SHALIKA PERIODS ON $\mathrm{GU}(2, 2)$

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ABSTRACT. In this paper we consider a certain Rankin-Selberg integral on a quasi-split similitude unitary group $\mathrm{GU}(2, 2)$, which is an analogue of Jacquet-Shalika's integral for the exterior square L -function for $\mathrm{GL}(2n)$ when $n = 2$. It indeed represents the twisted exterior square L -function, and we study the relationship between the existence of a pole at $s = 1$ and the non-vanishing of a unitary analogue of the Shalika period.

1. INTRODUCTION

To investigate relationships between periods of automorphic forms and leading terms of automorphic L -functions is one of the most interesting themes in number theory. A well-known theorem of Jacquet and Shalika [8, Theorem 1 in Section 8] is such an example. In the case of $\mathrm{GL}(4)$, which is related to our case in this paper, their theorem says that for an irreducible cuspidal unitary automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_F)$ whose central character is ξ^{-2} , the following two conditions are equivalent:

- (1) The Shalika period with respect to ξ does not vanish on the space of π .
- (2) The partial exterior square L -function $L^S\left(s, \pi, \bigwedge^2 \otimes \xi\right)$ has a simple pole at $s = 1$.

There are many applications of this theorem. It is natural to ask whether a similar statement holds for related groups. Jacquet and Martin considered the case of $\mathrm{GL}_2(D)$ where D is a quaternion algebra in [7] and proved an interesting theorem under some conditions. Later Gan and Takeda [4] completed the theorem, removing the conditions, and developed the theory much further. We also refer to Jiang, Nien and Qin [9] for the generalization of Jacquet-Martin theorem to $\mathrm{GL}_2(D)$ where D is an arbitrary division algebra.

In this paper we consider the case when the group in question is the quasi-split unitary group $\mathrm{GU}(2, 2)$. Then the Shalika period in this case is defined by (2.5) below. Our main theorem, Theorem 4.1, says that for an irreducible cuspidal unitary automorphic representation π of $\mathrm{GU}(2, 2)(\mathbb{A}_F)$, whose central character ω_π satisfies $\omega_\pi|_{\mathbb{A}_F^\times} = \xi^{-2}$, the following two conditions are equivalent:

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- (1) The Shalika period with respect to ξ does not vanish on the space of π .
- (2) π is globally generic and the partial twisted exterior square L -function $L^S\left(s, \pi, \bigwedge_t^2 \otimes \xi\right)$ has a simple pole at $s = 1$.

Here the twisted exterior square L -function is defined in 2.1.1. In the globally generic case, the meromorphic continuation of this L -function to the entire plane is known by Shahidi [13]. Recently Kim and Krishnamurthy [10] studied the lifting problem from $\mathrm{GU}(2, 2)$ to $\mathrm{GL}(6)$ in terms of this L -function by the Langlands-Shahidi method. It is of some interest that our theorem indeed is intimately related to the cuspidality condition for the image of the lifting constructed in [10]. Though our theorem looks very similar to the one in the case of $\mathrm{GL}(4)$, it is very interesting that the L -function appearing in the statement is a relevant but different one. We also note that, as Corollary 2.3 states, the Shalika period vanishes identically for π which is not globally generic.

Recall that our group $\mathrm{GU}(2, 2)$ is closely related to $\mathrm{GSO}(4, 2)$. In fact we have an isomorphism $\mathrm{PGU}(2, 2) \simeq \mathrm{PGSO}(4, 2)$. Thus we may consider the theta lifting from $\mathrm{GSp}(4)$ to $\mathrm{GU}(2, 2)$. Since both groups have corresponding Shimura varieties when the base field is totally real, this correspondence might be of some arithmetic interest. As in the $\mathrm{GL}(4)$ case, we expect that there is a close relationship between this theta lifting and the Shalika periods on $\mathrm{GU}(2, 2)$. We plan to investigate the relationship in the future.

Since our use of the mirabolic Eisenstein series in the integral representation is essential, it is hard for us to imagine that our theorem can be generalized to $\mathrm{GU}(n, n)$. Rather, it seems that our theorem might be generalized to groups of type D_n , as the accidental isomorphism above suggests. This looks consistent with the functorial consideration in Kim and Krishnamurthy [10, Section 3].

From the next section, we start towards the proof of our main theorem. We omit the details of the arguments of the proofs, which are similar to the ones in Jacquet and Shalika [8], and refer to [8]. We provide the details of the unfolding of the global zeta integral and the computation of the non-split unramified local integrals, since they are different from the ones in [8].

2. SETUP

Notation. Let F be a number field and let $\mathbb{A}_F$ be its ring of adeles. Let ψ be a non-trivial character of $\mathbb{A}_F/F$. Let E be a quadratic extension of F and let $\mathbb{A}_E$ be its ring of adeles. We take $\eta \in E^\times$ such that $\bar{\eta} = -\eta$, where $E \ni x \mapsto \bar{x} \in E$ denotes the unique non-trivial element of $\mathrm{Gal}(E/F)$.

