TUNNEL NUMBER ONE KNOTS, m -SMALL KNOTS AND THE MORIMOTO CONJECTURE

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ABSTRACT. In the present paper, we show that the Morimoto Conjecture on the super additivity of the tunnel numbers of knots in S^3 is true for knots K_1, K_2 in S^3 in which each K_i is either a tunnel number one or m -small, $i = 1, 2$. This extends two known results by Morimoto.

1. INTRODUCTION

For a knot K in S^3 , let $N(K)$ be a regular neighborhood of K in S^3 and $E(K) = \overline{S^3 - N(K)}$, the complement of K . An unknotting tunnel system for K is a collection $\Gamma = \{\gamma_1, \dots, \gamma_t\}$ of pairwise disjoint simple arcs properly embedded in $E(K)$ such that $\overline{E(K) - N(\Gamma)}$ is a handlebody, where $N(\Gamma)$ is a regular neighborhood of Γ in $E(K)$. Let $t(K) = \min\{|\Gamma| : \Gamma \text{ is an unknotting tunnel system for } K\}$, and call $t(K)$ the tunnel number of K . Clearly, an unknotting tunnel system $\Gamma = \{\gamma_1, \dots, \gamma_t\}$ for K gives rise to a Heegaard splitting $V \cup_S W$ for $E(K)$ with $g(S) = t + 1$, where V is the closure of a regular neighborhood of $(\partial E(K)) \cup \gamma_1 \cup \dots \cup \gamma_t$ in $E(K)$, $W = \overline{E(K) - V}$, and vice versa. Thus, the Heegaard genus $g(E(K))$ of $E(K)$ is $t(K) + 1$.

Given two knots $K_1, K_2 \subset S^3$, it is a natural question to ask: How does the sum of $t(K_1)$ and $t(K_2)$ compare with the tunnel number $t(K_1 \# K_2)$ of the connected sum $K_1 \# K_2$ of K_1 and K_2 ? It is not hard to check that $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$ always holds. Hence there are three possibilities for $K_1 \# K_2$: (1) Super additive: $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$; (2) Additive: $t(K_1 \# K_2) = t(K_1) + t(K_2)$; (3) Sub-additive: $t(K_1 \# K_2) \leq t(K_1) + t(K_2) - 1$.

Morimoto ([12], [13]) found infinitely many pairs of knots $K_1, K_2 \subset S^3$ such that $t(K_1 \# K_2) \leq t(K_1) + t(K_2) - 1$, and Kobayashi ([2]) generalized the results by showing that for any positive integer m , there exist knots $K_1, K_2 \subset S^3$ so that $t(K_1 \# K_2) \leq t(K_1) + t(K_2) - m$. On the other hand, it is well known that there exist infinitely many examples of $K_1, K_2 \subset S^3$ with $t(K_1 \# K_2) = t(K_1) + t(K_2)$. Moreover, Moriah-Rubinstein ([9]) and Morimoto-Sakuma-Yokota ([15]) showed that

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there exist infinitely many pairs of knots $K_1, K_2 \subset S^3$ satisfying $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$, so the bound is best possible. Refer to the survey paper [8] for more details. It is an interesting question to ask as to what kinds of knots have the super additivity of tunnel numbers.

A Heegaard splitting $V \cup_S W$ of $E(K)$ is *primitive* if there exists a spanning annulus A in V with $\partial A = \alpha \cup \gamma$, $\alpha \subset \partial_- V = \partial E(K)$ is a meridian of K and $\gamma \subset S$ meets an essential disk of W in one point. A knot K in S^3 is μ -*primitive* if $E(K)$ has a minimal primitive Heegaard splitting.

The results mentioned above lead to the following conjecture by Morimoto ([12] or [8]):

Conjecture 1.1. *Let K_1, K_2 be knots in S^3 . Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if neither K_1 nor K_2 is μ -primitive.*

Conjecture 1.1 is known to be true for tunnel number one knots in S^3 by Morimoto ([11]):

Theorem 1.2. *Let K_1, K_2 be two tunnel number one knots in S^3 . Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if neither K_1 nor K_2 is μ -primitive.*

A surface F properly embedded in $E(K)$ is *meridional* if $\partial F \neq \emptyset$ and each component of ∂F is a meridian of K . A knot K is *meridionally small* or *m-small* if there is no meridional essential surface in $E(K)$. A knot K is *small* if $E(K)$ contains no closed essential surface.

