

NONDEGENERACY OF THE BUBBLE IN THE CRITICAL CASE FOR NONLOCAL EQUATIONS

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ABSTRACT. We prove the nondegeneracy of the extremals of the fractional Sobolev inequality as solutions of a critical semilinear nonlocal equation involving the fractional Laplacian.

1. INTRODUCTION

This paper establishes the *linear nondegeneracy* property of the extremals of the optimal Hardy-Littlewood-Sobolev inequality, which states the existence of a positive number S such that for all $u \in C_0^\infty(\mathbb{R}^N)$ one has

$$(1.1) \quad S\|u\|_{L^{2^*}(\mathbb{R}^N)}^2 \leq \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}^2.$$

Here $0 < s < 1$, $N > 2s$ and $2^* = \frac{2N}{N-2s}$.

The validity of this inequality, without its optimal constant, traces back to Hardy and Littlewood [13, 14] and Sobolev [20]. For $s = 1$, (1.1) corresponds to the classical Sobolev inequality

$$(1.2) \quad S\|u\|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^2 \leq \|\nabla u\|_{L^2(\mathbb{R}^N)}^2.$$

Aubin [1] and Talenti [15] found the optimal constant and extremals for inequality (1.2). Indeed, equality is achieved precisely by the functions

$$(1.3) \quad w_{\mu,\xi}(x) = \alpha \left(\frac{\mu}{\mu^2 + |x - \xi|^2} \right)^{\frac{N-2}{2}},$$

which for the choice $\alpha = (N(N-2))^{\frac{N-2}{4}}$ solve the equation

$$(1.4) \quad -\Delta w = w^{\frac{N+2}{N-2}}, \quad w > 0 \quad \text{in } \mathbb{R}^N.$$

The solutions (1.3) are indeed the only ones of (1.4); see Caffarelli, Gidas and Spruk [2].

For $s \neq 1$, the optimal constant in inequality (1.1) was first found by Lieb in 1983 [18], while alternative proofs have been provided by Carlen and Loss [4] and by Frank and Lieb [10, 11]. Lieb established that the extremals correspond precisely to functions of the form

$$(1.5) \quad w_{\mu,\xi}(x) = \alpha \left(\frac{\mu}{\mu^2 + |x - \xi|^2} \right)^{\frac{N-2s}{2}}, \quad \alpha > 0,$$

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which for a suitable choice $\alpha = \alpha_{N,s}$ solve the equation

$$(1.6) \quad (-\Delta)^s w = w^{\frac{N+2s}{N-2s}}, \quad w > 0 \quad \text{in } \mathbb{R}^N.$$

Besides, under suitable decay assumptions, these are the only solutions of the equation; see Chen, Li and Ou [6], Li [16], and Li and Zhu [17].

In equation (1.6) and in what remains of this paper, we will always mean that the equation

$$(-\Delta)^s u = f \quad \text{in } \mathbb{R}^N$$

is satisfied if

$$(1.7) \quad u(x) = (-\Delta)^{-s} f(x) := \gamma \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-2s}} dy,$$

as long as f has enough decay for the integral to be well defined. The constant $\gamma = \gamma_{N,s} > 0$ is so that (1.7) defines an inverse for the operator whose Fourier multiplier is $|\xi|^{2s}$, namely,

$$|\xi|^{2s} \hat{u}(\xi) = \hat{f}(\xi).$$

There are other definitions of $(-\Delta)^s u$ which are equivalent to (1.7) under suitable assumptions; see [3].

When analyzing bubbling (blowing-up) concentration behavior of families of solutions to semilinear elliptic equations involving critical Sobolev growth such as Yamabe type problems, a crucial ingredient is understanding the *linear nondegeneracy* of the solutions (1.3) of equation (1.4). Let us observe that

$$(1.8) \quad w_{\mu,\xi}(x) = \mu^{\frac{N-2}{2}} w(\mu(x-\xi)), \quad w(x) = \alpha_N \left(\frac{1}{1+|x|^2} \right)^{\frac{N-2}{2}},$$

which actually reflects the invariance of the equation under the above scaling and translations. We say that the solution w is nondegenerate if the generators of these operations are the only nontrivial elements in the kernel of the associated linearized operator. This property of equation (1.4) is well-known; see for instance [19].