2.1. Similitude unitary group. Let J denote the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Then the similitude unitary group G of J is defined by

$$G = \{g \in \mathrm{GL}_4(E) \mid {}^t \bar{g} J g = \lambda(g) J, \lambda(g) \in \mathbb{G}_m(F)\}.$$

The center Z of G is given by $Z = \{a 1_4 \mid a \in E^\times\}$.

We regard G as an algebraic group over F . Hence for any algebra A over F , we have

$$G(A) = \{g \in \mathrm{GL}_4(E \otimes_F A) \mid {}^t \bar{g} J g = \lambda(g) J, \lambda(g) \in A^\times\}.$$

We note that when a place v of F splits in E as (w_1, w_2) , we have $E \otimes_F F_v = E_{w_1} \oplus E_{w_2}$ and

$$G(F_v) = \{(g, \lambda J {}^t g^{-1} J^{-1}) \in \mathrm{GL}_4(E_{w_1}) \times \mathrm{GL}_4(E_{w_2}) \mid g \in \mathrm{GL}_4(E_{w_1}), \lambda \in F_v^\times\}.$$

Thus by fixing an isomorphism $F_v \simeq E_{w_1}$, we have a group isomorphism

$$(2.1) \quad G(F_v) \ni (g, \lambda J {}^t g^{-1} J^{-1}) \xrightarrow{\sim} (g, \lambda) \in \mathrm{GL}_4(F_v) \times \mathrm{GL}_1(F_v).$$

2.1.1. L -group and the twisted exterior square map. Let ${}^L G$ be the L -group of G and ${}^L G^\circ$ be its connected component. Then we have

$${}^L G = {}^L G^\circ \rtimes \mathrm{Gal}(E/F), \quad \text{where } {}^L G^\circ = \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}),$$

and the action of the unique non-trivial element θ of $\mathrm{Gal}(E/F)$ is given by

$$\mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \ni (g, \lambda) \mapsto (J {}^t g^{-1} J^{-1}, \lambda \det g) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

We refer to Rogawski [12, 1.1] for the L -group computation.

The standard representation of $\mathrm{GL}_4(\mathbb{C})$ on $\mathbb{C}^4$ induces the representation $\bigwedge^2$ of $\mathrm{GL}_4(\mathbb{C})$ on $\bigwedge^2 \mathbb{C}^4$. By abuse of notation we also denote by $\bigwedge^2$ a six dimensional representation of ${}^L G^\circ = \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ given by $\bigwedge^2(g, \lambda) = \left(\bigwedge^2 g\right) \lambda$. We know that $\bigwedge^2$ and $\bigwedge^2 \circ \theta$ are isomorphic by the highest weight theory. Hence there exists $A : \bigwedge^2 \mathbb{C}^4 \rightarrow \bigwedge^2 \mathbb{C}^4$ such that $A^2 = 1$ and

$$\left(\bigwedge^2 \circ \theta\right)(g, \lambda) = A^{-1} \bigwedge^2(g, \lambda) A \quad \text{for any } (g, \lambda) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

Here we specify A to be the one satisfying $\mathrm{tr}(A) > 0$. We may extend $\bigwedge^2$ to a six dimensional representation $\bigwedge_t^2$ of ${}^L G$, which we call the twisted exterior square map, defined by

$$\bigwedge_t^2(g, \lambda, 1) = \bigwedge^2(g, \lambda) \quad \text{and} \quad \bigwedge_t^2(1, 1, \theta) = A.$$

We note that the other extension of $\bigwedge^2$ to ${}^L G$ is given by $\bigwedge_t^2 \cdot \mathrm{sgn}$, which is not isomorphic to $\bigwedge_t^2$. We recall that sgn denotes a character of ${}^L G$ defined by

$$\mathrm{sgn}(g, \lambda, 1) = 1 \quad \text{and} \quad \mathrm{sgn}(g, \lambda, \theta) = -1.$$

We refer to Kim and Krishnamurthy [10] for further information about the twisted exterior square map and its associated L -function.

2.1.2. Subgroups. Let B be the Borel subgroup of G with the Levi decomposition $B = TU$ where

$$T = \left\{ t(t_1, t_2; \lambda) = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & \lambda \bar{t}_1^{-1} & 0 \\ 0 & 0 & 0 & \lambda \bar{t}_2^{-1} \end{pmatrix} \mid t_1, t_2 \in E^\times, \lambda \in F^\times \right\},$$

$$U = \left\{ u(x, y; a, b) = \begin{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ -\bar{x} & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1_2 & \begin{pmatrix} a & y \\ \bar{y} & b \end{pmatrix} \\ 0 & 1_2 \end{pmatrix} \mid x, y \in E, a, b \in F \right\}.$$