Morimoto later ([12]) showed that Conjecture 1.1 is true for two prime knots in S^3 which are *m-small*:

Theorem 1.3. *Let K_1, K_2 be two prime m-small knots in S^3 . Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if neither K_1 nor K_2 is μ -primitive.*

Kobayashi-Rieck ([3]) further generalized Theorem 1.2 to *m-small* knots in general 3-manifolds. However, Kobayashi-Rieck ([4, 5]) recently showed that in S^3 there exist counterexamples to the above conjecture. Note that their counterexamples are non-prime knots, and tunnel number one knots in S^3 are prime. Moriah ([8]) proposed that Conjecture 1.1 be changed to:

Conjecture 1.4 (Morimoto Conjecture). *Let K_1, K_2 be two prime knots in S^3 . Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if neither K_1 nor K_2 is μ -primitive.*

In the present paper, we show that Conjecture 1.4 is true for knots K_1, K_2 in which each K_i is either tunnel number one or *m-small*, $i = 1, 2$:

Theorem 1.5. *Let K_1, K_2 be two knots in S^3 , each of which is either tunnel number one or m-small. Then $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ if and only if neither K_1 nor K_2 is μ -primitive.*

Clearly, Theorem 1.2 and Theorem 1.3 are direct consequences of Theorem 1.5. In fact, Theorem 1.5 can be obtained from the following theorem.

Theorem 1.6. *Let K_1, K_2 be two knots in S^3 , each of which is either tunnel number one or m-small. Then $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$, with equality if and only if either K_1 or K_2 is μ -primitive.*

The paper is organized as follows. In section 2, we first give some necessary preliminaries and review some known facts; then we prove a lemma which will be used in our proof of the main result. In section 3, we prove Theorem 1.6.

2. PRELIMINARIES

Let F be either a properly embedded connected surface in a 3-manifold M or a sub-surface of ∂M . We say F is *compressible* in M if there is an essential curve in F which bounds a disk in M or if F is a 2-sphere which bounds a 3-ball in M . Otherwise, F is *incompressible* in M . F is *bicompressible* in M if F is separating and F is compressible in the two complementary components. M is said to be *∂ -irreducible* if ∂M is incompressible. F is an *essential* surface in M if F is an incompressible surface in M and not parallel to a sub-surface of ∂M .

A *compression body* is a 3-manifold V obtained from a connected closed orientable surface S by attaching some 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components. We denote $S \times \{1\}$ by $\partial_+ V$ and $\partial V - \partial_+ V$ by $\partial_- V$. An essential disk for V is a compressing disk of $\partial_+ V$ in V . An incompressible annulus properly embedded in a compression body V is called *spanning* if one of its two boundary components is in $\partial_- V$ and the other is in $\partial_+ V$.

A *Heegaard splitting* of a 3-manifold M is a decomposition $V \cup_S W$ of M in which V and W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. S is called a *Heegaard surface* in M . The genus $g(S)$ of S is called the *genus* of the splitting $V \cup_S W$. We use $g(M)$ to denote the *Heegaard genus* of M , which is equal to the minimal genus of all Heegaard splittings of M . A Heegaard splitting $V \cup_S W$ for M is *minimal* if $g(S) = g(M)$. A Heegaard splitting $V \cup_S W$ is *reducible* (*weakly reducible*, or *stabilized*, respectively) if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ such that $\partial D_1 = \partial D_2$ ($\partial D_1 \cap \partial D_2 = \emptyset$ or $|\partial D_1 \cap \partial D_2| = 1$, respectively). Otherwise, $V \cup_S W$ is *irreducible* (*strongly irreducible*, *unstabilized*, respectively). It is a well-known theorem (called Haken's Lemma; see [1]) that any Heegaard splitting of a reducible 3-manifold is reducible.

