ONE-PARAMETER FAMILIES OF CIRCLE DIFFEOMORPHISMS WITH STRICTLY MONOTONE ROTATION NUMBER

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ABSTRACT. We show that if $f: S^1 \times S^1 \rightarrow S^1 \times S^1$ is C^2 , with $f(x, t) = (f_t(x), t)$, and the rotation number of f_t is equal to t for all $t \in S^1$, then f is topologically conjugate to the linear Dehn twist of the torus $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We prove a differentiability result where the assumption that the rotation number of f_t is t is weakened to say that the rotation number is strictly monotone in t .

1. INTRODUCTION

In this article we consider homeomorphisms and diffeomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. They are always assumed to preserve orientation. Following Poincaré, homeomorphisms of the circle may be classified according to their rotation number, which is defined as follows.

Let f be a homeomorphism of the circle, and let $\tilde{f}: \mathbb{R} \rightarrow \mathbb{R}$ be a lift of f (that is, $\pi(\tilde{f}(x)) = f(\pi(x))$ for all $x \in \mathbb{R}$, where π is the canonical projection). Define the translation number to be $\tau(\tilde{f}) = \lim_{n \rightarrow \infty} \frac{\tilde{f}^n(0)}{n}$; it is a fact that this limit will always exist. The value of $\tau(\tilde{f})$ does depend on the choice of lift, but for another lift $\tilde{f}'$ we will have $\tau(\tilde{f}) - \tau(\tilde{f}') \in \mathbb{Z}$. Therefore, we define the rotation number of f to be $\rho(f) = \pi(\tau(\tilde{f}))$; this does not depend on the choice of lift. We will write rotation numbers as if they were real numbers, understanding that when we write $\rho(f) = x$ we really mean $\rho(f) = x + \mathbb{Z}$.

Poincaré's classification theorem provides strong connections between the rotation number and the dynamics of f . It says that $\rho(f) = 0$ if and only if f has at least one fixed point (more generally, $\rho(f) \in \mathbb{Q}$ if and only if f has a periodic point) and that $\rho(f) \notin \mathbb{Q}$ if and only if f is topologically semi-conjugate to an irrational rotation. In fact, according to Denjoy's theorem, if $\rho(f) \notin \mathbb{Q}$ and f is a C^1 diffeomorphism such that the derivative f' has bounded variation (in particular, if f is C^2), then f is topologically conjugate to an irrational rotation. Note that a similar statement is false in the case of a rational rotation number (even under strong smoothness assumptions): having a rational rotation number implies that there exists a periodic orbit, not the much stronger statement that every orbit is

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periodic with the same period (which is equivalent to being conjugate to a rational rotation). For an excellent discussion of Poincaré's theory, see [2], [3], [7], and for Denjoy's theory, [2], [7].

A rotation number is a function $\rho: \text{Homeo}_+(S^1) \rightarrow \mathbb{R}/\mathbb{Z}$, where $\text{Homeo}_+(S^1)$ is the group of orientation-preserving homeomorphisms of the circle. Note that the function ρ is not a homomorphism. One can consider the restriction of ρ to the subgroups $\text{Diff}_+^r(S^1)$, $1 \leq r \leq \omega$. We may endow these with the C^r topology (for $\text{Homeo}_+(S^1)$, $r = 0$) and study the function ρ by investigating subspaces on which ρ takes on certain values. For example, $\rho^{-1}(\mathbb{Q})$ contains an open and dense subset of $\text{Homeo}_+(S^1)$, and $\rho^{-1}(\mathbb{Q}) \cap \text{Diff}_+^r(S^1)$ contains an open and dense subset of $\text{Diff}_+^r(S^1)$ for all $r \geq 1$ (see [4]).

One can gain some insight into the local structure of ρ by considering it on 1-parameter families of homeomorphisms of the circle. Whenever we have a 1-parameter family of homeomorphisms f_t , we will write $\underline{\rho}(t) := \rho(f_t)$.

One way to get a 1-parameter family is to let f be a homeomorphism and then define a family f_t by $f_t = R_t \circ f$, where R_t is rotation by t . The behavior of $\underline{\rho}(t)$ is very different when its value is rational than when its value is irrational. For any $\alpha \notin \mathbb{Q}$, $\underline{\rho}^{-1}(\alpha)$ will be a single point. This can be seen in the following way. Without loss of generality, $\rho(f) = \alpha$. It suffices to show that for arbitrarily small $t > 0$ and arbitrarily small $t < 0$, f_t has periodic points.

Since $\rho(f) = \alpha \notin \mathbb{Q}$, either f is conjugate to an irrational rotation, in which case every orbit is dense, or it is semi-conjugate to an irrational rotation, in which case there exists a unique invariant Cantor set on which the dynamics are minimal. In the first case, let $x \in S^1$ be arbitrary. Since the orbit of x accumulates on x on both sides and any finite subset of the orbit depends continuously on t , for arbitrarily small t (both positive and negative) composing by R_t will “close up” the orbit of x so that f_t will have periodic points. If f is merely semi-conjugate to R_α , let x be in the invariant Cantor set. Choose it to be an accumulation point from the left if we want to find small $t > 0$ such that f_t has periodic points, and similarly for $t < 0$. Once we find such an x , we argue as above.

