(\partial_x \mathcal{J}_\varepsilon u_\varepsilon)^2$ converges strongly to $(\partial_x u)^2$. We can therefore take the limit as $\varepsilon \rightarrow 0$ in (11), finding

$$\int_0^T \int_X \left[-(\partial_t \psi) u + (\partial_x^3 \psi + k \partial_x \psi)(u^2) - \frac{7}{2} (\partial_x \psi)(\partial_x u)^2 \right] dx dt = \int_X \psi(x, 0) \phi(x) dx.$$

Thus u is a weak solution and we are done. \square

3. EXISTENCE OF COMPACTLY SUPPORTED TRAVELING WAVES

Note that (3) can be rewritten as

$$(12) \quad u_t = \partial_x \left(2uu_{xx} - \frac{3}{2}u_x^2 + ku^2 \right).$$

Making the traveling wave *Ansatz* $u(x, t) = Q(s)$ where $s = x - ct$, we find

$$(13) \quad -cQ = 2Q\ddot{Q} - \frac{3}{2}\dot{Q}^2 + kQ^2.$$

We have integrated in s one time.

We rewrite (13) as a system

$$(14) \quad \begin{aligned} \dot{Q} &= P, \\ \dot{P} &= \frac{1}{2Q} \left(-cQ - kQ^2 + \frac{3}{2}P^2 \right). \end{aligned}$$

Our goal is to find a solution which is homoclinic to the origin and for which the duration of the trajectory is finite. We are able to find explicit formulae for such solutions. Notice that (14) has two equilibria: one at the origin (which is degenerate) and another at $Q = -c/k, P = 0$.

Notice that when $P < 0$ we have $\dot{Q} < 0$. Therefore we can assert that a solution which lies in the lower half of the QP -plane has a trajectory which is a graph over the Q axis. That is,

$$P(t) = f(Q(t))$$

for an as yet unspecified function f . We differentiate this relationship and use (14):

$$\dot{P} = f'(Q)\dot{Q} = f'(Q)P = f'(Q)f(Q) = \frac{1}{2} \frac{d}{dQ} (f^2(Q)).$$

The second equation in (14) then becomes

$$\frac{1}{2} \frac{d}{dQ} (f^2(Q)) = \frac{1}{2Q} \left(-cQ - kQ^2 + \frac{3}{2}f^2(Q) \right).$$

Letting $F = f^2$ and rearranging terms lead to the following ODE:

$$F' - \frac{3}{2Q}F = -c - kQ.$$

This can be solved explicitly for F :

$$F(Q) = 2cQ - 2kQ^2 - \beta_0 Q^{3/2},$$

where $\beta_0 \in \mathbf{R}$ is an arbitrary constant.

Thus the solution of (13) lies on the graph $P = -\sqrt{F(Q)}$, and therefore

$$(15) \quad \dot{Q} = -\sqrt{2cQ - 2kQ^2 - \beta_0 Q^{3/2}}.$$

This equation can be solved explicitly, though the details vary depending on the sign of k .

3.1. Traveling pulses and fronts for $k = -1$. Separating variables and integrating (15) give

$$\begin{aligned} & \int_A^{Q(s)} \frac{dQ}{\sqrt{2cQ + 2Q^2 - \beta_0 Q^{3/2}}} = -s \\ \Rightarrow \quad & \sqrt{2} \ln \left(\frac{-\sqrt{2}\beta_0/4 + \sqrt{2Q(s)} + \sqrt{2c + 2Q(s) - \beta_0\sqrt{Q(s)}}}{-\sqrt{2}\beta_0/4 + \sqrt{2A} + \sqrt{2c + 2A - \beta_0\sqrt{A}}} \right) = -s. \end{aligned}$$

If we take β_0 so that $Q(0) = A > 0$, we find

$$(16) \quad Q(s) = \frac{1}{4A} \left[c + A - (c - A) \cosh \left(\frac{\sqrt{2}}{2}s \right) \right]^2.$$

This function diverges exponentially quickly as $s \rightarrow \infty$ and so does not immediately give the profile of a compactly supported traveling wave for (3). Notice that this function has two zeros when $0 < A < c$ at

$$s = L := \pm\sqrt{2} \ln \left(\frac{c + A + 2\sqrt{cA}}{c - A} \right).$$

Since $Q(s)$ is nonnegative and in C^∞ , it follows that $\dot{Q}(\pm L) = 0$ and $\ddot{Q}(\pm L)$ is finite and positive. Thus we can cut the function Q off for $|s| \geq L$ and the resulting truncated function is still a solution of (13), since it is degenerate at $Q = 0$.

That is to say, we have a pulse solution to (13) of the form

$$Q(s) = Q_{c,A}^-(s) := \frac{\mathbf{1}_L(s)}{4A} \left[A + c + (A - c) \cosh \left(\frac{\sqrt{2}}{2} s \right) \right]^2.$$

Here $\mathbf{1}_L(s)$ is the characteristic function of $[-L, L]$. Clearly we require $L \leq L_0$.