It was shown by Scharlemann and Thompson [18] that any irreducible and ∂ -irreducible Heegaard splitting $M = V \cup_S W$ can be broken up into a series of strongly irreducible Heegaard splittings by rearranging the order of adding the 1- and 2-handles as

$$M = V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$

such that each $V_i \cup_{S_i} W_i$ is a strongly irreducible Heegaard splitting with $\partial_- W_i = F_i = \partial_- V_{i+1}$, $1 \leq i \leq m-1$, $\partial_- V_1 = \partial_- V$, $\partial_- W_m = \partial_- W$, each component of F_i is a closed incompressible surface of positive genus, only one component of $M_i = V_i \cup_{S_i} W_i$ is not a product, and none of the compression body V_i, W_{i-1} , $2 \leq i \leq m$, is trivial. Such a rearrangement of handles is called an *untelescoping* of the Heegaard splitting $V \cup_S W$. It is easy to see that $\chi(S) \leq \chi(S_i) - 2 \leq \chi(F_i) - 4$ for each i . There is a natural way to get a Heegaard splitting for M from an untelescoping which is called an *amalgamation* of the generalized Heegaard splitting ([16]).

Lemma 2.1 ([10]). *Let V be a non-trivial compression body and $\mathcal{A}$ be a collection of essential annuli properly embedded in V . Then there is an essential disk D in V with $D \cap \mathcal{A} = \emptyset$.*

The following lemma is a well-known fact (see [17]).

Lemma 2.2. *Let F be an incompressible surface properly embedded in a compression body V . Then $\overline{V - F}$ consists of compression bodies.*

Lemma 2.3 ([20]). *Let V be a non-trivial compression body and $\mathcal{A}$ be a collection of essential annuli properly embedded in V . If U is a component of $\overline{V - \mathcal{A}}$ with $U \cap \partial_- V \neq \emptyset$, then $\chi(U \cap \partial_- V) \geq \chi(U \cap \partial_+ V)$.*

Lemma 2.4 ([17]). *Let F be an incompressible surface (not a 2-sphere, a 2-disk or a projective plane space) properly embedded in $M = V \cup_S W$. If the Heegaard splitting $V \cup_S W$ is strongly irreducible, then F can be isotoped so that any component of $S \cap F$ is essential in both F and S .*

Lemma 2.5 ([19]). *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting. If α is an essential simple loop in S which bounds a disk D in M such that D is transverse to S , then α bounds an essential disk in V or W .*

The next lemma plays an essential role in the proof of the main result. Suppose the knot K is non-trivial in S^3 in the following.

Lemma 2.6. *Suppose Q is a connected bicompressible meridional surface in $E(K)$, and Q cuts $E(K)$ into two compression bodies N_1 and N_2 . If Q can be compressed into Q_i in N_i such that any component of Q_i is ∂ -parallel in $E(K)$ for $i = 1, 2$, then $E(K)$ has a primitive Heegaard splitting $V \cup_S W$ with $g(S) = 1 - \frac{1}{2}\chi(Q)$.*

Proof. Since Q_1 and Q_2 are meridional surfaces in $E(K)$, there are no disk components in Q_1 and Q_2 . By assumption, each component of Q_1 and Q_2 is a ∂ -parallel meridional annulus in $E(K)$ and $|Q_1| = |Q_2|$ (where $|\cdot|$ denotes the number of its component). Suppose $\mathcal{D}_i$ is a collection of pairwise disjoint compressing disks in N_i , by which Q is compressed into Q_i and N_i is cut into N_i^* , $i = 1, 2$. Let $A_1^1, \dots, A_1^n$ be all the components of Q_i , and let B_i^j be the subsurface of $\partial E(K)$ such that B_i^j is parallel to A_i^j and $\partial B_i^j = \partial A_i^j$, where $n = |Q_1|$, $j = 1, 2, \dots, n$, $i = 1, 2$. We divide it into two cases to discuss:

