ON LINEARLY COUPLED SCHRÖDINGER SYSTEMS

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ABSTRACT. We study the following system of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + u = f(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + v = g(v) + \lambda u, & x \in \mathbb{R}^N. \end{cases}$$

Under almost optimal assumptions on f and g , for small $\lambda > 0$, we obtain positive radial solutions and study their asymptotic behaviors as $\lambda \rightarrow 0$.

1. INTRODUCTION

We consider the following system of nonlinear Schrödinger equations:

$$(1.1) \quad \begin{cases} -\Delta u + u = f(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + v = g(v) + \lambda u, & x \in \mathbb{R}^N, \\ u(x), v(x) \rightarrow 0, & \text{as } |x| \rightarrow +\infty, \end{cases}$$

where $\lambda \geq 0$, $N \geq 3$, f and g are Carathéodory functions. Systems of this type arise in nonlinear optics (cf. [1]). It is well known that a solution $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of (1.1) is called a bound state. A bound state such that $(u, v) \neq (0, 0)$, respectively $u > 0$, $v > 0$, is called a nontrivial, respectively positive, bound state. A solution is called a ground state if $(u, v) \neq (0, 0)$ and its energy is minimal among the energy of all the nontrivial bound states of (1.1). A ground state satisfying $u > 0$ and $v > 0$ is called a positive ground state. Obviously, the bound states of (1.1) are the critical points of the functional $I_\lambda : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$(1.2) \quad \begin{aligned} I_\lambda(u, v) := & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + |v|^2) \, dx \\ & - \int_{\mathbb{R}^N} (F(u) + G(v)) \, dx - \lambda \int_{\mathbb{R}^N} uv \, dx \end{aligned}$$

if $I_\lambda \in C^1(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N), \mathbb{R})$. Here $F(u) := \int_0^u f(s) \, ds$ and $G(v) := \int_0^v g(s) \, ds$. Let $2^* = \frac{2N}{N-2}$. Assume that

(F₁) $f, g \in C(\mathbb{R}, \mathbb{R})$, and $\lim_{s \rightarrow 0} \frac{f(s)}{s} = \lim_{s \rightarrow 0} \frac{g(s)}{s} = 0$.

(F₂) There exists $p \in (1, 2^* - 1)$ such that

$$\limsup_{s \rightarrow +\infty} \frac{f(s)}{s^p} < +\infty \quad \text{and} \quad \limsup_{s \rightarrow +\infty} \frac{g(s)}{s^p} < +\infty.$$

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(F₃)

$$\sup_{s>0} \frac{2F(s)}{s^2} > 1 \quad \text{and} \quad \sup_{s>0} \frac{2G(s)}{s^2} > 1.$$

Here we are concerned with positive solutions of (1.1), and in the sequel we assume without loss of generality that

$$(1.3) \quad f(s) \equiv g(s) \equiv 0 \quad \text{for all } s \leq 0.$$

In the scalar case, assumptions (F₁)–(F₃) are called Berestycki-Lions conditions, which were introduced by Berestycki and Lions [7] to get a ground state solution for

$$(1.4) \quad -\Delta u + u = f(u), \quad u \in H^1(\mathbb{R}^N).$$

Other results about (1.4) can be seen in [5, 6, 11, 15] and the references therein. On the other hand, there have also been some results on the linearly coupled system (1.1) in the past several years. In the case of $N \leq 3$, $f(s) = g(s) = s^3$, Ambrosetti, Colorado and Ruiz [3] proved that (1.1) has multi-bump solitons for $\lambda > 0$ small enough. When $f(u)$ and $g(v)$ are replaced by $f(x, u) = (1 + c(x))|u|^{p-1}u$ and $g(x, v) = (1 + d(x))|v|^{p-1}v$ respectively, system (1.1) has been studied by Ambrosetti [2] with dimension $N = 1$ and Ambrosetti, Cerami and Ruiz [4] with dimension $N \geq 2$. Brezis and Lieb [8] consider some systems of equations

$$(1.5) \quad -\Delta u_i(x) = g^i(u(x)), \quad i = 1, \dots, n,$$

where the n functions $g^i : \mathbb{R}^n \rightarrow \mathbb{R}$ are the gradients of some function $G \in C^1(\mathbb{R}^n)$, namely $g^i(u) = \partial G(u) / \partial u_i$. Under some conditions on g^i (see (2.2), (2.3), (2.4) and (2.8) in [8]), they proved that (1.5) has a ground state solution which belongs to $H^1(\mathbb{R}^N) \cap W_{\text{loc}}^{2,q}(\mathbb{R}^N)$ for any $q < +\infty$ (see Theorem 2.2 and Theorem 2.3 in [8]). Recently, Byeon, Jeanjean and Maris [10] proved that ground state solutions of (1.5) obtained in [8] must be radial up to a translation. Note that (1.1) is a special case of (1.5), and (F₁)–(F₃) with $\lambda \in [0, 1)$ are consistent with those conditions on g^i in [8]; it follows that (1.1) produces a ground state solution $u, v \in H^1(\mathbb{R}^N) \cap W_{\text{loc}}^{2,q}(\mathbb{R}^N)$ for $\lambda \in [0, 1)$. Then some further properties of the ground states for (1.1) were studied by the authors in [12]. In particular, we showed the following result.