On the other hand, for a generic (countable intersection of open and dense sets) $f \in \text{Homeo}_+(S^1)$ or $\text{Diff}_+^r(S^1)$ in the C^r topology ($0 \leq r \leq \omega$), for every $\frac{p}{q} \in \mathbb{Q}$, $\underline{\rho}^{-1}(\frac{p}{q})$ is a nontrivial interval ([5], p. 37).

The most famous example where $\underline{\rho}^{-1}(\frac{p}{q})$ is a nontrivial interval for all $\frac{p}{q} \in \mathbb{Q}$, due to Arnold (as cited in Ghys [3], p. 358), is the following: $f_a(x) = x + a \cdot \sin(2\pi x)$, where a is chosen to be small enough so that f is a homeomorphism. As above, $(f_a)_t$ is defined to be $R_t \circ f_a$. For $\frac{p}{q} \in \mathbb{Q}$, the set of (t_0, a_0) in the (t, a) -plane such that $\rho((f_{a_0})_{t_0}) = \frac{p}{q}$ will be a strange “tongue” shape. These Arnold tongues are much studied and quite important.

In this paper, we consider a different type of 1-parameter family of circle homeomorphisms. We make the strong, rather unusual assumption that the rotation number is a *strictly* monotone function of the parameter; in particular, it has no “plateaus” when its value is rational, unlike in the above situations. We draw some strong conclusions in this context.

The first result says that in the special case where the rotation number is not just monotone but the identity, the behavior is highly constrained.

Theorem 1. *Suppose that $f: S^1 \times S^1 \rightarrow S^1 \times S^1$ is C^2 , with $f(x, t) = (f_t(x), t)$, and the rotation number of f_t is equal to t for all $t \in S^1$. Then for each t there is a conjugacy ϕ_t from f_t to R_t (i.e. $f_t = \phi_t^{-1} \circ R_t \circ \phi_t$) such that ϕ_t varies continuously with t . In particular, f is topologically conjugate to the linear Dehn twist of the torus $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

We will see, in the course of the proof, that when $t \in \mathbb{Q}$ the conjugacy between f_t and R_t is actually forced to be differentiable.

We will show by way of example that the result of Theorem 1 does not hold in general if we only assume that the rotation number is strictly monotone. However, interesting properties continue to hold in that generality, which we explore in Theorem 2.

Note that $\frac{\partial f_t^q}{\partial t}(x, t_0)$ (or $\frac{\partial f_t^q}{\partial t}|_{t=t_0}(x)$ if we wish to vary x and hold t constant) means $pr_1(\frac{\partial f_t^q}{\partial t}(x, t_0))$, where pr_1 is a projection onto the first component.

Theorem 2. *Suppose that $f: S^1 \times [a, b] \rightarrow S^1 \times [a, b]$ is C^2 , with $f(x, t) = (f_t(x), t)$, and the rotation number is a strictly monotone increasing function of t . Let t_0 be such that $\underline{\rho}(t_0) = \frac{p}{q} \in \mathbb{Q}$ (where this fraction is reduced). Then $\underline{\rho}$ is differentiable at t_0 . Moreover, if $\underline{\rho}'(t_0) \neq 0$, then $\frac{\partial f_t^q}{\partial t}(x, t_0) \neq 0$ for all $x \in S^1$. In that case,*

$$\underline{\rho}'(t_0) = \frac{1}{q \cdot \int_{S^1} \frac{1}{\frac{\partial f_t^q}{\partial t}(x, t_0)} dx}.$$

As we will discuss below, this relates to classic results about the derivative of the rotation number when the value of the rotation number is irrational (see Herman, [4]). It stands in sharp contrast to how the rotation number behaves at rational values in families given by composing by rotation by t ; see Matsumoto [6].

Remark 3. The arguments in this paper actually work in somewhat greater generality than we have assumed in Theorems 1 and 2. They imply the conclusion of Theorem 2 if we replace strict monotonicity of the rotation number with the following weaker condition:

($\star$) Whenever $\underline{\rho}(t_0) \in \mathbb{Q}$, there exists $\epsilon > 0$ such that either $\underline{\rho}(t) > \underline{\rho}(t_0)$ for $t \in (t_0, t_0 + \epsilon)$ and $\underline{\rho}(t) < \underline{\rho}(t_0)$ for $t \in (t_0 - \epsilon, t_0)$ or $\underline{\rho}(t) < \underline{\rho}(t_0)$ for $t \in (t_0, t_0 + \epsilon)$ and $\underline{\rho}(t) > \underline{\rho}(t_0)$ for $t \in (t_0 - \epsilon, t_0)$.