Case 1 ($n \geq 2$). Suppose there exist two different components $B_1^{i_0}$ and $B_1^{j_0}$ with $B_1^{i_0} \subset B_1^{j_0}$. Set $\mathcal{A}_1 = \{A_1^j : B_1^j \subset B_1^{j_0}, 1 \leq j \leq n, j \neq j_0\}$ and $\mathcal{A}_2 = Q_1 - (\mathcal{A}_1 \cup \{A_1^{j_0}\})$. We claim that $\mathcal{A}_2 = \emptyset$. Otherwise, since Q is connected, there must exist $A_1^{i_1} \in \mathcal{A}_1$, $A_1^{i_2} \in \mathcal{A}_2$, and $D_1, D_2 \in \mathcal{D}_1$ such that the two copies (obtained by compressing Q along $\mathcal{D}_1$) of D_k lie in $A_1^{i_k}$ and $A_1^{j_0}$, respectively, $k = 1, 2$. See Figure 1. This contradicts the assumption that Q is separating in $E(K)$. Thus $\mathcal{A}_2 = \emptyset$. Let N be the component of N_1^* which contains $A_1^{j_0}$; then N is a compression body and $A_1^{j_0} \subset \partial_+ N$. Hence $E(K)$ is homeomorphic to a compression body N , a contradiction.

FIGURE 1

Next assume that no components A_1^i, A_1^j of Q_1 are nested, i.e. $B_1^i \cap B_1^j = \emptyset$. Then $N_1^* = \bigcup_{j=1}^n B_1^j \times I$, where $B_1^j \times 0 = B_1^j$, $j = 1, 2, \dots, n$. By the same reasons, no two components of Q_2 are nested and $N_2^* = \bigcup_{j=1}^n B_2^j \times I$, where $B_2^j \times 0 = B_2^j$, $j = 1, 2, \dots, n$. Set $V = N_1 \cup (\bigcup_{j=1}^n B_2^j \times [0, \frac{1}{2}])$ and $W = \overline{E(K) - V}$. Then V is obtained from $\partial E(K) \times I$ by adding 1-handles whose co-cores are $\mathcal{D}_1$ and V is a compression body. Note that $W = \overline{N_2 - \bigcup_{i=1}^r B_2^i \times I} \cong N_2$ and W is a compression body. Let $S = \partial_+ V$; then $S = \partial_+ W$. Thus, S is a Heegaard surface of $E(K)$. Since $\partial E(K) \cap N_2$ are annuli, $2 - \chi(S) = 2 - \chi(\partial E(K) \cap N_2) - \chi(Q) = 2 - \chi(Q)$, i.e. $g(S) = \frac{1}{2}(2 - \chi(S)) = 1 - \frac{1}{2}\chi(Q)$. Let c be a component of ∂B_2^1 ; then $c \times I$ is a spanning annulus in V . Let α be an essential arc in $B_2^1 \times \frac{1}{2}$; then there exists an arc $\beta \subset A_2^1$ with $\beta \subset Q$ and $\partial\alpha = \partial\beta$, which is parallel to α in N_2 . Then α and β bound an essential disk E in W with $|\partial E \cap (c \times 1)| = 1$. Hence $c \times 1$ is longitude in W , which means that the Heegaard splitting $V \cup_S W$ is a primitive.

Case 2 ($n = 1$). If Q_1 is not ∂ -parallel in N_1 , then $E(K)$ is homeomorphic to the compression body N_1^* , a contradiction to the assumption. So Q_1 is ∂ -parallel in N_1 and Q_2 is ∂ -parallel in N_2 by the same reason. By the same arguments as in Case 1, there is a primitive Heegaard splitting $V \cup_S W$ with $g(S) = 1 - \frac{1}{2}\chi(Q)$. $\square$

3. THE PROOF OF THEOREM 1.6

Let K_1, K_2 be two knots in S^3 , and let $K = K_1 \# K_2$. Let A be the decomposing annulus in $E(K)$ which splits $E(K)$ into $E(K_1)$ and $E(K_2)$. Let A' be a properly embedded incompressible annulus in $E(K_1)$ with $\partial A' \subset A$. Then $\partial A'$ divides $\partial E(K_1)$ into two annuli, A_1 and A_2 . Say $A_1 \subset A$. We say that A' is *narrow* (respectively, *wide*) in $E(K_1)$ if A' is parallel to A_1 (respectively, A_2) in $E(K_1)$.