Theorem A (See [12, Theorem 1.3]). *Suppose $N \geq 3$ and f, g satisfy assumptions (F₁)–(F₃). Then for $\lambda \in (0, 1)$, (1.1) has a positive radially symmetric ground state (u_λ, v_λ) such that $u_\lambda, v_\lambda \in C^2(\mathbb{R}^N)$.*

Let $\lambda_n \in (0, 1)$, $n \in \mathbb{N}$, be any sequence with $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Then, passing to a subsequence, $(u_{\lambda_n}, v_{\lambda_n}) \rightarrow (\hat{u}, \hat{v})$ strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$, where one of the following conclusions holds:

(i) $\hat{u} \equiv 0$ and $\hat{v}$ is a positive radial ground state of

$$(1.6) \quad -\Delta v + v = g(v), \quad v \in H^1(\mathbb{R}^N).$$

(ii) $\hat{v} \equiv 0$ and $\hat{u}$ is a positive radial ground state of (1.4).

The main result of the current paper is the following.

Theorem 1.1. *Suppose $N \geq 3$ and f, g satisfy assumptions (F₁)–(F₃). Then there exists $\lambda_0 \in (0, 1)$ such that for any $0 < \lambda < \lambda_0$, (1.1) has a positive radially symmetric bound state $(u_\lambda, v_\lambda) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$.*

Furthermore, let $\lambda_n \in (0, \lambda_0)$, $n \in \mathbb{N}$, be any sequence with $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Then, passing to a subsequence, $(u_{\lambda_n}, v_{\lambda_n}) \rightarrow (U, V)$ strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$

as $n \rightarrow \infty$, where U is a positive radial ground state of (1.4) and V is a positive radial ground state of (1.6) respectively.

Remark 1.1. Obviously, for $\lambda > 0$ sufficiently small, bound states that are obtained in Theorem 1.1 are different from ground states obtained in Theorem A. That is, (1.1) has at least two positive bound states for $\lambda > 0$ sufficiently small.

Remark 1.2. Berestycki and Lions [7] showed that assumptions $(F_1) - (F_3)$ are almost optimal for the existence of ground states of (1.4) and (1.6). Therefore, assumptions $(F_1) - (F_3)$ in Theorem 1.1 are almost optimal.

The rest of this paper proves Theorem 1.1. Note that classical reduction approaches cannot be used because of the general nonlinearities f and g . Here we will adopt the ideas of Byeon and Jeanjean [9]. In order to construct a spike solution of the singularly perturbed nonlinear elliptic problem

$$-\varepsilon^2 \Delta u + V(x)u = f(u), \quad u \in H^1(\mathbb{R}^N),$$

for $\varepsilon > 0$ sufficiently small, Byeon and Jeanjean [9] developed a new variational approach. We will follow this variational approach to prove Theorem 1.1. However, this approach cannot be used directly, and we need some crucial modifications for our proof. Precisely, we will define a special mountain-pass value c_λ , where all paths are required to be bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ by the same constant which is independent of λ ; see Section 2.

We give some notation here. Throughout this paper, we denote the norm of $L^q(\mathbb{R}^N)$ by $|u|_q = (\int_{\mathbb{R}^N} |u|^q dx)^{\frac{1}{q}}$ and positive constants (possibly different) by C . The standard scalar product and norm of $H^1(\mathbb{R}^N)$ are given by

$$\langle u, v \rangle := \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) dx, \quad \|u\|^2 := \langle u, u \rangle.$$

Define

$$H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\}.$$

Denote $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with the norm $\|(u, v)\|^2 := \|u\|^2 + \|v\|^2$, and denote $H_r := H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. In the sequel it is understood that $\lambda \in [0, 1)$.

2. PROOF OF THEOREM 1.1

In this section, we consider the functional I_λ restricted to H_r . By Palais' Symmetric Criticality Principle, any critical points of $I_\lambda : H_r \rightarrow \mathbb{R}$ are radially symmetric bound states of (1.1). The energy functionals of (1.4) and (1.6) are given by

$$\begin{aligned} J_1(u) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx - \int_{\mathbb{R}^N} F(u) dx, \quad u \in H^1(\mathbb{R}^N), \\ J_2(v) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) dx - \int_{\mathbb{R}^N} G(v) dx, \quad v \in H^1(\mathbb{R}^N). \end{aligned}$$

Under assumptions $(F_1) - (F_3)$, Berestycki and Lions [7] proved that (1.4) (resp. (1.6)) has a positive ground state solution. They also showed that each solution U of (1.4) (resp. V of (1.6)) satisfies Pohozaev's identity

$$(2.1) \quad J_3(U) := \int_{\mathbb{R}^N} |\nabla U|^2 dx + 2^* \int_{\mathbb{R}^N} \frac{U^2}{2} - F(U) dx = 0$$

$$(2.2) \quad \left(\text{resp. } J_4(V) := \int_{\mathbb{R}^N} |\nabla V|^2 dx + 2^* \int_{\mathbb{R}^N} \frac{V^2}{2} - G(V) dx = 0 \right).$$

Later, Byeon, Jeanjean and Maris [10] proved that ground state solutions of (1.4) (resp. (1.6)) must be radial up to a translation. By (1.3), any nontrivial solutions of (1.4) (resp. (1.6)) must be positive. Define

$$\begin{aligned} S_1 &:= \{U \in H_r^1(\mathbb{R}^N) : U \text{ is a positive ground state of (1.4)}\}, \\ S_2 &:= \{V \in H_r^1(\mathbb{R}^N) : V \text{ is a positive ground state of (1.6)}\}, \\ X &:= S_1 \times S_2. \end{aligned}$$

Take fixed $U_0 \in S_1$ and $V_0 \in S_2$, and denote the least energy of (1.4) and (1.6) respectively by

$$M_1 := J_1(U_0) \quad \text{and} \quad M_2 := J_2(V_0) \quad \text{respectively.}$$

Then $M_1 > 0$ and $M_2 > 0$ (cf. [7]). Moreover, [14, Lemma 3.1] says that

$$(2.3) \quad M_1 = \inf_{\substack{u \in H^1(\mathbb{R}^N) \setminus \{0\} \\ J_3(u)=0}} J_1(u), \quad M_2 = \inf_{\substack{v \in H^1(\mathbb{R}^N) \setminus \{0\} \\ J_4(v)=0}} J_2(v).$$

