A REMARK ON INVARIANT SUBSPACES OF POSITIVE OPERATORS

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ABSTRACT. If S , T , R , and K are non-zero positive operators on a Banach lattice such that $S \leftrightarrow T \leftrightarrow R \leq K$, where “ $\leftrightarrow$ ” stands for the commutation relation, T is non-scalar, and K is compact, then S has an invariant subspace.

Throughout this note, X is a (real or complex) Banach lattice. For two operators S and T on X , the notation $S \leftrightarrow T$ means that S and T commute. A (norm closed) subspace Y of X is said to be invariant under an operator T in $L(X)$ if $\{0\} \neq Y \neq X$ and $TY \subseteq Y$. We follow the notation and terminology of [AA02].

There have been many extensions of Lomonosov’s theorem [Lom73] to positive operators; see Chapter 10 of [AA02] for a review of the subject. In particular, if $T \leftrightarrow R \geq K$ for some positive non-zero operators T , R , and K with T quasinilpotent and K compact, then T has an invariant subspace (even an invariant closed ideal). The condition $T \leftrightarrow R \geq K$ can be replaced with $T \leftrightarrow R \leq K$ or, even more generally, with $T \leftrightarrow R \geq C \leq K$ for some non-zero positive operator C ; in the latter case, T is said to be *compact friendly*. There have been several more recent similar extensions of Lomonosov’s theorem to positive quasinilpotent operators: [Drn01, IM04, AT05, ÇE07, FTT08, PT09, Ges09, FV09, DK11]. In this note we do not require that T be quasinilpotent. Our result was motivated by Theorem 3.5 of [ÇM11], where quasinilpotence is not required either.

Theorem 1. *Suppose that S , T , R , and K are non-zero positive operators on a Banach lattice such that $S \leftrightarrow T \leftrightarrow R \leq K$, T is non-scalar, and K is compact. Then S has an invariant subspace.*

Proof. Suppose that S has no invariant subspaces. Let $\tilde{S} = \sum_{n=0}^{\infty} t^n S^n$, where t is a positive real such that series converges. Then $\tilde{S} \geq I$, $\tilde{S} \geq tS$, and $\tilde{S}$ commutes with T .

Claim. For every $x > 0$, the vector $\tilde{S}x$ is quasi-interior; that is, the order ideal J generated by $\tilde{S}x$ is dense in X . Indeed, $\tilde{S}x \geq x > 0$, so that $J \neq \{0\}$. Note that J is invariant under S because for every $z \in J$ we have $|z| \leq \lambda \tilde{S}x$ for some $\lambda > 0$,

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so that

$$|Sz| \leq S|z| \leq \lambda S\tilde{S}x = \lambda \sum_{n=0}^{\infty} t^n S^{n+1}x = \frac{\lambda}{t} \sum_{n=0}^{\infty} t^{n+1} S^{n+1}x \leq \frac{\lambda}{t} \tilde{S}x.$$

Since S has no invariant subspaces, J has to be dense in X . This proves the claim.

Since $R \neq 0$, there exists $x_0 > 0$ such that $Rx_0 > 0$. By the claim, $\tilde{S}Rx_0$ is quasi-interior. Since R is positive and non-zero, it cannot vanish on a quasi-interior vector; hence $R\tilde{S}Rx_0 > 0$. Iterating this step, we get $R\tilde{S}R\tilde{S}Rx_0 > 0$. It follows that $R\tilde{S}R\tilde{S}R \neq 0$. Since $\tilde{S}R \leq \tilde{S}K$ and the latter operator is compact, $R\tilde{S}R\tilde{S}R$ is compact by Aliprantis-Burkinshaw's Cube Theorem [AA02, Theorem 2.34]. Hence, T commutes with a non-zero compact operator. Therefore, T has a hyperinvariant subspace: in case of a complex Banach lattice this follows from Lomonosov's Theorem, while in the case of a real Banach lattice we use Corollary 2.4 of [Sir05]. $\square$

Remark 2. We have, actually, proved more than stated: we proved that either S has an invariant closed ideal or T commutes with a non-zero compact operator and, therefore, has a hyperinvariant subspace. We would also like to point out that the assumption that T is positive is not really needed.

To put Theorem 1 in perspective, note that under the assumptions of the theorem, the following facts are well known:

- If both X and X^* have order continuous norm, then R is compact by the Dodds-Fremlin Theorem [AA02, Theorem 2.38], so that T has a hyperinvariant subspace by Lomonosov's Theorem.
- Note that R^3 is always compact by the Cube Theorem and $T \leftrightarrow R^3$. Thus, if $R^3 \neq 0$, then it follows immediately from Lomonosov's Theorem that T has a hyperinvariant subspace. On the other hand, if $R^3 = 0$, then $\ker R$ is a non-trivial subspace invariant under T . Hence, in any case, T has an invariant subspace.
- Note that T is compact-friendly. Therefore, if T is quasinilpotent at a positive vector, then Theorem 10.55 of [AA02] guarantees that S has an invariant closed ideal. The following result is an analogue of Theorem 10.55 in our setting.

Theorem 3. *Suppose that T , R , and K are non-zero positive operators on a Banach lattice X such that $T \leftrightarrow R \leq K$, T is non-scalar, and K is compact. If (S_n) is a sequence of positive operators commuting with T , then there is a subspace invariant under T , R , and all S_n 's.*

Proof. Let $S = T + R + \sum_{n=1}^{\infty} a_n S_n$, where (a_n) is a sequence of positive reals such that the series converges. Observe that S is a positive operator commuting with T . If S has an invariant closed ideal, then this ideal remains invariant under T , R , and each S_n because these operators are dominated by S . However, if S has no invariant closed subspaces, then T has a hyperinvariant subspace by Remark 2. $\square$

Example 4. $0 \leq R \leq K$, K is compact, R is not compact, and $R^2 = 0$.