Proof of Theorem 1.6. Let $V \cup_S W$ be a minimal genus Heegaard splitting of $E(K)$ with $\partial_- V = \partial E(K)$. Obviously A is essential in $E(K)$ and $g(S) = t(K) + 1$. We divide it into two cases:

Case 1. The Heegaard splitting $V \cup_S W$ is strongly irreducible.

In this case, by Lemma 2.4, we may assume that $S \cap A$ consists of essential loops in both S and A and that $|S \cap A|$ is minimal. Let $S_1 = S \cap E(K_1)$, $S_2 = S \cap E(K_2)$. Then each component of S_i is a meridional surface in $E(K_i)$, none of which is a disk. $A \cap V$ (respectively, $A \cap W$) consists of essential annuli in V (respectively, W). Let C_1 (respectively, C_2) be the manifolds obtained by cutting V (respectively, W) open along $V \cap A$ (respectively, $W \cap A$). By Lemma 2.2, each component of C_1 and C_2 is a compression body.

By Lemma 2.1, S_1 or S_2 is compressible in both V and W . Since $V \cup_S W$ is strongly irreducible, only one component of S_1 or S_2 , say P , is compressible in both V and W , and all the other components are incompressible in both V and W . We may assume that $P \subset S_1$. For $i = 1, 2$, let U_i be the component of C_i which contains P ; then P is compressible in both U_1 and U_2 . By maximally compressing P in U_i , we obtain the surface, denoted by P_i , each component of which is a meridional surface (not a disk) in $E(K_1)$. Since $V \cup_S W$ is strongly irreducible, every component of P_i is incompressible in $E(K_1)$ by Lemma 2.5. Then the following two claims are true:

Claim 1. $\chi(S_1) \leq -2t(K_1)$. Moreover, if $\chi(S_1) = -2t(K_1)$, then the knot K_1 is μ -primitive.

If $\chi(P) \geq -1$, then P is either a disk, an annulus, a twice-punctured disk, or a once-punctured torus, which implies that a component of ∂P bounds a disk in $E(K_1)$ and A is compressible in $E(K_1)$, a contradiction. So $\chi(P) \leq -2$. If one component of S_1 is essential in $E(K_1)$, then $t(K_1) = 1$ and $\chi(S_1) < \chi(P) \leq -2 = -2t(K_1)$; Claim 1 is true. If one component of P_1 or P_2 is essential in $E(K_1)$, then by assumption $t(K_1) = 1$ and $\chi(S_1) \leq \chi(P) < -2 = -2t(K_1)$; Claim 1 is true again. Thus we may assume that all incompressible components of S_1 , P_1 and P_2 are boundary-parallel annuli in $E(K_1)$. It is easy to see that $U_1 \cup_P U_2 \cong E(K_1)$. Then by Lemma 2.6 $E(K_1)$ has a primitive Heegaard splitting $V' \cup_{S'} W'$ with $g(S') = 1 - \frac{1}{2}\chi(P)$. So $1+t(K_1) \leq g(S') = 1 - \frac{1}{2}\chi(P) = 1 - \frac{1}{2}\chi(S_1)$ and $\chi(S_1) \leq -2t(K_1)$. Moreover, if $\chi(S_1) = -2t(K_1)$, then $V' \cup_{S'} W'$ is a minimal Heegaard splitting of $E(K_1)$ and K_1 is μ -primitive. Hence Claim 1 is true.

Claim 2. $\chi(S_2) \leq -2t(K_2)$.