Similarly, in Theorem 1, instead of assuming $\underline{\rho}(t) = t$, we could have assumed $\underline{\rho}$ is strictly monotone and has nonzero derivative when its value is rational. To extend Theorem 1 still more, we can replace strictly monotone with ($\star$), and we get conjugacy to the map of the torus that preserves the circles C_t of constant t , acting on C_t by $R_{\rho(f_t)}$. These generalizations of Theorem 1 rely on the fact that the rotation number is differentiable when its value is rational, even in the generality of ($\star$), by Theorem 2.

2. PROOF OF THEOREM 1

Throughout the proof, f is as in the statement of Theorem 1. We use g or g_t to refer to more general homeomorphisms or families of homeomorphisms of the circle.

Proof of Theorem 1. First suppose $t \notin \mathbb{Q}$. In that case, f_t is C^2 and has irrational rotation number t , so by Denjoy's theorem, f_t is conjugate to R_t . Moreover, since

the only homeomorphisms of the circle that commute with a given irrational rotation are themselves rotations, if ϕ_t and $\tilde{\phi}_t$ are two conjugacies from f_t to R_t , then they will differ by only a rotation. If we agree once and for all that the conjugacies should send 0 to 0, then we have a well defined ϕ_t for each irrational t .

Now what we must show is that these ϕ_t ($t \notin \mathbb{Q}$) vary continuously in t and that they extend continuously to conjugacies for $t \in \mathbb{Q}$. We note that the assumption that $\rho(t) = t$, instead of merely being strictly monotone, becomes important in Lemma 7, one of whose conditions is that the function ρ not have a derivative equal to 0 when its value is 0.

The following lemma implies that the ϕ_t ($t \notin \mathbb{Q}$) vary continuously in t .

Lemma 4. *Let $g, g': S^1 \rightarrow S^1$ be conjugate to irrational rotations, say $g = \phi^{-1} \circ R_\alpha \circ \phi$ and $g' = (\phi')^{-1} \circ R_{\alpha'} \circ \phi'$, where ϕ and ϕ' are chosen to send 0 to 0. For any $\epsilon > 0$, there is a $\delta > 0$ such that if $|g - g'|_{C^0} < \delta$, then $|\phi - \phi'|_{C^0} < \epsilon$.*

Proof. Conjugating g and g' by ϕ , we may assume that $g = R_\alpha$. There is no loss of generality, since conjugation by ϕ is a continuous mapping of $\text{Homeo}_+(S^1)$. Therefore, we have reduced to the case where $g = R_\alpha$, i.e. $\phi = id$.

Let $\epsilon > 0$. Choose N large enough so that the points $0, \alpha, \dots, N\alpha$ are ϵ -dense in S^1 in the sense that the intervals of $S^1 \setminus \{0, \dots, N\alpha\}$ all have length less than ϵ . If g' is sufficiently close to R_α , we will have $|\alpha - \alpha'| < \frac{\epsilon}{N}$, which implies that the intervals of $S^1 \setminus \{0, \dots, N\alpha'\}$ have length less than 2ϵ .

First we show that ϕ' is close to the identity on the points $(g')^n(0), 0 \leq n \leq N$. By the choice of α' ,

$$|n\alpha - n\alpha'| < \epsilon$$

when $0 \leq n \leq N$. If g' is close enough to R_α , then we will have

$$|n\alpha - (g')^n(0)| < \epsilon$$

for all $0 \leq n \leq N$. Therefore,

$$|(g')^n(0) - n\alpha'| < 2\epsilon.$$

Now we show that ϕ' is close to the identity everywhere. Let $x \in S^1$, and let $0 \leq m, n \leq N$ be such that $((g')^m(0), (g')^n(0))$ is one of the intervals of $S^1 \setminus \{0, \dots, (g')^N(0)\}$ and $x \in [(g')^m(0), (g')^n(0)]$. Then $m\alpha' \leq \phi'(x) \leq n\alpha'$. Since $(g')^m(0)$ is within 2ϵ of $m\alpha'$ and $(g')^n(0)$ is within 2ϵ of $n\alpha'$,

$$|\phi'(x) - x| < 2\epsilon + (n - m)\alpha'.$$

Recall that $(n - m)\alpha' < 2\epsilon$, so

$$|\phi'(x) - x| < 4\epsilon.$$

Since ϵ can be made arbitrarily small, we are done. $\square$

Remark 5. This reasoning also applies if g' is conjugate to a rational rotation. In that case, it says that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|g - g'|_{C^0} < \delta$, then $|\phi - \phi'|_{C^0} < \epsilon$, where ϕ' is any conjugacy of g' to a rotation that sends 0 to 0.

It is necessary to assume that g is conjugate to an irrational rotation. The following statement is false: for any circle homeomorphism g which is conjugate to a rotation, for any $\epsilon > 0$, there exists a δ such that if g' is a circle homeomorphism conjugate to a rotation satisfying $|g - g'|_{C^0} < \delta$, then there exist circle homeomorphisms ϕ and ϕ' such that $g = \phi^{-1} \circ R_{\rho(g)} \circ \phi$, $g' = (\phi')^{-1} \circ R_{\rho(g')} \circ \phi'$, and $|\phi - \phi'|_{C^0} < \epsilon$.