Let P be the Siegel parabolic subgroup of G . Then P has the Levi decomposition $P = MN$ where

$$M = \left\{ \begin{pmatrix} h & 0 \\ 0 & \lambda \cdot {}^t\bar{h}^{-1} \end{pmatrix} \mid h \in \mathrm{GL}_2(E), \lambda \in F^\times \right\},$$

$$N = \left\{ n(X) = \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} \mid X \in \mathrm{Herm}_2(E) \right\}.$$

Here

$$\mathrm{Herm}_2(E) = \{Y \in \mathrm{Mat}_{2 \times 2}(E) \mid Y = {}^t\bar{Y}\}.$$

The Shalika subgroup R of G is defined by

$$R = \{n h^* \mid n \in N, h \in \mathrm{GL}_2(F)\},$$

where

$$h^* = \begin{pmatrix} h & 0 \\ 0 & \det h \cdot {}^t\bar{h}^{-1} \end{pmatrix} \quad \text{for } h \in \mathrm{GL}_2(F).$$

2.2. The global zeta integral and the unitary Shalika period. Let π be an irreducible cuspidal unitary representation of $G(\mathbb{A}_F)$ acting on its space of cusp forms V_π by the right regular representation. Let ω_π be the central character of π .

Let us introduce an Eisenstein series on $\mathrm{GL}(2)$. Let ξ be an unitary idele class character of F . For Φ in $\mathcal{S}(\mathbb{A}_F^2)$, the space of Schwartz-Bruhat functions on $\mathbb{A}_F^2$, and $s \in \mathbb{C}$, let

$$(2.2) \quad f(h, s) = f_{\Phi, \xi}(h, s) = \xi(\det h) |\det h|^s \int_{\mathbb{A}_F^\times} \Phi((0, t)h) |t|^{2s} \xi^2 \omega_\pi(t) d^\times t.$$

Then the corresponding Eisenstein series

$$E(h, s) = E_{\Phi, \xi}(h, s) = \sum_{\gamma \in B_2(F) \backslash \mathrm{GL}_2(F)} f(\gamma h, s)$$

converges for $\mathrm{Re}(s) > 1$. Here B_2 denotes the group of upper triangular matrices in $\mathrm{GL}(2)$. Indeed the Poisson summation formula implies that the Eisenstein series has a meromorphic continuation to $\mathbb{C}$. We refer to Jacquet [5, Section 19] for the analytic properties of the Eisenstein series $E(h, s)$.

We recall that any character of $N(\mathbb{A}_F)/N(F)$ is written uniquely as

$$\psi_H(n(X)) = \psi[\mathrm{tr}(HX)]$$

for $H \in \mathrm{Herm}_2(E)$. By abuse of notation let us denote the character ψ_{H_0} by ψ where

$$H_0 = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}.$$

Let M_ψ be the stabilizer of ψ in the Levi component M , i.e.

$$M_\psi = \left\{ \begin{pmatrix} h & 0 \\ 0 & \lambda \cdot {}^t\bar{h}^{-1} \end{pmatrix} \mid {}^t\bar{h} H_0 h = \lambda \cdot H_0 \right\}.$$

Then we have $M_\psi = Z \cdot \{h^* \mid h \in \mathrm{GL}(2)\}$. Thus we consider a global zeta integral defined by

$$(2.3) \quad Z(s, \varphi) = Z(s, \varphi, \Phi, \xi)$$

$$= \int_{\mathbb{A}_F^\times \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{N(F) \backslash N(\mathbb{A}_F)} \varphi(n h^*) \psi(n) E(h, s) dn dh,$$

where $\varphi \in V_\pi$. This is clearly a $\mathrm{GU}(2, 2)$ -variant of Jacquet-Shalika's integral in [8]. For each unitary idele class character ξ of F satisfying $\xi^2 \cdot \omega_\pi|_{\mathbb{A}_F^\times} = 1$, let ψ_ξ be a character of $R(\mathbb{A}_F)$ defined by

$$(2.4) \quad \psi_\xi(nh^*) = \xi(\det h) \psi(n).$$

Then the unitary Shalika period of type ξ is a linear form on V_π defined by

$$(2.5) \quad S_\xi(\varphi) = \int_{\mathbb{A}_F^\times \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{N(F) \backslash N(\mathbb{A}_F)} \varphi(nh^*) \psi_\xi(nh^*) \, dn \, dh.$$

Proposition 2.1. *The global integral $Z(s, \varphi)$ is holomorphic at $s = 1$ if $\xi^2 \cdot \omega_\pi|_{\mathbb{A}_F^\times} \neq 1$. If on the contrary $\xi^2 \cdot \omega_\pi|_{\mathbb{A}_F^\times} = 1$, the integral $Z(s, \varphi)$ has a simple pole at $s = 1$ when and only when $S_\xi(\varphi) \neq 0$.*

Proof. The proof of Proposition 1 in Jacquet and Shalika [8, Section 5] applies step by step, *mutatis mutandis*, to our zeta integral $Z(s, \varphi)$. $\square$