Assume that every component of S_2 is a boundary-parallel annulus in $E(K_2)$. Note that the manifolds obtained by cutting $E(K_2)$ open along S_2 consist of compression bodies. Precisely, they are solid tori, since each positive boundary is a torus and the negative boundary is empty. Therefore, $E(K_2)$ is a solid torus and K_2 is a trivial knot, a contradiction. So at least one component of S_2 is essential in $E(K_2)$, the essential component is not an annulus and $t(K_2) = 1$ by assumption. Since any closed surface in S^3 is separating, $\chi(S_2)$ is even and $\chi(S_2) \leq -2 \leq -2t(K_2)$. Hence Claim 2 is true.

By Claim 1 and Claim 2, $2t(K_1 \# K_2) = -\chi(S) = -\chi(S_1) - \chi(S_2) \geq 2t(K_1) + 2t(K_2)$, i.e. $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$, and if $t(K_1 \# K_2) = t(K_1) + t(K_2)$, then $\chi(S_1) = -2t(K_1)$ and K_1 is μ -primitive. We complete the proof of Case 1.

Case 2. The Heegaard splitting $V \cup_S W$ is weakly reducible.

Since $E(K)$ is irreducible, by Haken's lemma, $V \cup_S W$ is irreducible. Then by Scharlemann-Thompson's theorem in [18], there exists an untelescoping

$$(V'_1 \cup_{S'_1} W'_1) \cup_{F'_1} (V'_2 \cup_{S'_2} W'_2) \cup_{F'_2} \cdots \cup_{F'_{m-1}} (V'_m \cup_{S'_m} W'_m)$$

of $V \cup_S W$ such that each $V'_i \cup_{S'_i} W'_i$ ($1 \leq i \leq m$) is a strongly irreducible Heegaard splitting and each F'_i ($1 \leq i \leq m-1$) is incompressible in $E(K)$. Let $\mathcal{F} = \bigcup_{i=1}^{m-1} F'_i$ and $\mathcal{S} = \bigcup_{i=1}^m S'_i$. Since both $\mathcal{F}$ and A are incompressible in $E(K)$, we may assume that $\mathcal{F} \cap A$ consists of essential loops in both $\mathcal{F}$ and A . Since each $V'_i \cup_{S'_i} W'_i$ is strongly irreducible, by Lemma 2.4, we may further assume $\mathcal{S} \cap A$ consists of essential loops in both $\mathcal{S}$ and A , and $|A \cap (\mathcal{F} \cup \mathcal{S})|$ is minimal. Let $V'_j \cup_{S'_j} W'_j$ be the innermost component among all $V'_i \cup_{S'_i} W'_i$ with $S'_i \cap A \neq \emptyset$, which means that S'_j bounds a 3-manifold M' with $\partial E(K) \cap M' = \emptyset$ and $(\mathcal{S} \cap M') \cap A = S'_j \cap A \neq \emptyset$. Set $S_i^0 = S'_j \cap E(K_i)$ for $i = 1, 2$. Then by Lemma 2.3, for $i = 1, 2$, the following inequality holds:

$$\begin{aligned} & \chi(\mathcal{S} \cap E(K_i)) - \chi(\mathcal{F} \cap E(K_i)) \\ (*) \quad &= \sum_{k=1}^{j-1} (\chi(S'_k \cap E(K_i)) - \chi(F'_k \cap E(K_i))) \\ & \quad + \sum_{k=j}^{m-1} (\chi(S'_{k+1} \cap E(K_i)) - \chi(F'_k \cap E(K_i))) + \chi(S'_j \cap E(K_i)) \\ & \leq \chi(S'_j \cap E(K_i)). \end{aligned}$$

By Lemma 2.1, S_1^0 or S_2^0 is compressible in both V_j' and W_j' . Since $V_j' \cup_{S_j'} W_j'$ is strongly irreducible, only one component, say P^0 , of S_1^0 or S_2^0 is compressible in both V_j' and W_j' , and all the other components are incompressible in both V_j' and W_j' . We may assume that $P^0 \subset S_1^0$. By Lemma 2.2, any component of $\overline{E(K)} - (\mathcal{S} \cup \mathcal{F} \cup A)$ is a compression body. Then we have the following two claims:

Claim 3. $\chi(\mathcal{S} \cap E(K_2)) - \chi(\mathcal{F} \cap E(K_2)) \leq -2t(K_2)$.