Without loss of generality, we assume that

$$(2.4) \quad M_1 \leq M_2.$$

Lemma 2.1. *X is compact in H_r , and there exist constants $C_2 > C_1 > 0$ such that*

$$C_1 \leq \|U\|, \|V\| \leq C_2, \quad \forall (U, V) \in X.$$

Proof. By [9, Proposition 1], we see that S_1 and S_2 are both compact in $H_r^1(\mathbb{R}^N)$. Since $M_1 > 0$ and $M_2 > 0$, Lemma 2.1 follows trivially. $\square$

For $t, s > 0$, we define $U_{0,t}(x) := U_0(\frac{x}{t})$ and $V_{0,s}(x) := V_0(\frac{x}{s})$. Then by (2.1) we have

$$\begin{aligned} J_1(U_{0,t}) &= \int_{\mathbb{R}^N} \frac{t^{N-2}}{2} |\nabla U_0|^2 + \frac{t^N}{2} U_0^2 - t^N F(U_0) dx \\ (2.5) \quad &= \left(\frac{t^{N-2}}{2} - \frac{(N-2)t^N}{2N} \right) \int_{\mathbb{R}^N} |\nabla U_0|^2 dx. \end{aligned}$$

Note that

$$(2.6) \quad J_1(U_{0,1}) = \max_{t>0} J_1(U_{0,t}) = J_1(U_0) = M_1.$$

It is easily seen that there exists $0 < t_0 < 1 < t_1$ such that

$$(2.7) \quad J_1(U_{0,t}) \leq M_1/4 \quad \text{for } t \in (0, t_0] \cup [t_1, \infty).$$

Similarly, there exists $0 < s_0 < 1 < s_1$ such that

$$(2.8) \quad J_2(V_{0,s}) \leq M_1/4 \quad \text{for } s \in (0, s_0] \cup [s_1, \infty).$$

Define

$$\begin{aligned} \tilde{\gamma}_1(t) &:= U_{0,t} \text{ for } 0 < t \leq t_1, \quad \tilde{\gamma}_1(0) := 0; \\ \tilde{\gamma}_2(s) &:= V_{0,s} \text{ for } 0 < s \leq s_1, \quad \tilde{\gamma}_2(0) := 0; \\ \tilde{\gamma}(t, s) &:= (\tilde{\gamma}_1(t), \tilde{\gamma}_2(s)). \end{aligned}$$

Then there exists a constant $\mathcal{C} > 0$ such that

$$(2.9) \quad \max_{(t,s) \in [0,t_1] \times [0,s_1]} \|\tilde{\gamma}(t,s)\| \leq \mathcal{C}.$$

Denote $Q := [0, t_1] \times [0, s_1]$ for convenience. Recalling C_2 in Lemma 2.1, we define

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{(t,s) \in Q} I_\lambda(\gamma(t,s)), \quad d_\lambda := \max_{(t,s) \in Q} I_\lambda(\tilde{\gamma}(t,s)),$$

where

$$(2.10) \quad \begin{aligned} \Gamma := & \left\{ \gamma \in C(Q, H_r) : \max_{(t,s) \in Q} \|\gamma(t,s)\| \leq 2C_2 + \mathcal{C}, \right. \\ & \left. \gamma(t,s) = \tilde{\gamma}(t,s) \quad \text{for } (t,s) \in Q \setminus (t_0, t_1) \times (s_0, s_1) \right\}. \end{aligned}$$

Lemma 2.2. $\lim_{\lambda \rightarrow 0} c_\lambda = \lim_{\lambda \rightarrow 0} d_\lambda = c_0 = d_0 = M_1 + M_2.$

Proof. Recalling $\lambda \geq 0$ and the definition of I_λ in (1.2), we have $I_\lambda(\tilde{\gamma}(t,s)) \leq I_0(\tilde{\gamma}(t,s))$, and so

$$(2.11) \quad \begin{aligned} d_\lambda &\leq d_0 = \max_{(t,s) \in Q} I_0(\tilde{\gamma}(t,s)) = \max_{t \in [0,t_1]} J_1(\tilde{\gamma}_1(t)) + \max_{s \in [0,s_1]} J_2(\tilde{\gamma}_2(s)) \\ &= J_1(\tilde{\gamma}_1(1)) + J_2(\tilde{\gamma}_2(1)) = J_1(U_0) + J_2(V_0) = M_1 + M_2. \end{aligned}$$

Note that $\tilde{\gamma} \in \Gamma$; we have $c_\lambda \leq d_\lambda$, that is,

$$(2.12) \quad \limsup_{\lambda \rightarrow 0} c_\lambda \leq \liminf_{\lambda \rightarrow 0} d_\lambda \leq \limsup_{\lambda \rightarrow 0} d_\lambda \leq d_0, \quad c_0 \leq d_0.$$

On the other hand, for any $\gamma(t,s) = (\gamma_1(t,s), \gamma_2(t,s)) \in \Gamma$, we define $\Upsilon(\gamma) : [t_0, t_1] \times [s_0, s_1] \rightarrow \mathbb{R}^2$ by

$$\Upsilon(\gamma)(t,s) := (J_5(\gamma_1(t,s)) - J_6(\gamma_2(t,s)), J_5(\gamma_1(t,s)) + J_6(\gamma_2(t,s)) - 2),$$

where $J_5, J_6 : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ are defined by

$$J_5(u) := \begin{cases} \frac{2^* \int_{\mathbb{R}^N} F(u)}{\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2^*}{2} u^2}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0, \end{cases}, \quad J_6(u) := \begin{cases} \frac{2^* \int_{\mathbb{R}^N} G(u)}{\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{2^*}{2} u^2}, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