The purpose of this note is to establish the validity of this property for the solutions (1.5) of the fractional critical equation (1.6). We observe now that

$$(1.9) \quad w_{\mu,\xi}(x) = \mu^{\frac{N-2s}{2}} w(\mu(x-\xi)), \quad w(x) = \alpha_{N,s} \left(\frac{1}{1+|x|^2} \right)^{\frac{N-2s}{2}}.$$

Differentiating the equation

$$(-\Delta)^s w_{\mu,\xi} = w_{\mu,\xi}^p \quad \text{in } \mathbb{R}^N, \quad p = \frac{N+2s}{N-2s},$$

with respect of the parameters at $\mu = 1$, $\xi = 0$, we see that the functions

$$(1.10) \quad \partial_\mu w_{\mu,\xi} = \frac{N-2s}{2} w + x \cdot \nabla w, \quad \partial_{\xi_i} w_{\mu,\xi} = -\partial_{x_i} w$$

annihilate the linearized operator around w ; namely, they satisfy the equation

$$(1.11) \quad (-\Delta)^s \phi = p w^{p-1} \phi \quad \text{in } \mathbb{R}^N.$$

Our main result states the nondegeneracy of the solution w in the following sense.

Theorem 1.1. *The solution*

$$w(x) = \alpha_{N,s} \left(\frac{1}{1 + |x|^2} \right)^{\frac{N-2s}{2}}$$

of equation (1.5) is nondegenerate in the sense that all bounded solutions of equation (1.11) are linear combinations of the functions

$$(1.12) \quad \frac{N-2s}{2} w + x \cdot \nabla w, \quad \partial_{x_i} w, \quad 1 \leq i \leq N.$$

We shall devote the rest of this paper to the proof of Theorem 1.1. Semilinear equations involving fractional operators have attracted substantial research in recent years. In particular, the issue of nondegeneracy of least energy solutions in the related equation $(-\Delta)^s u + u - u^p = 0$, $p > 1$, has been treated by Frank and Lenzmann [8], Frank, Lenzmann and Silvestre [9] and Fall and Valdinoci [7]. See also [5], [12].

2. PROOF OF THEOREM 1.1

Let ϕ be a bounded solution of equation (1.11). Then ϕ solves the integral equation

$$(2.1) \quad \phi(x) = \gamma \int_{\mathbb{R}^N} \frac{pw(y)^{p-1}\phi(y)}{|x-y|^{N-2s}} dy, \quad \forall x \in \mathbb{R}^N,$$

as we note that the integral is well defined thanks to the decay of w .

The proof of the theorem is based on transforming equation (2.1) into the sphere by means of stereographic projection. Let \mathbb{S}^N be the sphere (contained in \mathbb{R}^{N+1}) and $S : \mathbb{R}^N \rightarrow \mathbb{S}^N \setminus \{pole\}$,

$$S(x) = \left(\frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right),$$

be the stereographic projection. We have the following well-known formula for the Jacobian of the stereographic projection:

$$J_S(x) = \left(\frac{2}{1 + |x|^2} \right)^N = cw(x)^{\frac{N-2s}{2N}} = cw(x)^{\frac{1}{p^*+1}}.$$

Here and later $c > 0$ is a constant that depends on N, s . For $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ define $\tilde{\varphi} : \mathbb{S}^N \rightarrow \mathbb{R}$ by

$$(2.2) \quad \varphi(x) = J(x)^{\frac{N+2s}{2N}} \tilde{\varphi}(S(x)).$$

Then we have

$$(2.3) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)\psi(y)}{|x-y|^{N-2s}} dx dy = \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{\tilde{\varphi}(\omega)\tilde{\psi}(\eta)}{|\omega-\eta|^{N-2s}} d\omega d\eta,$$

where $|\omega - \eta|$ is the Euclidean distance in \mathbb{R}^{N+1} from ω to η ; see [10].

We rewrite (2.1) as

$$(2.4) \quad \phi(x) = c \int_{\mathbb{R}^N} \frac{J(y)^{\frac{2s}{N}} \phi(y)}{|x-y|^{N-2s}} dy, \quad \text{for all } x \in \mathbb{R}^N.$$

Let $\psi \in C_c^\infty(\mathbb{R}^N)$. Multiplying equation (2.4) by $\psi J^{-\frac{2s}{N}}$ and integrating in \mathbb{R}^N , we get

$$\int_{\mathbb{R}^N} J(x)^{-\frac{2s}{N}} \phi(x) \psi(x) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{J(y)^{\frac{2s}{N}} \phi(y) J(x)^{-\frac{2s}{N}} \psi(x)}{|x-y|^{N-2s}} dy dx.$$