If one component of S_2^0 is essential in $E(K_2)$, then by assumption $t(K_2) = 1$. Since any closed surface in S^3 is separating, $\chi(S_2^0)$ is even, and if there is a component of S_2^0 which is not an annulus, then by the inequality (*), $\chi(\mathcal{S} \cap E(K_2)) - \chi(\mathcal{F} \cap E(K_2)) \leq \chi(S_j' \cap E(K_2)) = \chi(S_2^0) \leq -2 = -2t(K_2)$. If any component of S_2^0 is a boundary-parallel annulus in $E(K_2)$, S_2^0 must be nested wide annuli in $E(K_2)$ since $|A \cap (\mathcal{F} \cup \mathcal{S})|$ is minimal. Let B be the innermost wide annulus among S_2^0 ; then ∂B bounds an annulus, denoted by A^* , in A . Since $V_j' \cup_{S_j'} W_j'$ and B are innermost, B and A^* bound a compression body, denoted by C , of $\overline{E(K)} - (\mathcal{S} \cup \mathcal{F} \cup A)$ with $\partial_+ C = B \cup A^*$. If $\partial_- C = \emptyset$, then $E(K_2) \cong C$ is a solid torus and K_2 is a trivial knot, a contradiction. Hence $\partial_- C \neq \emptyset$, C is a torus $\times I$ and $\partial_- C$ is boundary parallel in $E(K_2)$. Note that $\partial_- C \subset \mathcal{F}$, $\partial_- C$ bounds a sub-untelescoping of $(V_1' \cup_{S_1'} W_1') \cup_{F_1'} (V_2' \cup_{S_2'} W_2') \cup_{F_2'} \cdots \cup_{F_{m-1}'} (V_m' \cup_{S_m'} W_m')$, denoted by $(V_{k+1}^* \cup_{S_{k+1}^*} W_{k+1}^*) \cup_{F_{k+1}^*} (V_{k+2}^* \cup_{S_{k+2}^*} W_{k+2}^*) \cup_{F_{k+2}^*} \cdots \cup_{F_{m-1}^*} (V_m^* \cup_{S_m^*} W_m^*)$, which can be amalgamated to a Heegaard splitting $V^* \cup_{S^*} W^*$ of $E(K_2)$, where $V_i^* \cup_{S_i^*} W_i^*$ is a component of $V_i' \cup_{S_i'} W_i'$ for $k+1 \leq i \leq m$. After the amalgamation, we have $\chi(\mathcal{S} \cap E(K_2)) - \chi(\mathcal{F} \cap E(K_2)) = \chi(S^*) \leq -2t(K_2)$. Hence Claim 3 is true.

Claim 4. $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) \leq -2t(K_1)$. Moreover, if $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) = -2t(K_1)$, then K_1 is μ -primitive.

For $i = 1, 2$, let U_i^0 be the component of $\overline{E(K)} - (\mathcal{S} \cup \mathcal{F} \cup A)$ which contains P^0 . Then P^0 is compressible in both U_1^0 and U_2^0 , and by maximally compressing P^0 in U_i^0 , we obtain a surface, denoted by P_i^0 , each component of which is a meridional surface (not a disk) in $E(K_1)$. Since $V_j' \cup_{S_j'} W_j'$ is strongly irreducible, any component of P_1^0 and P_2^0 is incompressible in $E(K_1)$ by Lemma 2.5. If one component of S_1^0 , P_1^0 or P_2^0 , say P^* , is essential in $E(K_1)$, then $t(K_1) = 1$. By the inequality (*), $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) \leq \chi(S_j' \cap E(K_1)) \leq \chi(S_1^0) \leq \chi(P^*) - 2 < -2 = -2t(K_1)$, and Claim 4 is true. So we may assume that any incompressible component of S_1^0 is a boundary-parallel annulus in $E(K_1)$, and any component with boundary of both P_1^0 and P_2^0 is a boundary-parallel annulus in $E(K_1)$.