We may construct a counterexample as follows. Let $\psi_\epsilon: S^1 \rightarrow S^1$ be the homeomorphism which is linear on the intervals $[0, \frac{1}{4})$, $[\frac{1}{4}, \frac{1}{4} + \epsilon)$, $[\frac{1}{4} + \epsilon, \frac{3}{4} - \epsilon)$, $[\frac{3}{4} - \epsilon, \frac{3}{4})$, and $[\frac{3}{4}, 1)$, with

$$\begin{aligned}\psi_\epsilon([0, \tfrac{1}{4})) &= [0, \tfrac{1}{8}), \\ \psi_\epsilon([\tfrac{1}{4}, \tfrac{1}{4} + \epsilon)) &= [\tfrac{1}{8}, \tfrac{3}{8}), \\ \psi_\epsilon([\tfrac{1}{4} + \epsilon, \tfrac{3}{4} - \epsilon)) &= [\tfrac{3}{8}, \tfrac{5}{8}), \\ \psi_\epsilon([\tfrac{3}{4} - \epsilon, \tfrac{3}{4})) &= [\tfrac{5}{8}, \tfrac{7}{8}), \\ \psi_\epsilon([\tfrac{3}{4}, 1)) &= [\tfrac{7}{8}, 1).\end{aligned}$$

For ϵ sufficiently small, we will have $\psi_\epsilon^{-1} \circ R_{\frac{1}{2}} \circ \psi_\epsilon \approx R_{\frac{1}{2}}$. To see this, consider $\psi(x) = \lim_{\epsilon \rightarrow 0} \psi_\epsilon^{-1}(x)$. ψ is no longer a homeomorphism: it will send $[\frac{1}{8}, \frac{3}{8}]$ to $\{\frac{1}{4}\}$ and $[\frac{5}{8}, \frac{7}{8}]$ to $\{\frac{3}{4}\}$. Nonetheless, we have $\psi \circ R_{\frac{1}{2}} = R_{\frac{1}{2}} \circ \psi$.

Even though $\psi_\epsilon^{-1} \circ R_{\frac{1}{2}} \circ \psi_\epsilon \approx R_{\frac{1}{2}}$, ψ_ϵ is not close to any homeomorphism that commutes with $R_{\frac{1}{2}}$, since $\psi_\epsilon \circ R_{\frac{1}{2}}(\frac{1}{4}) = \frac{7}{8}$, which is far from $R_{\frac{1}{2}} \circ \psi_\epsilon(\frac{1}{4}) = \frac{5}{8}$.

Let $g = R_{\frac{1}{2}}$. Let ϵ be small enough that $\psi_\epsilon^{-1} \circ R_{\frac{1}{2}} \circ \psi_\epsilon$ is close to $R_{\frac{1}{2}}$, and let $\alpha' \notin \mathbb{Q}$ be close enough to $\frac{1}{2}$ that $g' = \psi_\epsilon^{-1} \circ R_{\alpha'} \circ \psi_\epsilon$ is still close to $R_{\frac{1}{2}}$. Despite the fact that g' is arbitrarily close to g , no conjugacy for g' is close to any conjugacy for g (since g' is conjugate to an irrational rotation, ψ_ϵ is essentially the only conjugacy for g').

The following lemma shows that $f_0 = id$.

Lemma 6. *If a homeomorphism g of the circle has $\rho(g) = 0$ and there exist homeomorphisms arbitrarily close to g with a positive rotation number, as well as homeomorphisms arbitrarily close with a negative rotation number, then $g = id$.*

Proof. By assumption, $\rho(g) = 0$, so g has fixed points. The intuition of the lemma is as follows. If $g \neq id$, then either its graph crosses the diagonal $y = x$ or does not cross but is tangent to the diagonal at some points.

If the graph crosses the diagonal, then under slight perturbations it will still have fixed points, hence still have rotation number 0. If it is tangent to the diagonal but does not cross, then under small perturbations we may increase or decrease (but not both) the rotation number. The reason is that in this situation the graph either does not go below or does not go above the diagonal. Suppose it does not go below the diagonal. Then by pushing the graph upward slightly we make the rotation number positive, but if we push it downward the rotation number will still be 0. Similarly, if the graph does not go above the diagonal, then by applying a small perturbation, we may decrease, but not increase, the rotation number. The details are left as an exercise for the reader. $\square$

The following lemma implies that $\frac{\partial f_t}{\partial t}(x, t_0)$ is greater than 0 for all x , since $\underline{\rho}'(0) = 1$.