Let $\tilde{\phi}, \tilde{\psi}$ be defined by (2.2). Then, using (2.3) we obtain

$$\int_{\mathbb{R}^N} J(x)^{-\frac{2s}{N}} \phi(x) \psi(x) dx = \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{J(S^{-1}(\omega))^{\frac{2s}{N}} \tilde{\phi}(\omega) J(S^{-1}(\eta))^{-\frac{2s}{N}} \tilde{\psi}(\eta)}{|\omega-\eta|^{N-2s}} d\omega d\eta.$$

But

$$\int_{\mathbb{R}^N} J(x)^{-\frac{2s}{N}} \phi(x) \psi(x) dx = \int_{\mathbb{R}^N} J(x) \tilde{\phi}(S(x)) \tilde{\psi}(S(x)) dx = \int_{\mathbb{S}^N} \tilde{\phi}(\omega) \tilde{\psi}(\omega) d\omega,$$

so replacing $\tilde{\psi}$ by a new test function of the form $\tilde{\psi} J^{\frac{2s}{N}} \circ S^{-1}$, we get

$$\int_{\mathbb{S}^N} J^{\frac{2s}{N}} \circ S^{-1} \tilde{\phi} \tilde{\psi} = c \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{J^{\frac{2s}{N}} \circ S^{-1}(\omega) \tilde{\phi}(\omega) \tilde{\psi}(\eta)}{|\omega-\eta|^{N-2s}} d\omega d\eta.$$

Since $\tilde{\psi}$ is arbitrary (smooth with support away from the pole), the function

$$(2.5) \quad h = J^{\frac{2s}{N}} \circ S^{-1} \tilde{\phi}$$

satisfies the integral equation

$$(2.6) \quad ah(\omega) = \int_{\mathbb{S}^N} \frac{h(\eta)}{|\omega-\eta|^{N-2s}} d\eta,$$

for all $\omega \neq \text{pole}$, where $a > 0$ depends only on N, s . Note that h is a bounded function. Indeed, in terms of ϕ this is equivalent to the estimate

$$(2.7) \quad |\phi(x)| \leq \frac{C}{|x|^{N-2s}} \quad \text{for all } |x| \geq 1.$$

We observe that if $v : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies $|v(x)| \leq C/(1+|x|)^\nu$ and $\nu \geq 0$, then

$$(2.8) \quad \left| \int_{\mathbb{R}^N} \frac{J(y)^{\frac{2s}{N}}}{|x-y|^{N-2s}} v(y) dy \right| \leq \frac{C}{(1+|x|)^{\min(\nu+2s, N-2s)}}.$$

Hence, starting from the hypothesis that ϕ is bounded, we get estimate (2.7) after a finite number of applications of (2.8). This implies that h is bounded. Then equation (2.6) shows that h has a continuous extension to all \mathbb{S}^N .

Let T denote the linear integral operator defined by the right hand side of (2.6). Then T is a self-adjoint compact operator on $L^2(\mathbb{S}^N)$ whose spectral decomposition can be described in terms of the spaces H_l , $l \geq 0$, that consist of restrictions to \mathbb{S}^N of homogeneous harmonic polynomials in \mathbb{R}^{N+1} of degree l . Then $L^2(\mathbb{S}^N)$ is the closure of the direct sum of H_l , $l \geq 0$, and the elements in H_l are eigenvectors of T with eigenvalue

$$(2.9) \quad e_l = \kappa_N 2^\alpha (-1)^l \frac{\Gamma(1-\alpha) \Gamma(N/2-\alpha)}{\Gamma(-l+1-\alpha) \Gamma(l+N-\alpha)},$$

where $\alpha = N/2 - s$ and

$$\kappa_N = \begin{cases} 2\pi^{1/2} & \text{if } N = 1, \\ 2^{2(N-1)} \pi^{(N-1)/2} \frac{\Gamma((N-1)/2) \Gamma(N/2)}{(N-2)!} & \text{if } N \geq 2. \end{cases}$$

Moreover these are the only eigenvalues of T ; see e.g. [10]. Formula (2.9) and standard identities of the gamma function give

$$\frac{e_{l+1}}{e_l} = \frac{l + \alpha}{l + N - \alpha} < 1.$$

We note that if ϕ is any of the functions in (1.12), then the corresponding function h belongs to H_1 and is a nontrivial solution of (2.6). Thus the number a appearing in (2.6) must be equal to e_1 (this could also be deduced from an explicit formula for a). Therefore, if h is a solution of (2.6), then $h \in H_1$. Since the dimension of H_1 is $N + 1$, H_1 is generated by functions obtained by the transformation (2.5) applied to the functions in (1.12). The proof is concluded. \square

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