By **(F₁)**-**(F₂)** and the Sobolev inequality $|u|_{p+1}^{p+1} \leq C\|u\|^{p+1}$, it is easy to prove that J_5, J_6 are continuous. Similarly as in (2.5) it is easily seen that

$$\begin{aligned} \Upsilon(\tilde{\gamma})(t,s) = & \left(\frac{2^* t^2 \int_{\mathbb{R}^N} F(U_0)}{\int_{\mathbb{R}^N} |\nabla U_0|^2 + \frac{2^*}{2} t^2 U_0^2} - \frac{2^* s^2 \int_{\mathbb{R}^N} G(V_0)}{\int_{\mathbb{R}^N} |\nabla V_0|^2 + \frac{2^*}{2} s^2 V_0^2}, \right. \\ & \left. \frac{2^* t^2 \int_{\mathbb{R}^N} F(U_0)}{\int_{\mathbb{R}^N} |\nabla U_0|^2 + \frac{2^*}{2} t^2 U_0^2} + \frac{2^* s^2 \int_{\mathbb{R}^N} G(V_0)}{\int_{\mathbb{R}^N} |\nabla V_0|^2 + \frac{2^*}{2} s^2 V_0^2} - 2 \right). \end{aligned}$$

Recalling (2.1)-(2.2) and $J_3(U_0) = J_4(V_0) = 0$, we have $\Upsilon(\tilde{\gamma})(1,1) = (0,0)$. By a direct computation, one gets that $\deg(\Upsilon(\tilde{\gamma}), [t_0, t_1] \times [s_0, s_1], (0,0)) = 1$. By (2.10) we see that for any $(t,s) \in \partial([t_0, t_1] \times [s_0, s_1])$, $\Upsilon(\gamma)(t,s) = \Upsilon(\tilde{\gamma})(t,s) \neq (0,0)$. Therefore, $\deg(\Upsilon(\gamma), [t_0, t_1] \times [s_0, s_1], (0,0))$ is well defined and

$$\deg(\Upsilon(\gamma), [t_0, t_1] \times [s_0, s_1], (0,0)) = \deg(\Upsilon(\tilde{\gamma}), [t_0, t_1] \times [s_0, s_1], (0,0)) = 1.$$

Then there exists $(t_2, s_2) \in [t_0, t_1] \times [s_0, s_1]$ such that $\Upsilon(\gamma)(t_2, s_2) = (0, 0)$, that is, $J_5(\gamma_1(t_2, s_2)) = J_6(\gamma_2(t_2, s_2)) = 1$. This implies $J_3(\gamma_1(t_2, s_2)) = J_4(\gamma_2(t_2, s_2)) = 0$ and $\gamma_i(t_2, s_2) \neq 0$ for $i = 1, 2$. Combining these with (2.3), we have

$$\begin{aligned} \max_{(t,s) \in Q} I_0(\gamma(t, s)) &\geq I_0(\gamma(t_2, s_2)) = J_1(\gamma_1(t_2, s_2)) + J_2(\gamma_2(t_2, s_2)) \\ (2.13) \qquad \qquad \qquad &\geq M_1 + M_2 = d_0. \end{aligned}$$

Therefore, $c_0 \geq d_0$. By (2.12) one gets that $c_0 = d_0$.

Assume by contradiction that $\liminf_{\lambda \rightarrow 0} c_\lambda < d_0$. Then there exists $\varepsilon > 0$, $\lambda_n \rightarrow 0$ and $\gamma_n = (\gamma_{n,1}, \gamma_{n,2}) \in \Gamma$ such that

$$\max_{(t,s) \in Q} I_{\lambda_n}(\gamma_n(t, s)) \leq d_0 - 2\varepsilon.$$

By the definition of Γ in (2.10), there exists n_0 large enough such that

$$\max_{(t,s) \in Q} \lambda_n \left| \int_{\mathbb{R}^N} \gamma_{n,1}(t, s) \gamma_{n,2}(t, s) dx \right| \leq C \lambda_n \leq \varepsilon, \quad \forall n \geq n_0,$$

and so

$$\max_{(t,s) \in Q} I_0(\gamma_n(t, s)) \leq \max_{(t,s) \in Q} I_{\lambda_n}(\gamma_n(t, s)) + \varepsilon \leq d_0 - \varepsilon, \quad \forall n \geq n_0,$$

a contradiction with (2.13). Therefore, $\liminf_{\lambda \rightarrow 0} c_\lambda \geq d_0$. Combining this with (2.12), we complete the proof. $\square$

Recalling that $X = S_1 \times S_2$, we define

$$X^\delta := \{(u, v) \in H_r : \text{dist}((u, v), X) \leq \delta\}, \quad I_\lambda^d := \{(u, v) \in H_r : I_\lambda(u, v) \leq d\}.$$

Lemma 2.3. *Let C_1 be as in Lemma 2.1. For a small $\delta \in (0, C_1/2)$, there exist constants $0 < \sigma < 1$ and $\lambda_1 \in (0, 1)$ such that $\|I'_\lambda(u, v)\| \geq \sigma$ for any $(u, v) \in I_\lambda^{d_\lambda} \cap (X^\delta \setminus X^{\delta/2})$ and any $\lambda \in (0, \lambda_1)$.*

Proof. Assume by contradiction that there exist $\lambda_n \rightarrow 0$ and $(u_n, v_n) \in I_{\lambda_n}^{d_{\lambda_n}} \cap (X^\delta \setminus X^{\delta/2})$ such that $\|I'_{\lambda_n}(u_n, v_n)\| \rightarrow 0$. By Lemma 2.1, $\{(u_n, v_n), n \geq 1\}$ are uniformly bounded in H_r . Recall that the Sobolev embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for any $q \in (2, 2^*)$. Up to a subsequence, we may assume that $(u_n, v_n) \rightarrow (U, V)$ weakly in H_r and strongly in $L^q(\mathbb{R}^N)$, $q \in (2, 2^*)$. Then $I'_0(U, V) = 0$; that is, U (resp. V) is a solution of (1.4) (resp. (1.6)). Moreover, by $(F_1) - (F_2)$ we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n) u_n dx = \int_{\mathbb{R}^N} f(U) U dx.$$