Lemma 7. *Suppose $g: S^1 \times [a, b] \rightarrow S^1 \times [a, b]$ is C^2 , with $g(x, t) = (g_t(x), t)$, and the rotation number is a strictly monotone increasing function of t , with $\underline{\rho}(t_0) = 0$. Suppose it is not the case that the derivative $\underline{\rho}'(t_0)$ exists and is equal to 0. Then $\frac{\partial g_t}{\partial t}(x, t_0)$ is greater than 0 for all x .*

Proof. We must show that the function $\frac{\partial g_t}{\partial t}|_{t=t_0}(x)$ is strictly greater than 0. By Lemma 6, $g_{t_0} = id$. For any $t > t_0$, $g_t(x) > x$ for all x , and similarly for $t < t_0$; this must be true because $\underline{\rho}$ is strictly monotone increasing. Therefore, for any x , $\frac{\partial g_t}{\partial t}|_{t=t_0}(x) \geq 0$. It remains to show that $\frac{\partial g_t}{\partial t}|_{t=t_0}(x) \geq 0$ never vanishes.

Suppose, by way of contradiction, that for some x_0 , $\frac{\partial g_t}{\partial t}|_{t=t_0}(x_0) = 0$. Since the function $\frac{\partial g_t}{\partial t}|_{t=t_0}$ is nonnegative and C^1 (because g is C^2), it will have a minimum at x_0 . Therefore, for any $\epsilon > 0$ there exists $\delta > 0$ such that when $|x - x_0| < \delta$, $\frac{\partial g_t}{\partial t}|_{t=t_0}(x) < \epsilon|x - x_0|$.

Since it is not the case that $\underline{\rho}'(t_0)$ exists and is equal to 0 and $\underline{\rho}$ is monotone increasing, there exist $\eta > 0$ and t arbitrarily close to t_0 such that $\frac{\underline{\rho}(t) - \underline{\rho}(t_0)}{t - t_0} > \eta$. Without loss of generality there are $t > t_0$ arbitrarily close to t_0 with this property. We may rewrite it as $\underline{\rho}(t) > \eta(t - t_0)$. Let $Q = \frac{2}{\eta} + 3(t - t_0)$, and choose a δ that works for $\epsilon = \frac{1}{Q}$. For t sufficiently close to t_0 , we have that for all x ,

$$g_t(x) - x \in \left(\left(\frac{\partial g_t}{\partial t}|_{t=t_0}(x) - \frac{\delta}{Q} \right)(t - t_0), \left(\frac{\partial g_t}{\partial t}|_{t=t_0}(x) + \frac{\delta}{Q} \right)(t - t_0) \right).$$

Fix a $t > t_0$ which is this close to t_0 and which has the property that $\underline{\rho}(t) > \eta(t - t_0)$.

Consider what iterates of g_t do to x_0 . We have

$$g_t(x_0) - x_0 < \frac{\delta}{Q}(t - t_0).$$

Since this is less than δ ,

$$g_t^2(x_0) - x_0 < \left(\frac{\delta}{Q} + \frac{2\delta}{Q} \right)(t - t_0) = \frac{3\delta}{Q}(t - t_0).$$

This pattern will continue; in general, we will have

$$g_t^n(x_0) - x_0 < \frac{2n-1}{Q}\delta(t - t_0)$$

as long as $\frac{2n-1}{Q}\delta(t - t_0) < \delta$. This says that when $\frac{2n-1}{Q}\delta(t - t_0) < \delta$, i.e. when $n < \frac{Q}{2(t-t_0)} + \frac{1}{2}$, or (substituting in the value $Q = \frac{2}{\eta} + 3(t - t_0)$) when $n < \frac{1}{\eta(t-t_0)} + 2$, then

$$(1) \quad g_t^n(x_0) - x_0 < \delta.$$

Now notice that since $\underline{\rho}(t) > \eta(t - t_0)$, there is an integer $n \leq \frac{1}{\eta(t-t_0)} + 1$ such that after applying g_t n times starting at x_0 , we will have made a full circle. But equation (2) says that after applying g_t this many times we will not have moved farther than δ ; this is a contradiction. $\square$

In the following lemma, note that t plays two roles: in ϕ_t and as a dummy variable in $\frac{\partial f_t}{\partial t}(s, 0)$.

Lemma 8. *As $t \rightarrow 0$ ($t \notin \mathbb{Q}$), the conjugacies ϕ_t approach the function $x \mapsto \int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds$. Therefore, the conjugacies extend continuously to $t = 0$.*

Proof. We must show that for every x , $\phi_t(x) \rightarrow \int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds$ as $t \rightarrow 0$.

Choose t small enough so that for any $x \in S^1$,

$$\int_x^{f_t(x)} \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds \in ((1 - \epsilon)t, (1 + \epsilon)t).$$

Let n be such that $nt \leq \phi_t(x) \leq (n + 1)t$. Note that

$$\int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds \approx \int_0^{f_t^n(0)} \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds,$$

and the latter integral can be broken into integrals of the form $\int_{f_t^i(0)}^{f_t^{i+1}(0)} \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds$, each of which lies in $((1 - \epsilon)t, (1 + \epsilon)t$. In this way, we find that

$$\frac{1}{(1 + \epsilon)t} \int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds - 1 < n < \frac{1}{(1 - \epsilon)t} \int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds.$$

Now note that $\phi_t(x)$ lies between nt and $(n + 1)t$, so it differs from nt by at most t . Also, nt satisfies

$$\frac{1}{1 + \epsilon} \int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds - t < nt < \frac{1}{1 - \epsilon} \int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds.$$