We make the following observations about $Q_{c,A}^-$:

- The maximum of $Q_{c,A}^-$ is A ; thus for any speed $c > 0$ there are compactly supported traveling waves with any amplitude $A \in (0, c)$.
- As $A \rightarrow c^-$, notice that $L \rightarrow +\infty$.
- $\lim_{s \rightarrow L^-} \frac{d^2}{ds^2} Q_{c,A}^-(s) = \lim_{s \rightarrow -L^+} \frac{d^2}{ds^2} Q_{c,A}^-(s) = c$.

There are also front solutions of (3) with speed c if we allow $X = \mathbf{R}$. These correspond to the two heteroclinic solutions of (14) connecting the origin to the equilibrium at $(c, 0)$. These are given by

$$Q_{c,\text{front}}^-(s) = \begin{cases} c \left[1 - e^{\frac{\sqrt{2}}{2}s} \right]^2, & \text{if } s < 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$Q_{c,\text{back}}^-(s) = Q_{c,\text{front}}^-(-s).$$

Observe that $\lim_{s \rightarrow 0^-} \frac{d^2}{ds^2} Q_{c,\text{front}}^-(s) = \lim_{s \rightarrow 0^+} \frac{d^2}{ds^2} Q_{c,\text{back}}^-(s) = c$. We plot the various pulse and front solutions in Figures 1 and 2.

3.2. Traveling pulses for $k = 1$. We can likewise compute explicit formulae for pulses when $k = 1$. The details are only slightly different than above, and so we only provide the end result:

$$Q_{c,A}^+(s) := \frac{\mathbf{1}_M(s)}{4A} \left[A - c + (A + c) \cos \left(\frac{\sqrt{2}}{2} s \right) \right]^2.$$

Here $\mathbf{1}_M(s)$ is the characteristic function of $[-M, M]$ and

$$M := \sqrt{2} \arccos \left(\frac{c - A}{c + A} \right).$$

Clearly we require $M \leq L_0$.

We make the following observations about $Q_{c,A}^+$:

- The maximum of $Q_{c,A}^+$ is A , and unlike when $k = -1$, there is no upper limit on A . For any speed $c > 0$ there are compactly supported traveling waves with any amplitude $A > 0$.

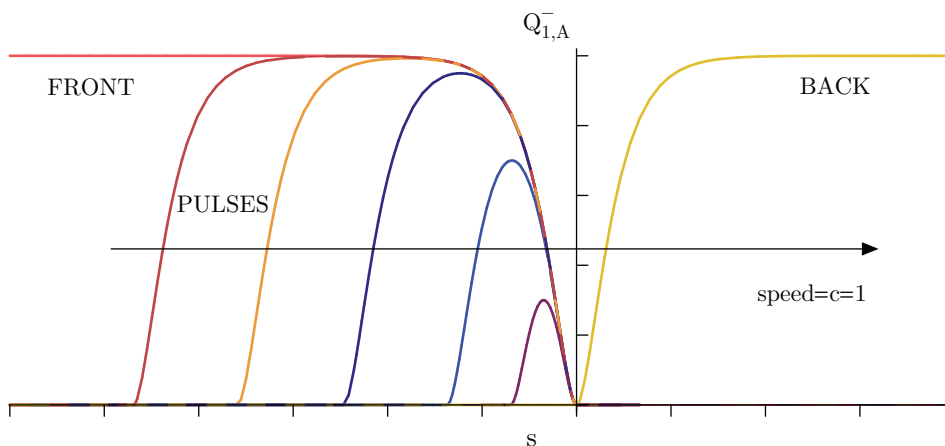


FIGURE 1. Profiles for traveling waves, including the front and back, of (3) when $k = -1$. Here, all waves move with the same speed $c = 1$ but vary in amplitude. All traveling waves are asymptotic to the same parabola as they approach 0. Also observe that pulses approach the front solution as their amplitude approaches 1.

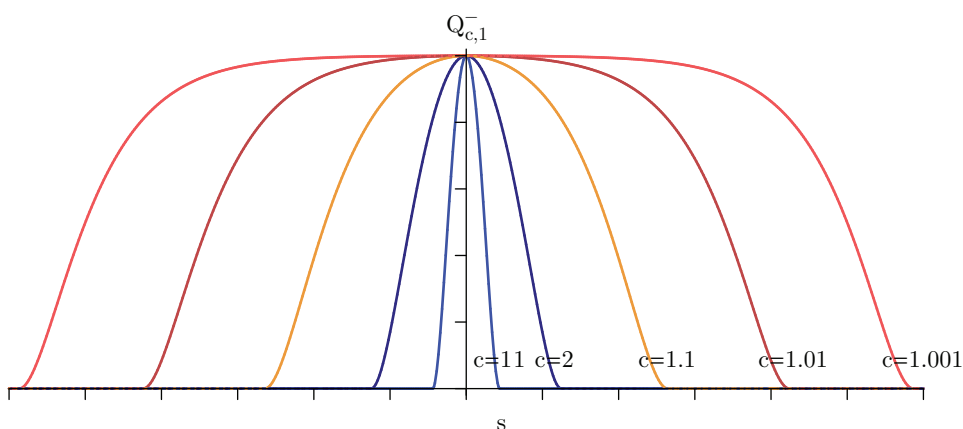


FIGURE 2. Profiles for traveling waves of (3) when $k = -1$. Here, all waves have the same amplitude $A = 1$ but vary in speed.