2.3. Unfolding. Let us unfold the global zeta integral (2.3) and prove the basic identity (2.6) below.

2.3.1. Whittaker function. Let us define a non-degenerate character τ of $U(\mathbb{A}_F)$ by

$$\tau[u(x, y; a, b)] = \psi(\mathrm{tr}_{E/F}(\eta x) + b) \quad \text{for } u(x, y; a, b) \in U(\mathbb{A}_F).$$

Then the Whittaker function for $\varphi \in V_\pi$ is defined by

$$W_\varphi(g) = \int_{U(F) \backslash U(\mathbb{A}_F)} \tau(u) \varphi(ug) \, du.$$

2.3.2. Basic identity. Let us define a homomorphism $\bar{u} : \mathbb{G}_a \rightarrow G$ by

$$\bar{u}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & z & 0 & 1 \end{pmatrix}.$$

We also define a homomorphism $\iota : \mathrm{GL}(2) \rightarrow G$ by

$$\iota \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & a_{12} \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{21} & a_{22} & 0 \\ a_{21} & 0 & 0 & a_{22} \end{pmatrix}.$$

Then we have the following basic identity for the global zeta integral $Z(s, \varphi)$ defined by (2.3).

Proposition 2.2. *We have*

$$(2.6) \quad Z(s, \varphi, \Phi, \xi) = \int_{U_2(\mathbb{A}_F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{\mathbb{A}_F} W_\phi(\bar{u}(z) \iota(h)) \Phi((0, 1)h) \xi(\det h) |\det h|^s \, dz \, dh.$$

Here U_2 denotes the unipotent radical of B_2 and $\phi = \pi(w^{-1})\varphi$ where

$$(2.7) \quad w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Proof. Since

$$w^{-1}u(0, y; a, b)w = u(-y, 0; a, 0)\bar{u}(-b), \quad w^{-1}h^*w = \iota(h)$$

and $\varphi(g) = \phi(w^{-1}gw)$ for $g \in G(\mathbb{A}_F)$, we have

$$Z(s, \varphi) = \int_{\mathbb{A}_F^\times \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{E \backslash \mathbb{A}_E} \int_{F^2 \backslash \mathbb{A}_F^2} \phi[u(x, 0; a, 0)\bar{u}(z)\iota(h)] \psi(\mathrm{tr}_{E/F}(\eta x)) E(h, s) da dz dx dh.$$

By telescoping, we have

$$Z(s, \varphi) = \int_{\mathbb{A}_F^\times B_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{E \backslash \mathbb{A}_E} \int_{F^2 \backslash \mathbb{A}_F^2} \phi[u(x, 0; a, 0)\bar{u}(z)\iota(h)] \psi(\mathrm{tr}_{E/F}(\eta x)) f(h, s) da dz dx dh.$$

Since

$$\mathbb{A}_F^\times B_2(F) \subset \mathbb{A}_F^\times T_2(F) U_2(\mathbb{A}_F) \subset \mathrm{GL}_2(\mathbb{A}_F),$$

where T_2 denotes the group of diagonal matrices in $\mathrm{GL}(2)$, we have

$$Z(s, \varphi) = \int_{\mathbb{A}_F^\times T_2(F) U_2(\mathbb{A}_F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{E \backslash \mathbb{A}_E} \int_{F^3 \backslash \mathbb{A}_F^3} \phi[u(x, c; a, 0)\bar{u}(z)\iota(h)] \psi(\mathrm{tr}_{E/F}(\eta x)) f(h, s) da dc dz dx dh.$$

By the Poisson summation formula, we have

$$\phi(g) = \sum_{\alpha \in F} \int_{F \backslash \mathbb{A}_F} \psi(-2\alpha\eta^2 c') \phi(u(0, \eta c'; 0, 0,)g) dc'.$$

Hence

$$Z(s, \varphi) = \int_{\mathbb{A}_F^\times T_2(F) U_2(\mathbb{A}_F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{E^2 \backslash \mathbb{A}_E^2} \int_{F^2 \backslash \mathbb{A}_F^2} f(h, s) \sum_{\alpha \in F} \phi[u(x, y; a, 0)\bar{u}(z)\iota(h)] \psi[\mathrm{tr}_{E/F}(\eta(x - \alpha y))] da dz dx dy dh,$$

where $y = c + \eta c'$. Since

$$u(x, y; a, 0)\bar{u}(z) = \bar{u}(-\alpha) u(x - \alpha y, y; a + \alpha y\bar{y}, 0)\bar{u}(\alpha + z),$$

we have

$$Z(s, \varphi) = \int_{\mathbb{A}_F^\times T_2(F) U_2(\mathbb{A}_F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{E^2 \backslash \mathbb{A}_E^2} \int_{F \backslash \mathbb{A}_F} \int_{\mathbb{A}_F} f(h, s) \phi[u(0, y; a, 0)u(x, 0; 0, 0)\bar{u}(z)\iota(h)] \psi[\mathrm{tr}_{E/F}(\eta x)] dz da dx dy dh.$$