This is the case in Example 5.19 of [AB06]; it is one of the few classical examples showing that the Dodds-Fremlin Theorem may fail when X^* is not order continuous. Here is the example. Put $X = \ell_1 \oplus L_2$. Let $(e_i)_{i=1}^{\infty}$ stand for the unit vector basis

of ℓ_1 , $(r_i)_{i=1}^\infty$ stand for the sequence of the Rademacher functions in L_2 , and $r_0 = \mathbb{1}$ stand for the constant one function in L_2 . Recall that the sequence $(r_i)_{i=0}^\infty$ is an orthonormal sequence in L_2 . Note also that $r_i^+ = \frac{1}{2}(r_i + \mathbb{1})$ for all i . We define $R_0, K_0: \ell_1 \rightarrow L_2$ via $K_0 e_i = \mathbb{1}$ and $R_0 e_i = r_i^+$ for all $i \geq 1$. It is easy to see that both operators are bounded, K_0 is compact, R_0 is not compact, and $0 \leq R_0 \leq K_0$. Now put $R = \begin{bmatrix} 0 & 0 \\ R_0 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 0 \\ K_0 & 0 \end{bmatrix}$. Then R and K are two operators on X with $0 \leq R \leq K$, K is compact, R is not compact, and $R^2 = 0$.

Example 5. With R and K as in Example 4, we will construct T such that T commutes with R but not with K . Put $T = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$, where $P: \ell_1 \rightarrow \ell_1$ is the left shift: $P e_i = e_{i-1}$ if $i > 1$ and $P e_1 = 0$, and $Q: L_2 \rightarrow L_2$ is defined as follows. Put $Q \mathbb{1} = \mathbb{1}$, $Q r_1 = -\mathbb{1}$, $Q r_i = r_{i-1}$ for $i > 1$ and define Q arbitrarily on the orthogonal complement of the closed span of $(r_i)_{i=0}^\infty$ in L_2 . Using the fact that $r_i^+ = \frac{1}{2}(r_i + \mathbb{1})$ we see that Q acts as a left shift on the sequence $(r_i^+)_{i=1}^\infty$. It is easy to see that T commutes with R because for every $\sum_{i=1}^\infty \alpha_i e_i$ in ℓ_1 and every $f \in L_2$ we have $TR(\sum_{i=1}^\infty \alpha_i e_i, f) = (0, \sum_{i=1}^\infty \alpha_{i+1} r_i^+) = RT(\sum_{i=1}^\infty \alpha_i e_i, f)$. However, T does not commute with K because $TK(e_1, 0) = (0, \mathbb{1})$, while $KT(e_1, 0) = (0, 0)$. Note that T is not positive.

Example 6. We construct three non-zero *positive* operators T , R , and K such that $0 \leq R \leq K$, K is compact, R is not compact, and T commutes with R but not with K . In particular, the operators K , R , and T , together with any positive operator S which commutes with T , satisfy the assumptions of Theorem 1.

We construct R and K similarly to Example 4. We again put $X = \ell_1 \oplus L_2$, but this time we consider ℓ_1 indexed by $\mathbb{N} \cup \{0\}$, so that the unit basis now starts with e_0 . Again, we define $R = \begin{bmatrix} 0 & 0 \\ R_0 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 0 \\ K_0 & 0 \end{bmatrix}$, where $R_0 e_i = r_i^+$ and $K_0 e_i = \mathbb{1}$ for all $i = 0, 1, 2, \dots$ (recall that $r_0 = \mathbb{1}$). We still have $0 \leq R \leq K$, K is compact, R is not compact, and $R^2 = 0$. Put $T = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$, where $P: \ell_1 \rightarrow \ell_1$ and $Q: L_2 \rightarrow L_2$ are defined as follows. Fix a positive real parameter α . For $f \in L_2$, put

$$(Qf)(t) = f\left(\frac{t}{2}\right) + 2\alpha \int_{\frac{1}{2}}^1 f, \quad t \in [0, 1].$$

It is easy to see that $Q \mathbb{1} = (1 + \alpha) \mathbb{1}$, $Q r_1 = (1 - \alpha) \mathbb{1}$, and $Q r_i = r_{i-1}$ for $i > 1$. It follows from $r_i^+ = \frac{1}{2}(r_i + \mathbb{1})$ that $Q r_1^+ = \mathbb{1}$ and $Q r_i^+ = r_{i-1}^+ + \frac{\alpha}{2} \mathbb{1}$ whenever $i > 1$. Now we define P so that the action of P on $(e_i)_{i=0}^\infty$ matches the action of Q on $(r_i^+)_{i=0}^\infty$, namely,

$$P e_i = \begin{cases} (1 + \alpha) e_0 & i = 0, \\ e_0 & i = 1, \\ e_{i-1} + \frac{\alpha}{2} e_0 & i > 1. \end{cases}$$

Clearly, Q and P are positive; hence so is T . It is easy to verify that T commutes with R . However, T does not commute with K as $TK(e_1, 0) = T(0, \mathbb{1}) = (0, (1 + \alpha) \mathbb{1})$, while $KT(e_1, 0) = K(e_0, 0) = (0, \mathbb{1})$.

Note that $(0, \mathbb{1})$ is an eigenvector of T ; it follows that T has a hyperinvariant subspace. Also, if $\alpha = 1$, then T commutes with the compact positive operator C defined by $C(x, f) = (0, (\int_0^1 f) \mathbb{1})$. We do not know whether T commutes with a compact operator when $\alpha \neq 1$.

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