If both P_1^0 and P_2^0 have no closed surface component, then $U_1^0 \cup_{P^0} U_2^0 \cong E(K_1)$ and P^0 satisfies the conditions of Lemma 2.6 in $E(K_1)$. By Lemma 2.6, $E(K_1)$ has a primitive Heegaard splitting $V^0 \cup_{S^0} W^0$ with $g(S^0) = 1 - \frac{1}{2}\chi(P^0)$. So $1 + t(K_1) \leq g(S^0) = 1 - \frac{1}{2}\chi(P^0) = 1 - \frac{1}{2}\chi(S_1^0)$ and $\chi(S_1^0) \leq -2t(K_1)$. By the inequality (*), $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) \leq \chi(S_j' \cap E(K_1)) \leq \chi(S_1^0) \leq -2t(K_1)$, and if $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) = -2t(K_1)$, then $V^0 \cup_{S^0} W^0$ is a minimal Heegaard splitting of $E(K_1)$ and K_1 is μ -primitive, and Claim 4 is true.

If P_1^0 or P_2^0 has closed surface components, then those closed components must be parallel to $\mathcal{F}$ since a closed incompressible surface in a compression body is ∂ -parallel in the compression body. Let $\mathcal{F}_1$ be the components of $\mathcal{F}$ which are contained in $E(K_1)$. $\mathcal{F}_1$ bounds a sub-untelescoping of the foregoing untelescoping. By the amalgamation of the sub-untelescoping and $V_j' \cup_{S_j'} W_j'$ along $\mathcal{F}_1$, denote the resulting Heegaard splitting by $V_j'' \cup_{S_j''} W_j''$. By the amalgamation, we can assume that $S_j' \cap E(K_2) = S_j'' \cap E(K_2)$, the incompressible components of $S_j' \cap E(K_1)$ are the same as the incompressible components $S_j'' \cap E(K_1)$ and only one component, say P^* , of $S_j'' \cap E(K_1)$ is compressible in both V_j'' and W_j'' .

For $i = 1, 2$, set $S_i^* = S_j'' \cap E(K_i)$. Denote P_1^* (P_2^* , respectively) as the resulting components by maximally compressing P^* in V_j'' (W_j'' , respectively). Then we may assume that any incompressible component of $S_j'' \cap E(K_1)$ and any component of both P_1^* and P_2^* are boundary-parallel annuli in $E(K_1)$. Hence P^* satisfies the conditions of Lemma 2.6 in $E(K_1)$. Then by Lemma 2.6, $E(K_1)$ has a primitive Heegaard splitting $V^0 \cup_{S^0} W^0$ with $g(S^0) = 1 - \frac{1}{2}\chi(P^*)$. So $1 + t(K_1) \leq g(S^0) = 1 - \frac{1}{2}\chi(P^*) = 1 - \frac{1}{2}\chi(S_1^*)$ and $\chi(S_1^*) \leq -2t(K_1)$. By the inequality (*), $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) \leq \chi(S_j'' \cap E(K_1)) \leq \chi(S_1^*) \leq -2t(K_1)$, and if $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) = -2t(K_1)$, then $V^0 \cup_{S^0} W^0$ is a minimal Heegaard splitting of $E(K_1)$ and K_1 is μ -primitive. Hence Claim 4 is true.

By Claim 3 and Claim 4, $-2t(K_1 \# K_2) = \chi(\mathcal{S}) = \chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) + \chi(\mathcal{S} \cap E(K_2)) - \chi(\mathcal{F} \cap E(K_2)) \leq -2t(K_1) - 2t(K_2)$, i.e. $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$, and if $t(K_1 \# K_2) = t(K_1) + t(K_2)$, then $\chi(\mathcal{S} \cap E(K_1)) - \chi(\mathcal{F} \cap E(K_1)) = -2t(K_1)$ and K_1 is μ -primitive. This completes the proof of Case 2 and the proof of Theorem 1.6. $\square$

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