When t and ϵ go to 0, the difference between nt and $\int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds$, and hence also the difference between $\phi_t(x)$ and $\int_0^x \frac{1}{\frac{\partial f_t}{\partial t}(s, 0)} ds$, goes to 0. $\square$

To finish the proof of Theorem 1, observe that there's nothing special about $t = 0$ as opposed to any other rational number. For any $t = \frac{p}{q}$ (a reduced fraction), we may consider the function f^q . By Lemma 6, this is the identity on the circle $\{t = \frac{p}{q}\}$, and nearby it will look just like f did near $t = 0$. In particular, we can look at $\frac{\partial f_t^q}{\partial t}|_{t=\frac{p}{q}}$; this will be strictly greater than 0, and we will get a function that the conjugacies must approach as $t \rightarrow \frac{p}{q}$, $t \notin \mathbb{Q}$. (Note that for $t \notin \mathbb{Q}$, the conjugacy ϕ_t from f_t to R_t is also the conjugacy from f_t^q to R_{qt} .)

Therefore, we have a family of conjugacies ϕ_t defined for every t which is continuous in t , as desired. $\square$

We note that the above result does not hold if we merely require the rotation number to be strictly monotone in t .

Example. Define a family f_t parametrized by $t \in S^1$ by

$$f_t = \begin{cases} id, & t = 0, \\ \phi_t^{-1} \circ R_{\underline{\rho}(t)} \circ \phi_t, & t \neq 0, \end{cases}$$

where $\underline{\rho}: S^1 \rightarrow S^1$ is 1-1, sends 0 to 0, and is “exponentially flat” about 0 (i.e. there exists $\epsilon > 0$ such that for all $|t| < \epsilon$, $|\underline{\rho}(t)| < e^{-\frac{1}{|t|}}$), and ϕ_t is defined to be

$$\phi_t(x) = x + \frac{1}{4\pi} \sin\left(\frac{1}{t}\right) \sin(2\pi x)$$

for $t \neq 0$.

Due to the exponential flatness of $\underline{\rho}$, the family f_t is C^∞ . The given conjugacies do not vary continuously as $t \rightarrow 0$, so it looks as if f may not be topologically conjugate to the map $(x, t) \mapsto (R_{\underline{\rho}(t)}(x), t)$, which itself is conjugate to the map $(\frac{1}{0} \frac{1}{1})$ of the torus. Let us see why this is true: f is not conjugate to $(\frac{1}{0} \frac{1}{1})$.

Let us denote by C_t the circle $\{(x, t) : x \in S^1\}$; we will call these “horizontal circles”. Suppose, by way of contradiction, that ψ is a conjugacy such that $f = \psi^{-1} \circ (\frac{1}{0} \frac{1}{1}) \circ \psi$. Then there are two possibilities: ψ sends C_t to $C_{\underline{\rho}(t)}$ or $C_{-\underline{\rho}(t)}$. The reason is the following. The torus is foliated by horizontal circles on which f acts. The circles on which the rotation number is irrational are dense in this foliation, so where they are sent determines the conjugacy. Let t be such that $\underline{\rho}(t) \notin \mathbb{Q}$, and consider where $(0, t)$ is sent by ψ . If $\psi(0, t) \in C_u$, then the whole f_t -orbit of $(0, t)$ must be sent into C_u , since $(\frac{1}{0} \frac{1}{1})$ preserves horizontal circles. Since the orbit of $(0, t)$ is dense in C_t , the image of C_t is C_u . The dynamics of f on C_t must be conjugate to the dynamics of $(\frac{1}{0} \frac{1}{1})$ on C_u . This implies that $u = \underline{\rho}(t)$ or $-\underline{\rho}(t)$.

Suppose ψ sends every C_t to $C_{\underline{\rho}(t)}$. When $\underline{\rho}(t) \notin \mathbb{Q}$, we know how ψ maps C_t to $C_{\underline{\rho}(t)}$: up to a rotation, ψ_t agrees with ϕ_t (since rotations are the only maps that commute with an irrational rotation). Define a function $k(t) = \psi_t(0)$; note that, since ψ is assumed to be continuous, k is continuous. When $\underline{\rho}(t) \notin \mathbb{Q}$, we have $\psi_t(x) = k(t) + \phi_t(x)$. In fact, even if $\underline{\rho}(t_0) \in \mathbb{Q}$ ($\underline{\rho}(t_0) \neq 0$), we can choose $t \notin \mathbb{Q} \rightarrow t_0$. As $t \rightarrow t_0$, $k(t) \rightarrow k(t_0)$ and $\phi_t \rightarrow \phi_{t_0}$, so $\psi_t \rightarrow k(t_0) + \phi_{t_0} = \psi_{t_0}$. Therefore, for all x and $t \neq 0$,

$$\psi_t(x) = k(t) + \phi_t(x).$$

But notice that

$$\psi_t(\frac{1}{4}) = k(t) + \frac{1}{4} + \frac{1}{4\pi} \sin(\frac{1}{t}).$$

Since $k(t)$ approaches a limit as $t \rightarrow 0$ and $\frac{1}{4\pi} \sin(\frac{1}{t})$ does not, $\psi_t(\frac{1}{4})$ does not approach a limit as $t \rightarrow 0$. Therefore, ψ is not continuous, a contradiction.