By $I'_{\lambda_n}(u_n, v_n)(u_n, 0) \rightarrow 0$, we get that

$$\|u_n\|^2 = \int_{\mathbb{R}^N} f(u_n) u_n dx + o(1) = \int_{\mathbb{R}^N} f(U) U dx + o(1) = \|U\|^2 + o(1),$$

and so $u_n \rightarrow U$ strongly in $H_r^1(\mathbb{R}^N)$. Similarly, $v_n \rightarrow V$ strongly in $H_r^1(\mathbb{R}^N)$, and so $(U, V) \in X^\delta$, which implies that $U \not\equiv 0$ and $V \not\equiv 0$. By (1.3) and the strong maximum principle we have $U > 0$ and $V > 0$. Recalling Lemma 2.2 and the definition of M_1, M_2 , we have

$$\begin{aligned} M_1 + M_2 &\leq J_1(U) + J_2(V) = I_0(U, V) = \lim_{n \rightarrow \infty} I_{\lambda_n}(u_n, v_n) \\ &\leq \lim_{n \rightarrow \infty} d_{\lambda_n} = M_1 + M_2. \end{aligned}$$

This means $J_1(U) = M_1$ and $J_2(V) = M_2$, that is, $U \in S_1$ and $V \in S_2$. So $(U, V) \in X$, which contradicts $(u_n, v_n) \notin X^{\delta/2}$ for any $n \geq 1$. This completes the proof. $\square$

From now on, we fix a small $\delta \in (0, \min\{C/2, C_1/2\})$ and corresponding $0 < \sigma < 1$ and $\lambda_1 > 0$ such that conclusions in Lemma 2.3 hold.

Lemma 2.4. *There exists $\lambda_2 \in (0, \lambda_1]$ and $\alpha > 0$ such that for any $\lambda \in (0, \lambda_2)$,*

$$I_\lambda(\tilde{\gamma}(t, s)) \geq c_\lambda - \alpha \quad \text{implies that} \quad \tilde{\gamma}(t, s) \in X^{\delta/2}.$$

Proof. Assume by contradiction that there exist $\lambda_n \rightarrow 0$, $\alpha_n \rightarrow 0$ and $(t_n, s_n) \in Q$ such that

$$(2.14) \quad I_{\lambda_n}(\tilde{\gamma}(t_n, s_n)) \geq c_{\lambda_n} - \alpha_n \quad \text{and} \quad \tilde{\gamma}(t_n, s_n) \notin X^{\delta/2}.$$

Passing to a subsequence, we may assume that $(t_n, s_n) \rightarrow (\tilde{t}, \tilde{s}) \in Q$. Then by Lemma 2.2 we have

$$I_0(\tilde{\gamma}(\tilde{t}, \tilde{s})) \geq \lim_{n \rightarrow \infty} c_{\lambda_n} = M_1 + M_2.$$

Combining this with (2.5)-(2.6) and (2.11), it is easy to see that $(\tilde{t}, \tilde{s}) = (1, 1)$. Hence,

$$\lim_{n \rightarrow \infty} \|\tilde{\gamma}(t_n, s_n) - \tilde{\gamma}(1, 1)\| = 0.$$

However, $\tilde{\gamma}(1, 1) = (U_0, V_0) \in X$, which is a contradiction to (2.14). $\square$

Let

$$(2.15) \quad \alpha_0 := \min \left\{ \frac{\alpha}{2}, \frac{M_1}{4}, \frac{1}{8} \delta \sigma^2 \right\},$$

where δ, σ are seen in Lemma 2.3. By Lemma 2.2 there exists $\lambda_0 \in (0, \lambda_2]$ such that

$$(2.16) \quad |c_\lambda - d_\lambda| < \alpha_0, \quad |c_\lambda - (M_1 + M_2)| < \alpha_0, \quad \forall \lambda \in (0, \lambda_0).$$

Lemma 2.5. *For fixed $\lambda \in (0, \lambda_0)$, there exists $\{(u_n, v_n)\}_{n=1}^\infty \subset X^\delta \cap I_\lambda^{d_\lambda}$ such that*

$$I'_\lambda(u_n, v_n) \rightarrow 0 \quad \text{in } H_r, \quad \text{as } n \rightarrow \infty.$$

Proof. Fix a $\lambda \in (0, \lambda_0)$. Assume by contradiction that there exists $0 < l(\lambda) < 1$ such that $\|I'_\lambda(u, v)\| \geq l(\lambda)$ on $X^\delta \cap I_\lambda^{d_\lambda}$. Then there exists a pseudo-gradient vector field T_λ in H_r which is defined on a neighborhood Z_λ of $X^\delta \cap I_\lambda^{d_\lambda}$ (cf. [16]) such that for any $(u, v) \in Z_\lambda$ there holds

$$\begin{aligned} \|T_\lambda(u, v)\| &\leq 2 \min\{1, \|I'_\lambda(u, v)\|\}, \\ \langle I'_\lambda(u, v), T_\lambda(u, v) \rangle &\geq \min\{1, \|I'_\lambda(u, v)\|\} \|I'_\lambda(u, v)\|. \end{aligned}$$