By the Poisson summation formula and the cuspidality of ϕ , we have

$$\begin{aligned} & \int_{E \backslash \mathbb{A}_E} \int_{F \backslash \mathbb{A}_F} \phi(u(0, y; a, 0)g) da dy \\ &= \sum_{\beta \in F^\times} \int_{E \backslash \mathbb{A}_E} \int_{F^2 \backslash \mathbb{A}_F^2} \psi(\beta b) \phi(u(0, y; a, b)g) da db dy. \end{aligned}$$

Since

$$\begin{aligned} & u(0, y; a, b) u(x, 0; 0, 0) \bar{u}(z) \iota(h) \\ &= \iota \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} u(0, \beta^{-1}y; \beta^{-1}a, \beta^{-1}b) u(x, 0; 0, 0) \bar{u}(\beta z) \iota \left(\begin{pmatrix} \beta^{-1} & 0 \\ 0 & 1 \end{pmatrix} h \right) \end{aligned}$$

and

$$\mathbb{A}_F^\times U_2(\mathbb{A}_F) \backslash \mathbb{A}_F^\times T_2(F) U_2(\mathbb{A}_F) \stackrel{\text{bijective}}{\approx} \left\{ \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \mid \beta \in F^\times \right\},$$

we have

$$(2.8) \quad Z(s, \varphi) = \int_{\mathbb{A}_F^\times U_2(\mathbb{A}_F) \backslash \mathrm{GL}_2(\mathbb{A}_F)} \int_{\mathbb{A}_F} W_\phi(\bar{u}(z) \iota(h)) f(h, s) dz dh.$$

Finally by plugging (2.2) into (2.8), the basic identity (2.6) holds. $\square$

Corollary 2.3. *Suppose that $\xi^2 \cdot \omega_\pi|_{\mathbb{A}_F^\times} = 1$. Then $S_\xi(\varphi) = 0$ for all $\varphi \in V_\pi$ unless π is globally generic, i.e. $W_\phi \neq 0$ for some $\phi \in V_\pi$.*

Proof. If $W_\phi = 0$ for all $\phi \in V_\pi$, we have $Z(s, \varphi) = 0$ for all $\varphi \in V_\pi$ by Proposition 2.2. In particular $Z(s, \varphi)$ is entire and hence $S_\xi(\varphi) = 0$ by Proposition 2.1. $\square$

3. LOCAL ZETA INTEGRALS

Let v be a place of F . When v is finite, let $\mathcal{O}_v$ be the integer ring of F_v and $\mathcal{O}_{E_v}$ be that of E_v . Let ϖ_v be a prime element of F_v , and we denote by q_v the cardinality of the residue field $\mathcal{O}_v/\varpi_v \mathcal{O}_v$.

According to the basic identity (2.6), we shall study the local zeta integral defined by

$$(3.1) \quad Z(s, W_v, \Phi_v, \xi_v) = \int_{U_2(F_v) \backslash \mathrm{GL}_2(F_v)} \int_{F_v} W_v(\bar{u}(z) \iota(h)) \Phi_v((0, 1) h) \xi_v(\det h) |\det h|^s dz dh,$$

where W_v is an element of $\mathcal{W}(\pi_v, \tau_v)$, the space of Whittaker functions for an irreducible unitary generic representation π_v of $G(F_v)$ with respect to the non-degenerate character τ_v , $\Phi_v \in \mathcal{S}(F_v^2)$, and ξ_v is a unitary character of $F_v^\times$.

When a place v splits in F , by the isomorphism (2.1), we have $\pi_v \simeq \sigma_v \otimes \chi_v$, where σ_v is an irreducible unitary generic representation of $\mathrm{GL}_4(F_v)$ and χ_v is a unitary character of $F_v^\times$. Hence we may rewrite (3.1) as

$$(3.2) \quad Z(s, W'_v \otimes \chi_v, \Phi_v, \xi_v) = \int_{U_2(F_v) \backslash \mathrm{GL}_2(F_v)} \int_{F_v} W'_v(\bar{u}(z) \iota(h)) \Phi_v((0, 1) h) \xi_v \chi_v(\det h) |\det h|^s dz dh,$$

where W'_v is an element of the space of Whittaker functions for σ_v with respect to a non-degenerate character τ'_v . Hence $W'_v(ug) = \tau'_v(u)^{-1} W'_v(g)$ for $u \in U(F_v)$ and $g \in \mathrm{GL}_4(F_v)$, where τ'_v is given by

$$\tau'_v \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x_{43} & 1 \end{pmatrix} = \psi(-\alpha x_{12} - \alpha x_{43} + x_{24})$$

when $\eta = (\alpha, -\alpha) \in F_v \oplus F_v$. Let us take the Weyl group elements w_i ($1 \leq i \leq 3$), where

$$w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad w_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the group $B^\circ = w_1 B w_1^{-1}$ becomes the group of upper triangular matrices in $\mathrm{GL}_4(F_v)$. The function $W_v^\circ(g) = W'_v(w_1 g)$ is the Whittaker function of $\mathrm{GL}_4(F_v)$ with respect to the non-degenerate character τ_v° of the unipotent radical of B° given by

$$\tau_v^\circ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \psi(-\alpha x_{12} - \alpha x_{34} + x_{23}).$$