Now suppose that ψ sends C_t to $C_{-\underline{\rho}(t)}$. Arguing as above, for some continuous function $k(t)$ we will have $\psi_t(x) = k(t) - \phi_t(x)$ for all x and $t \neq 0$. But as above, if we plug in $x = \frac{1}{4}$ and let $t \rightarrow 0$, $\psi_t(\frac{1}{4})$ does not approach a limit, contradicting the assumption that ψ is continuous.

Therefore, we have a C^∞ family of circle diffeomorphisms parametrized by $t \in S^1$ with rotation number a strictly monotone increasing injective function $S^1 \rightarrow S^1$, conjugate at each t to a rotation by $\underline{\rho}(t)$ but not globally conjugate to $(\frac{1}{0} \frac{1}{1})$.

3. PROOF OF THEOREM 2

Before we prove Theorem 2, we remark that the subject of the derivative of the rotation number in 1-parameter families has a long and rich history. In [4], Herman essentially showed that the function $\rho: \text{Diff}_+^1(S^1) \rightarrow \mathbb{R}/\mathbb{Z}$ is (Fréchet) differentiable at irrational rotations. The derivative is the map

$$\phi \in C^1(S^1) \mapsto \int_{S^1} \phi(x) dx \in \mathbb{R}.$$

He used this to derive a result originally due to Brunovský [1] that if f_t is a family of homeomorphisms of the circle varying in a C^1 way, and if f_0 is an irrational rotation, then $\underline{\rho}(t)$ is differentiable at $t = 0$, with derivative $\int_{S^1} \frac{\partial f_t}{\partial t}(x, 0) dx$. As

a corollary, he showed that if f_0 is conjugate to an irrational rotation, say $f_0 = \phi^{-1} \circ R_\alpha \circ \phi$, then $\underline{\rho}(t)$ is still differentiable at $t = 0$, with derivative

$$\underline{\rho}'(0) = \int_{S^1} \phi'(f_0(\phi^{-1}(x))) \cdot \frac{\partial f_t}{\partial t}(\phi^{-1}(x), 0) dx$$

(his notation was slightly different).

Recently, Matsumoto [6] showed the following: Let f be a real-analytic diffeomorphism of the circle and let $f_t = R_t \circ f$. Suppose $\underline{\rho}(t) = \frac{p}{q}$ and t is an endpoint of a nontrivial interval on which $\underline{\rho}$ is constantly equal to $\frac{p}{q}$ (recall that for a generic $f \in \text{Diff}_+^\omega(S^1)$, $\underline{\rho}^{-1}(\frac{p}{q})$ will be a nontrivial interval for all $\frac{p}{q} \in \mathbb{Q}$). Then

$$\limsup_{t' \rightarrow t} \frac{\underline{\rho}(t') - \underline{\rho}(t)}{t' - t} = \infty.$$

The hypotheses of Theorem 2 rule out this kind of behavior.

Proof of Theorem 2. We begin by supposing that $\underline{\rho}(t_0) = 0$; the general case of $\underline{\rho}(t_0) = \frac{p}{q}$ is not much harder. If $\underline{\rho}'(t_0) = 0$, then there is nothing to show. Otherwise, a priori either $\underline{\rho}'(t_0)$ exists and is nonzero or it does not exist. We must show that only the first possibility can happen and the value of the derivative is given by the formula in the theorem.

By Lemma 7, $\frac{\partial f_t}{\partial t}(x, t_0)$ is greater than 0 for all x . We want to show that

$$\lim_{t \rightarrow t_0} \frac{\underline{\rho}(t)}{t - t_0} = \frac{1}{\int_{S^1} \frac{1}{\frac{\partial f_t}{\partial t}(x, t_0)} dx}.$$

We demonstrate this limit for t approaching t_0 from above, with the case of t approaching t_0 from below being similar.

We want to define a “rotation time” T_t , meaning basically how many times we have to apply f_t before we make a full circle. Let $\tilde{f}_t$ be the lift of f_t with translation number $\tau(\tilde{f}_t) \in (0, 1)$. Define T_t to be the smallest integer such that $\tilde{f}_t^{T_t}(0) \geq 1$.