Let η_λ be a Lipschitz continuous function on H_r such that $0 \leq \eta_\lambda \leq 1$, $\eta_\lambda \equiv 1$ on $X^\delta \cap I_\lambda^{d_\lambda}$ and $\eta_\lambda \equiv 0$ on $H_r \setminus Z_\lambda$. Let ξ_λ be a Lipschitz continuous function on $\mathbb{R}$ such that $0 \leq \xi_\lambda \leq 1$, $\xi_\lambda(l) \equiv 1$ if $|l - c_\lambda| \leq \frac{\alpha}{2}$ and $\xi_\lambda(l) \equiv 0$ if $|l - c_\lambda| \geq \alpha$. Let

$$e_\lambda(u, v) = \begin{cases} -\eta_\lambda(u, v) \xi_\lambda(I_\lambda(u, v)) T_\lambda(u, v) & \text{if } (u, v) \in Z_\lambda, \\ 0 & \text{if } (u, v) \in H_r \setminus Z_\lambda. \end{cases}$$

Then there exists a global solution $\psi_\lambda : H_r \times [0, +\infty) \rightarrow H_r$ for the initial value problem

$$\begin{cases} \frac{d}{d\theta} \psi_\lambda(u, v, \theta) = e_\lambda(\psi_\lambda(u, v, \theta)), \\ \psi_\lambda(u, v, 0) = (u, v). \end{cases}$$

It is easy to see that ψ_λ has the following properties:

- (1) $\psi_\lambda(u, v, \theta) = (u, v)$ if $\theta = 0$ or $(u, v) \in H_r \setminus Z_\lambda$ or $|I_\lambda(u, v) - c_\lambda| \geq \alpha$.
- (2) $\|\frac{d}{d\theta}\psi_\lambda(u, v, \theta)\| \leq 2$.
- (3) $\frac{d}{d\theta}I_\lambda(\psi_\lambda(u, v, \theta)) = \langle I'_\lambda(\psi_\lambda(u, v, \theta)), e_\lambda(\psi_\lambda(u, v, \theta)) \rangle \leq 0$.

Step 1. For any $(t, s) \in Q$, we claim that there exists $\theta_{t,s} \in [0, +\infty)$ such that $\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}) \in I_\lambda^{c_\lambda - \alpha_0}$, where α_0 is seen in (2.15).

Assume by contradiction that there exists $(t, s) \in Q$ such that

$$I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta)) > c_\lambda - \alpha_0,$$

for any $\theta \geq 0$. Note that $\alpha_0 < \alpha$; we see from Lemma 2.4 that $\tilde{\gamma}(t, s) \in X^{\delta/2}$. Note that $I_\lambda(\tilde{\gamma}(t, s)) \leq d_\lambda < c_\lambda + \alpha_0$; we see from the property (3) that

$$c_\lambda - \alpha_0 < I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta)) \leq d_\lambda < c_\lambda + \alpha_0, \quad \forall \theta \geq 0.$$

This implies $\xi_\lambda(I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta))) \equiv 1$. If $\psi_\lambda(\tilde{\gamma}(t, s), \theta) \in X^\delta$ for all $\theta \geq 0$, then $\eta_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta)) \equiv 1$, and $\|I'_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta))\| \geq l(\lambda)$ for all $\theta > 0$. Then

$$I_\lambda \left(\psi_\lambda \left(\tilde{\gamma}(t, s), \frac{\alpha}{l(\lambda)^2} \right) \right) \leq c_\lambda + \frac{\alpha}{2} - \int_0^{\frac{\alpha}{l(\lambda)^2}} l(\lambda)^2 dt \leq c_\lambda - \frac{\alpha}{2},$$

a contradiction. Thus, there exists $\theta_{t,s} > 0$ such that $\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}) \notin X^\delta$. Note that $\tilde{\gamma}(t, s) \in X^{\delta/2}$; there exist $0 < \theta_{t,s}^1 < \theta_{t,s}^2 \leq \theta_{t,s}$ such that $\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}^1) \in \partial X^{\delta/2}$, $\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}^2) \in \partial X^\delta$ and $\psi_\lambda(\tilde{\gamma}(t, s), \theta) \in X^\delta \setminus X^{\delta/2}$ for all $\theta \in (\theta_{t,s}^1, \theta_{t,s}^2)$. Then by Lemma 2.3 we have $\|I'_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta))\| \geq \sigma$ for all $\theta \in (\theta_{t,s}^1, \theta_{t,s}^2)$. Then using property (2) we have

$$\delta/2 \leq \|\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}^2) - \psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}^1)\| \leq 2|\theta_{t,s}^2 - \theta_{t,s}^1|,$$

that is, $\theta_{t,s}^2 - \theta_{t,s}^1 \geq \delta/4$. This implies

$$\begin{aligned} I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}^2)) &\leq I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}^1)) + \int_{\theta_{t,s}^1}^{\theta_{t,s}^2} \frac{d}{d\theta} I_\lambda(\psi_\lambda(u, v, \theta)) d\theta \\ &< c_\lambda + \alpha_0 - \sigma^2(\theta_{t,s}^2 - \theta_{t,s}^1) \leq c_\lambda + \alpha_0 - \frac{1}{4}\delta\sigma^2 \\ &\leq c_\lambda - \alpha_0, \end{aligned}$$

which is a contradiction.

By Step 1 we can define $T(t, s) := \inf\{\theta \geq 0 : I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta)) \leq c_\lambda - \alpha_0\}$ and let $\gamma(t, s) := \psi_\lambda(\tilde{\gamma}(t, s), T(t, s))$. Then $I_\lambda(\gamma(t, s)) \leq c_\lambda - \alpha_0$ for all $(t, s) \in Q$.

Step 2. We shall prove that $\gamma(t, s) \in \Gamma$.