Then, by a straightforward computation, we have

$$(3.3) \quad Z(s, \sigma_v(w_2) W'_v \otimes \chi_v, \Phi_v, \xi_v) = \int_{U_2(F_v) \backslash \mathrm{GL}_2(F_v)} \int_{F_v} W_v^\circ \left(w_3 \begin{pmatrix} 1_2 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right) \Phi_v((0, 1) h) \xi_v \chi_v(\det h) |\det h|^s dz dh.$$

Since we have

$$W_v^\circ \left(w_3 \begin{pmatrix} 1_2 & \begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \\ 0 & 1_2 \end{pmatrix} g \right) = \psi(\alpha x + \alpha w) W_v^\circ(w_3 g) \quad \text{for any } g \in \mathrm{GL}_4(F_v),$$

we may rewrite the right hand side of (3.3) as

$$\int_{U_2(F_v) \backslash \mathrm{GL}_2(F_v)} \int_{\mathfrak{p}(F_v) \backslash \mathrm{Mat}_{2 \times 2}(F_v)} W_v^\circ \left(w_3 \begin{pmatrix} 1_2 & Z \\ 0 & 1_2 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right) \cdot \Phi_v((0, 1) h) \xi_v \chi_v(\det h) \psi(-\mathrm{tr}(\alpha Z)) |\det h|^s dZ dh,$$

where $\mathfrak{p}(F_v)$ denotes the space of upper triangular matrices in $\mathrm{Mat}_{2 \times 2}(F_v)$. This integral is nothing but the local integral considered by Jacquet and Shalika [8, p. 203] in the case of $\mathrm{GL}(4)$.

Hereafter until the end of this section, we suppress the subscript v from the notation, since the setting is entirely local.

3.1. Convergence and non-vanishing at $s = 1$ of local zeta integrals.

3.1.1. Estimates for the Whittaker functions. From Jacquet and Langlands [6, Section 8], we recall that a continuous function on a locally compact abelian group is called finite when its translates span a finite dimensional vector space. For a local field F , the set of finite functions on $(F^\times)^2$ is indeed the set of finite linear combinations of functions of the form

$$(3.4) \quad \mu(a_1, a_2) = \prod_{i=1}^2 \mu_i(a_i) |a_i|^{u_i} (\log|a_i|)^{n_i},$$

where μ_1, μ_2 are unitary characters; u_1, u_2 are real; and n_1, n_2 are non-negative integers.

Let B_0 denote the subgroup of the Borel subgroup B given by $B_0 = T_0 U$ where

$$T_0 = \{t(\lambda, t; \lambda) \mid t \in E^\times, \lambda \in F^\times\};$$

i.e. B_0 consists of elements of B whose $(3, 3)$ -entry is 1. Let δ_{B_0} denote the modulus function of the group B_0 . Let K be a maximal compact subgroup of G .

Then by the same argument which leads to Proposition 3 in Jacquet and Shalika [8, Section 4], replacing the mirabolic subgroup of GL_n there by B_0 in our case, we have the following proposition.

Proposition 3.1. *There exists a finite set X of finite functions μ of the form (3.4), with $u_i \geq 0$ ($i = 1, 2$) such that for any $W \in \mathcal{W}(\pi, \tau)$, there exist functions ϕ_μ in the Schwartz space $\mathcal{S}(F^2 \times K)$ such that*

$$W(tk) = \delta_{B_0}^{\frac{1}{2}}(t) \sum_{\mu \in X} \phi_\mu(a_1, a_2, k) \mu(a_1, a_2)$$

for $k \in K$ and $t = \mathrm{diag}(a_1, a_2, 1, a_1 a_2^{-1})$.

3.1.2. Convergence and non-vanishing. By exactly the same argument as in the proof of Proposition 1 in Jacquet and Shalika [8, Section 7], we obtain the following proposition from Proposition 3.1.

Proposition 3.2. *For given π , there exists $\varrho > 0$ such that the zeta integral $Z(s, W, \Phi, \xi)$ converges absolutely in the half plane $\mathrm{Re}(s) > 1 - \varrho$.*

Again, by the same argument as in the proof of Proposition 3 in Jacquet and Shalika [8, Section 7], we have the following proposition.