- As $A \rightarrow \infty$, notice that $M \rightarrow \sqrt{2} \arccos(-1) = \sqrt{2}\pi$ monotonically. Thus pulses have a maximum width in this setting.
- Similarly, for any amplitude $A > 0$, there is a standing pulse ($c = 0$) which is of maximum width.
- As before $\lim_{s \rightarrow M^-} \frac{d^2}{ds^2} Q_{c,A}^+(s) = \lim_{s \rightarrow -M^+} \frac{d^2}{ds^2} Q_{c,A}^+(s) = c$.

We plot the various pulse solutions in Figures 3 and 4.

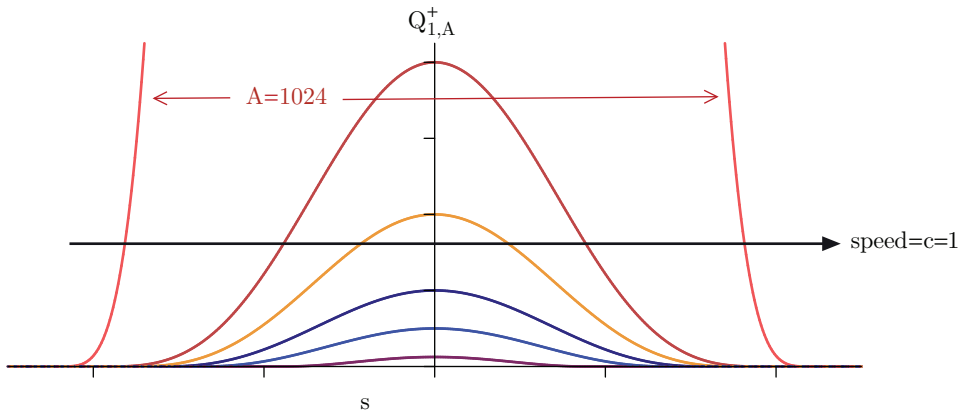


FIGURE 3. Profiles for traveling waves when $k = 1$. Here, all waves move with the same speed $c = 1$ but vary in amplitude. Observe that the large amplitude solution has width approaching $4.4 \sim \sqrt{2}\pi$.

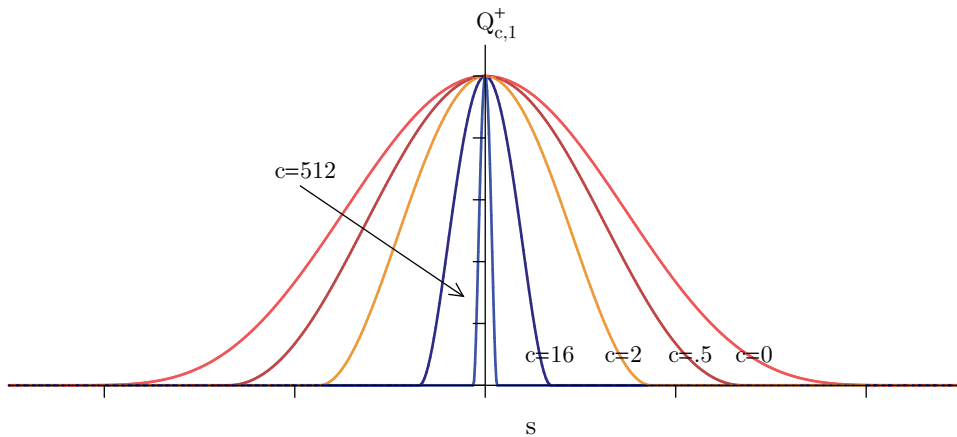


FIGURE 4. Profiles for a variety of traveling waves for the $k = 1$ equation when $A = 1$ and c varies. Note that the $c = 0$ wave has width exactly $4.4 \sim \sqrt{2}\pi$.

Remark 2. Note that for any $A \in \mathbf{R}$ and $c \in \mathbf{R}$,

$$\tilde{Q}(s) := \frac{1}{4A} \left[A - c + (A + c) \cos \left(\frac{\sqrt{2}}{2}s \right) \right]^2$$

gives the profile of a periodic traveling wave for (3) when $k = 1$ and the period of \tilde{Q} is compatible with the width of the domain X . When $c > 0$ and $A < 0$, these periodic waves are negative valued and are strictly bounded away from zero; in particular, we have $\tilde{Q} \in [\frac{c^2}{4A}, A]$.

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