For any $(t, s) \in Q \setminus (t_0, t_1) \times (s_0, s_1)$, by (2.4), (2.6)-(2.8) and (2.15)-(2.16), we have

$$\begin{aligned} I_\lambda(\tilde{\gamma}(t, s)) &\leq I_0(\tilde{\gamma}(t, s)) = J_1(\tilde{\gamma}_1(t)) + J_2(\tilde{\gamma}_2(s)) \\ &\leq M_1/4 + M_2 \leq M_1 + M_2 - 3\alpha_0 < c_\lambda - \alpha_0, \end{aligned}$$

which implies that $T(t, s) = 0$ and so $\gamma(t, s) = \tilde{\gamma}(t, s)$.

From the definition of Γ in (2.10), it suffices to prove that $\|\gamma(t, s)\| \leq 2C_2 + \mathcal{C}$ for all $(t, s) \in Q$ and $T(t, s)$ is continuous with respect to (t, s) .

For any $(t, s) \in Q$, if $I_\lambda(\tilde{\gamma}(t, s)) \leq c_\lambda - \alpha_0$, we have $T(t, s) = 0$ and so $\gamma(t, s) = \tilde{\gamma}(t, s)$, and by (2.9) we see that $\|\gamma(t, s)\| \leq \mathcal{C} < 2C_2 + \mathcal{C}$.

If $I_\lambda(\tilde{\gamma}(t, s)) > c_\lambda - \alpha_0$, then $\tilde{\gamma}(t, s) \in X^{\delta/2}$ and

$$c_\lambda - \alpha_0 < I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta)) \leq d_\lambda < c_\lambda + \alpha_0, \quad \forall \theta \in [0, T(t, s)).$$

This implies $\xi_\lambda(I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta))) \equiv 1$ for $\theta \in [0, T(t, s))$. If $\psi_\lambda(\tilde{\gamma}(t, s), T(t, s)) \notin X^\delta$, then there exists $0 < \theta_{t,s}^1 < \theta_{t,s}^2 < T(t, s)$ as above. Then we can prove that $I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), \theta_{t,s}^2)) \leq c_\lambda - \alpha_0$ as above, which contradicts the definition of $T(t, s)$. Therefore, $\gamma(t, s) = \psi_\lambda(\tilde{\gamma}(t, s), T(t, s)) \in X^\delta$. Then there exists $(u, v) \in X$ such that $\|\gamma(t, s) - (u, v)\| \leq \delta \leq \mathcal{C}/2$. By Lemma 2.1 we have

$$\|\gamma(t, s)\| \leq \|(u, v)\| + \mathcal{C}/2 \leq 2C_2 + \mathcal{C}.$$

To prove the continuity of $T(t, s)$, we fix any $(\tilde{t}, \tilde{s}) \in Q$. Assume that $I_\lambda(\gamma(\tilde{t}, \tilde{s})) < c_\lambda - \alpha_0$ first. Then $T(\tilde{t}, \tilde{s}) = 0$ from the definition of $T(t, s)$. So $I_\lambda(\tilde{\gamma}(\tilde{t}, \tilde{s})) < c_\lambda - \alpha_0$. By the continuity of $\tilde{\gamma}$, there exists $\tau > 0$ such that for any $(t, s) \in (\tilde{t} - \tau, \tilde{t} + \tau) \times (\tilde{s} - \tau, \tilde{s} + \tau) \cap Q$, we have $I_\lambda(\tilde{\gamma}(t, s)) < c_\lambda - \alpha_0$, that is, $T(t, s) = 0$, and T is continuous at $(\tilde{t}, \tilde{s})$. Now we assume that $I_\lambda(\gamma(\tilde{t}, \tilde{s})) = c_\lambda - \alpha_0$. Then from the previous proof we see that $\gamma(\tilde{t}, \tilde{s}) = \psi_\lambda(\tilde{\gamma}(\tilde{t}, \tilde{s}), T(\tilde{t}, \tilde{s})) \in X^\delta$, and so

$$\|I'_\lambda(\psi_\lambda(\tilde{\gamma}(\tilde{t}, \tilde{s}), T(\tilde{t}, \tilde{s})))\| \geq l(\lambda) > 0.$$

Then for any $\omega > 0$, we have $I_\lambda(\psi_\lambda(\tilde{\gamma}(\tilde{t}, \tilde{s}), T(\tilde{t}, \tilde{s}) + \omega)) < c_\lambda - \alpha_0$. By the continuity of ψ_λ , there exists $\tau > 0$ such that for any $(t, s) \in (\tilde{t} - \tau, \tilde{t} + \tau) \times (\tilde{s} - \tau, \tilde{s} + \tau) \cap Q$, we have $I_\lambda(\psi_\lambda(\tilde{\gamma}(t, s), T(\tilde{t}, \tilde{s}) + \omega)) < c_\lambda - \alpha_0$, so $T(t, s) \leq T(\tilde{t}, \tilde{s}) + \omega$. It follows that

$$0 \leq \limsup_{(t,s) \rightarrow (\tilde{t}, \tilde{s})} T(t, s) \leq T(\tilde{t}, \tilde{s}).$$

If $T(\tilde{t}, \tilde{s}) = 0$, we immediately have

$$\lim_{(t,s) \rightarrow (\tilde{t}, \tilde{s})} T(t, s) = T(\tilde{t}, \tilde{s}).$$

If $T(\tilde{t}, \tilde{s}) > 0$, then for any $0 < \omega < T(\tilde{t}, \tilde{s})$, similarly we have $I_\lambda(\psi_\lambda(\tilde{\gamma}(\tilde{t}, \tilde{s}), T(\tilde{t}, \tilde{s}) - \omega)) > c_\lambda - \alpha_0$. By the continuity of ψ_λ again, we see that

$$\liminf_{(t,s) \rightarrow (\tilde{t}, \tilde{s})} T(t, s) \geq T(\tilde{t}, \tilde{s}).$$

So T is continuous at $(\tilde{t}, \tilde{s})$. This completes the proof of Step 2.