Proposition 3.3. *There exist W and Φ such that*

$$Z(1, W, \Phi, \xi) \neq 0.$$

3.2. Unramified computations. We shall compute unramified local integrals over finite places. We note that by the Iwasawa decomposition, we may write (3.1) as

$$(3.5) \quad Z(s, W, \Phi, \xi) = \int_{T_2(F)} \int_{K_2} \int_F \xi(\det tk) |\det tk|^s \delta_{B_2}(t)^{-1} \\ W(\bar{u}(z) \iota(tk)) \Phi((0, 1) tk) dz dt dk,$$

where $K_2 = \mathrm{GL}_2(\mathcal{O})$. We normalize the measures so that

$$\int_{T_2(\mathcal{O})} dt = \int_{K_2} dk = \int_{\mathcal{O}} dz = 1.$$

Suppose that all the data involved in the local integral (3.5) are unramified, meaning that

- π is unramified,
- ξ is unramified,
- E/F is not a ramified quadratic extension,
- the conductor of the character ψ is $\mathcal{O}$ and the conductor of the character $x \mapsto \psi(\mathrm{tr}_{E/F}(\eta x))$ is $\mathcal{O}_E$,
- Φ is the characteristic function of $\mathcal{O} \oplus \mathcal{O}$,
- W is the unique unramified Whittaker function for π such that $W(1) = 1$.

Proposition 3.4. *We have*

$$(3.6) \quad Z(s, W, \Phi, \xi) = L\left(s, \pi, \bigwedge_t^2 \otimes \xi\right).$$

Proof. When v is split, we have

$$Z(s, W' \otimes \chi, \Phi, \xi) = L\left(s, \bigwedge^2(\sigma) \otimes \chi\xi\right)$$

by Proposition 2 in Jacquet and Shalika [8, Section 7], and hence the equality (3.6) holds.

Suppose that v is inert. We have

$$Z(s, W, \Phi, \xi) = \int_{T_2(F)} \int_F W\left(\bar{u}(z) \iota \begin{pmatrix} ab & 0 \\ 0 & b \end{pmatrix}\right) \Phi(0, b) \xi(ab^2) |a|^{s-1} |b|^{2s} dz d^\times a d^\times b.$$

By considering the support of Φ , the integral is supported on $b \in F^\times \cap \mathcal{O}$, and we have

$$(3.7) \quad Z(s, W, \Phi, \xi) = \int_{F^\times \cap \mathcal{O}} (\omega_\pi \xi^2)(b) |b|^{2s} \int_{F^\times} \xi(a) |a|^{s-2} \int_F W\left(\iota \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \bar{u}(z)\right) dz d^\times a d^\times b.$$

We note that

$$I = \int_{F \setminus \mathcal{O}} W\left(\iota \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \bar{u}(z)\right) dz = 0$$

by an argument in Roberts and Schmidt [11, Proof of Lemma 4.1.1]. For the convenience of the reader, we shall recollect their argument here. Since

$$\iota \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \bar{u}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & az^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & az^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z \end{pmatrix} w^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & z^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where w is as (2.7), we have

$$I = \int_{F \setminus \mathcal{O}} \psi(-az^{-1}) W(t(a, az^{-1}; a) w^{-1}) dz.$$

Since W is unramified, we have

$$I = \int_{\mathcal{O}_E} \int_{F \setminus \mathcal{O}} \psi(-az^{-1}) W(t(a, az^{-1}; a) w^{-1} u(0, y; 0, 0)) dz dy.$$

Hence

$$I = \int_{F \setminus \mathcal{O}} \psi(-az^{-1}) W(t(a, az^{-1}; a)) \left(\int_{\mathcal{O}_E} \psi(z \operatorname{tr}_{E/F}(\eta y)) dy \right) dz = 0$$

since the inner integral vanishes.

Thus the integral (3.7) becomes

$$Z(s, W, \Phi, \xi) = L(2s, \xi^2 \cdot \omega_\pi|_{F^\times}) \int_{F^\times \cap \mathcal{O}} W(t(a, a; a)) \xi(a) |a|^{s-2} d^\times a,$$

and our remaining task is to evaluate

$$Z_1 = \sum_{n=0}^{\infty} W(t(\varpi^n, \varpi^n; \varpi^n)) \xi(\varpi)^n q^{-n(s-2)}.$$

We shall evaluate the infinite sum Z_1 by the Casselman-Shalika formula [2]. We refer the reader to Tamir [15] and to Gan and Hundley [3] for the interpretation of the Casselman-Shalika formula in terms of an appropriate dual group in some non-split cases. Since the relative root system involved in our case is reduced as in [3], we shall closely follow their lucid computation.