To show that

$$\lim_{t \downarrow t_0} \frac{\underline{\rho}(t)}{t - t_0} = \frac{1}{\int_{S^1} \frac{1}{\frac{\partial f_t}{\partial t}(x, t_0)} dx},$$

it is equivalent to show that

$$\lim_{t \downarrow t_0} \frac{t - t_0}{\underline{\rho}(t)} = \int_{S^1} \frac{1}{\frac{\partial f_t}{\partial t}(x, t_0)} dx.$$

Notice that $T_t \in [\frac{1}{\underline{\rho}(t)}, \frac{1}{\underline{\rho}(t)} + 1]$, so

$$\lim_{t \downarrow t_0} \frac{t - t_0}{\underline{\rho}(t)} = \lim_{t \downarrow t_0} (t - t_0) T_t$$

(provided either of these limits exists), and it suffices to show that

$$(2) \quad \lim_{t \downarrow t_0} (t - t_0) T_t = \int_{S^1} \frac{1}{\frac{\partial f_t}{\partial t}(x, t_0)} dx.$$

We will use the notation $\frac{\partial f_t}{\partial t}|_{t=t_0}(x)$ for $\frac{\partial f_t}{\partial t}(x, t_0)$ to emphasize that x is being varied. Let us consider Riemann sums for the integral $\int_{S^1} \frac{1}{\frac{\partial f_t}{\partial t}|_{t=t_0}(x)} dx$. Specifically, let

$$S_N = \sum_{n=0}^{N-1} \frac{1}{\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N})} \cdot \frac{1}{N}.$$

Since $\frac{\partial f_t}{\partial t}|_{t=t_0}$ is strictly greater than 0 and continuous, $\frac{1}{\frac{\partial f_t}{\partial t}|_{t=t_0}}$ is continuous, so $S_N \rightarrow \int_{S^1} \frac{1}{\frac{\partial f_t}{\partial t}|_{t=t_0}(x)} dx$ as $N \rightarrow \infty$. It suffices to show that for sufficiently large N and t close enough to t_0 , $(t - t_0)T_t$ is within ϵ of S_N , since (if N is large enough) S_N will be within ϵ of $\int_{S^1} \frac{1}{\frac{\partial f_t}{\partial t}|_{t=t_0}(x)} dx$.

For any $\eta > 0$, for sufficiently large N , if we divide the circle into N equal intervals $I_0 = [0, \frac{1}{N})$, $\dots$, $I_{N-1} = [\frac{N-1}{N}, 1)$, then for any $x \in I_n$,

$$\frac{\partial f_t}{\partial t}|_{t=t_0}(x) \in \left(\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N}) - \eta, \frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N}) + \eta \right).$$

Furthermore, for any $\theta > 0$ we can choose t close enough to t_0 so that for all x ,

$$f_t(x) - x \in \left(\left(\frac{\partial f_t}{\partial t}|_{t=t_0}(x) - \theta \right)(t - t_0), \left(\frac{\partial f_t}{\partial t}|_{t=t_0}(x) + \theta \right)(t - t_0) \right).$$

Let us consider the orbit of 0 under f_t . We would like estimates on the number of iterates contained in each I_n ; let us call this $(T_t)_n$. We have

$$(T_t)_n \leq \frac{1}{\left(\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N}) - \eta - \theta \right)(t - t_0)N}.$$

Further, for any $\iota > 0$ we can choose t small enough so that

$$\frac{1 - \iota}{\left(\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N}) + \eta + \theta \right)(t - t_0)N} \leq (T_t)_n.$$

The number ι comes in because we have to consider how far to the right of $\frac{n}{N}$ the first iterate in I_n is. Multiplying everything by $t - t_0$, we get

$$\frac{1 - \iota}{\left(\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N}) + \eta + \theta \right)N} \leq (t - t_0)(T_t)_n \leq \frac{1}{\left(\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N}) - \eta - \theta \right)N}.$$

For any $\kappa > 0$, we can therefore say that

$$(t - t_0)(T_t)_n \in \left((1 - \kappa) \frac{1}{\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N})} \cdot \frac{1}{N}, (1 + \kappa) \frac{1}{\frac{\partial f_t}{\partial t}|_{t=t_0}(\frac{n}{N})} \cdot \frac{1}{N} \right),$$

provided we made η, θ , and ι small enough. This says that

$$(t - t_0)T_t = (t - t_0)(T_t)_0 + \dots + (t - t_0)(T_t)_{N-1} \in ((1 - \kappa)S_N, (1 + \kappa)S_N),$$

and since S_N is bounded above (independent of N), for sufficiently large N and t close enough to t_0 we have $(t - t_0)T_t \in (S_N - \epsilon, S_N + \epsilon)$.

Finally, we must address the case where $\rho(t_0) = \frac{p}{q} \neq 0$ (assume this fraction is reduced). Then we consider the family g_t , where $g_t = f_t^q$ for all t . Then g_{t_0} has rotation number 0. The above reasoning applied to the family g_t says that

$$\frac{d}{dt}\bigg|_{t=t_0} \rho(g_t) = \frac{1}{\int_{S^1} \frac{1}{\frac{\partial g_t}{\partial t}(x, t_0)} dx}.$$

But $\frac{\partial g_t}{\partial t}(x, t_0) = \frac{\partial f_t^q}{\partial t}(x, t_0)$ and $\frac{d}{dt}\big|_{t=t_0} \rho(g_t) = q \cdot \frac{d}{dt}\big|_{t=t_0} \rho(f_t)$, yielding the desired formula. $\square$

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