Now, we have proved that $\gamma(t, s) \in \Gamma$ and $\max_{(t,s) \in Q} I_\lambda(\gamma(t, s)) \leq c_\lambda - \alpha_0$, which contradicts the definition of c_λ . This completes the proof. $\square$

Lemma 2.6. *Let $\lambda \in (0, 1)$. Assume that (u, v) is a nontrivial solution of (1.1); then $u > 0$ and $v > 0$.*

Proof. It is easily seen from (1.1) that both $u \not\equiv 0$ and $v \not\equiv 0$. Assume by contradiction that $\{x \in \mathbb{R}^N : u(x) < 0\}$ is not empty. By a standard regularity theory, we see that $u, v \in W_{\text{loc}}^{2,q}(\mathbb{R}^N)$ for any $q < +\infty$, which implies

$$-\Delta u + u = f(u) + \lambda v, \quad \text{a.e. } x \in \mathbb{R}^N,$$

and $u, v \in C_{\text{loc}}^{1,\alpha}(\mathbb{R}^N)$, $0 < \alpha < 1$. Therefore, $\Delta u \in C(\mathbb{R}^N)$.

Let $u(x_1) = \inf_{\mathbb{R}^N} u(x) < 0$. We claim that $\Delta u(x_1) \geq 0$. Indeed, if $\Delta u(x_1) < 0$, there exists $r > 0$ such that

$$u(x) < 0, \quad \Delta u(x) < 0, \quad \forall x \in B(x_1, r),$$

which implies from the maximum principle (cf. [13]) that $\inf_{B(x_1, r)} u(x) \geq \inf_{\partial B(x_1, r)} u(x)$. Note that $u(x_1) = \inf_{\mathbb{R}^N} u(x)$; we obtain that $u \equiv \text{const}$ in $B(x, r)$ from the strong maximum principle (cf. [13]), so $\Delta u(x_1) = 0$, which is a contradiction.

Thus $\Delta u(x_1) \geq 0$. Note that $f(s) \equiv 0$ for $s \leq 0$, and if we have $f(u(x_1)) = 0$, then $-\Delta u(x_1) + u(x_1) = \lambda v(x_1) \leq u(x_1) < \lambda u(x_1)$, that is, $v(x_1) < \inf_{\mathbb{R}^N} u(x) < 0$, so $\inf_{\mathbb{R}^N} v(x) < \inf_{\mathbb{R}^N} u(x)$. Similarly, we can obtain that $\inf_{\mathbb{R}^N} u(x) < \inf_{\mathbb{R}^N} v(x)$, which is a contradiction. Hence, $\{x \in \mathbb{R}^N : u(x) < 0\} = \emptyset$, and $u \geq 0$. Similarly, $v \geq 0$. Then by the strong maximum principle (cf. [13]), we see that $u, v > 0$. $\square$

Proof of Theorem 1.1. First we fix any $\lambda \in (0, \lambda_0)$. By Lemma 2.5 there exists $\{(u_n, v_n)\}_{n=1}^\infty \subset X^\delta \cap I_\lambda^{d_\lambda}$ such that

$$I'_\lambda(u_n, v_n) \rightarrow 0 \quad \text{in } H_r, \quad \text{as } n \rightarrow \infty.$$

By Lemma 2.1, $\{(u_n, v_n), n \geq 1\}$ are uniformly bounded in H_r . Up to a subsequence, we may assume that $(u_n, v_n) \rightarrow (u_\lambda, v_\lambda)$ weakly in H_r and strongly in $L^q(\mathbb{R}^N)$, $q \in (2, 2^*)$. Then $I'_\lambda(u_\lambda, v_\lambda) = 0$ and (u_λ, v_λ) is a solution of (1.1). Moreover, by $(F_1) - (F_2)$ we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(u_n)u_n + g(v_n)v_n \, dx = \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda + g(v_\lambda)v_\lambda \, dx.$$

By $I'_\lambda(u_n, v_n)(u_n, v_n) = o(1)$ we have

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla u_n|^2 + |\nabla v_n|^2 + u_n^2 + v_n^2 - 2\lambda u_n v_n \, dx \\ &= \int_{\mathbb{R}^N} f(u_n)u_n + g(v_n)v_n \, dx + o(1) \\ &= \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda + g(v_\lambda)v_\lambda \, dx + o(1) \\ &= \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |\nabla v_\lambda|^2 + u_\lambda^2 + v_\lambda^2 - 2\lambda u_\lambda v_\lambda \, dx + o(1). \end{aligned}$$

Therefore, $(u_n, v_n) \rightarrow (u_\lambda, v_\lambda)$ strongly in H_r , and so $(u_\lambda, v_\lambda) \in X^\delta$, which implies that $u_\lambda \not\equiv 0$ and $v_\lambda \not\equiv 0$. By Lemma 2.6 we have $u_\lambda > 0$ and $v_\lambda > 0$. Therefore, (u_λ, v_λ) is a positive radially symmetric solution of (1.1). Then by the standard regularity theory, we see that $u_\lambda, v_\lambda \in C^2(\mathbb{R}^N)$. Moreover, $I_\lambda(u_\lambda, v_\lambda) \leq d_\lambda$.

Let $\lambda_n \in (0, \lambda_0)$, $n \in \mathbb{N}$, be any sequence with $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Then by repeating the proof of Lemma 2.3 and passing to a subsequence, $(u_{\lambda_n}, v_{\lambda_n}) \rightarrow (U, V)$ strongly in H_r as $n \rightarrow \infty$, where $U \in S_1$ and $V \in S_2$. That is, U is a positive radial ground state of (1.4), and V is a positive radial ground state of (1.6) respectively. This completes the proof. $\square$

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