First we note that the group G contains a connected F -split subgroup H whose root system is equal to the relative root system of G and whose maximal torus is the maximal F -split subtorus S of T . Namely we may take

$$H = \mathrm{GSp}_4(F) = \{g \in \mathrm{GL}_4(F) \mid {}^t g J g = \lambda(g) J, \lambda(g) \in \mathbb{G}_m(F)\}.$$

Suppose that π is a spherical representation corresponding to an unramified character ν of T . Then the Satake parameter t_π of π is given by

$$t_\pi = (s_\pi, \theta) \in {}^L G^\circ \rtimes \mathrm{Gal}(E/F),$$

where

$$(3.8) \quad s_\pi = (\mathrm{diag}(a, b, 1, 1), \lambda) \in \mathrm{GL}_4(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}).$$

Here $a = \nu(t(\varpi, 1; \varpi))$, $b = \nu(t(1, \varpi; \varpi))$ and $\lambda = \nu(t(1, 1; \varpi))$. Let $\varsigma : S \rightarrow T$ denote the inclusion. Then ς induces a homomorphism $\varsigma^* : {}^L T^\circ \rightarrow {}^L S^\circ$ on the dual side. Let us define $\overline{s}_\pi \in {}^L S^\circ$ by $\overline{s}_\pi = \varsigma^*(s_\pi)$. Let

$$\Lambda^+ = \{\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3 \mid k_1 \geq k_2 \geq 0\}$$

and let $\varpi^{\mathbf{k}} = t(\varpi^{k_1+k_2+k_3}, \varpi^{k_1+k_3}, \varpi^{k_1+k_2+2k_3})$ for $\mathbf{k} \in \Lambda^+$. Then the Casselman-Shalika formula reads

$$(3.9) \quad W(\varpi^{\mathbf{k}}) = \mathrm{tr}(\overline{s}_\pi|_{V_{\mathbf{k}}}) \cdot \delta_B(\varpi^{\mathbf{k}})^{\frac{1}{2}}.$$

Here $V_{\mathbf{k}}$ is the irreducible representation of ${}^L H^\circ = \mathrm{GSp}_4(\mathbb{C})$ obtained from an element of $X^*({}^L S^\circ) \simeq X_*(S)$ in the dominant chamber corresponding to $\mathbf{k}$.

Let $\nu' = \nu|_S$. Let π' be the spherical representation of the F -split group H corresponding to ν' and let $s_{\pi'} \in {}^L S^\circ$ be its Satake parameter. Then it is easily seen that $\overline{s}_\pi = s_{\pi'}$. Hence by (3.9) we have

$$W(\varpi^{\mathbf{k}}) \cdot \delta_B(\varpi^{\mathbf{k}})^{-\frac{1}{2}} = W_H(\varpi^{\mathbf{k}}) \cdot \delta_{B'}(\varpi^{\mathbf{k}})^{-\frac{1}{2}},$$

where W_H denotes the normalized unramified Whittaker function for π' and B' is the Borel subgroup of H . Thus, in particular,

$$W(\varpi^{(n,0;0)}) = W_H(\varpi^{(n,0;0)}) \cdot q^{-\frac{n}{2}} \quad \text{for } n \geq 0,$$

and hence we have

$$Z_1 = \sum_{n=0}^{\infty} W_H(\varpi^{(n,0;0)}) \xi(\varpi)^n q^{-n(s-\frac{3}{2})} = L(s, \pi', \mathrm{spin} \otimes \xi),$$

as in Bump [1, 3.3].

Thus we have shown that

$$Z(s, W, \Phi, \xi) = L(2s, \xi^2 \cdot \omega_\pi|_{F^\times}) L(s, \pi', \mathrm{spin} \otimes \xi).$$

We note that the semi-simple conjugacy class of $\bigwedge_t^2(t_\pi)$ in $\mathrm{GL}_6(\mathbb{C})$ for t_π as in (3.8) is given by

$$\mathrm{diag}(\lambda, \lambda a, \lambda b, \lambda ab, \lambda \alpha, \lambda \beta),$$

where α and β are roots of $X^2 - ab$. Thus the equality (3.6) holds. $\square$

Here we remark that Sugano [14] obtained a different integral representation for the same L -function for π which is not globally generic but attached to a holomorphic cuspidal eigenform.

4. MAIN THEOREM

We return to the global situation. Let π be an irreducible cuspidal unitary representation of $G(\mathbb{A}_F)$ whose central character is ω_π . Suppose that π is globally generic. Let S be a finite set of places of F consisting of infinite places and the finite places which are not unramified in the sense of 3.2. We set

$$L^S\left(s, \pi, \bigwedge_t^2 \otimes \xi\right) = \prod_{v \notin S} L\left(s, \pi_v, \bigwedge_t^2 \otimes \xi_v\right).$$

We shall simply write $L^S(s)$ for this partial L -function. Also, let $K = \prod_v K_v$, where K_v is a maximal compact subgroup of $G(F_v)$ for each place v of F .

Then our main result is the following theorem.

Theorem 4.1. *The partial L -function $L^S(s)$ extends to a half plane $\mathrm{Re}(s) > 1 - \varrho$ with $\varrho > 0$ as a meromorphic function. It has a pole at $s = 1$ if and only if $\xi^2 \cdot \omega_\pi|_{\mathbb{A}_F^\times} = 1$ and there exists a K -finite vector φ in the space of π such that the period integral $S_\xi(\varphi)$ defined by (2.5) does not vanish.*

Proof. The theorem is proved by the same argument as in the proof of Theorem 1 in Jacquet and Shalika [8, Section 8]. $\square$

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