SOME RESULTS ON 3-CORES

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ABSTRACT. We prove that if $u(n)$ denotes the number of representations of a nonnegative integer n in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, and $a_3(n)$ is the number of 3-cores of n , then $u(12n + 4) = 6a_3(n)$. With the help of a classical result by L. Lorenz in 1871, we also deduce that

$$a_3(n) = d_{1,3}(3n + 1) - d_{2,3}(3n + 1),$$

where $d_{r,3}(n)$ is the number of divisors of n congruent to $r \pmod{3}$, a result proved earlier by Granville and Ono by using the theory of modular forms and by Hirschhorn and Sellers with the help of elementary generating function manipulations.

1. INTRODUCTION

Throughout this paper, we assume that $|q| < 1$ and use the standard notation

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n).$$

A partition λ is said to be a t -core if and only if it has no hook numbers that are multiples of t or if and only if λ has no rim hooks that are multiples of t . If $a_t(n)$ denotes the number of partitions of n that are t -cores, then the generating function for $a_t(n)$ is given by [6, Equation (2.1)]

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t; q^t)_\infty}{(q; q)_\infty}.$$

In particular, for $t = 3$,

$$(1.1) \quad \sum_{n=0}^{\infty} a_3(n) q^n = \frac{(q^3; q^3)_\infty}{(q; q)_\infty}.$$

We also note from [6, Equation (7.1b)] that, for t odd, the generating function for $asc_t(n)$, the number of t -cores that are self-conjugate, is given by

$$(1.2) \quad \sum_{n=0}^{\infty} asc_t(n) q^n = \frac{(-q; q^2)_\infty (q^{2t}; q^{2t})_\infty^{(t-1)/2}}{(-q^t; q^{2t})_\infty}.$$

By using the theory of modular forms, Granville and Ono [7] proved that

$$(1.3) \quad a_3(n) = d_{1,3}(3n + 1) - d_{2,3}(3n + 1),$$

where $d_{r,3}(n)$ is the number of divisors of n congruent to $r \pmod{3}$.

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Again, Baruah and Berndt [1] used a modular equation of Ramanujan to prove that

$$(1.4) \quad a_3(4n+1) = a_3(n), \text{ for all } n \geq 0.$$

Though not explicitly written in [1], from the same modular equation it follows that

$$(1.5) \quad a_3(4n+3) = 0, \text{ for all } n \geq 0.$$

Hirschhorn and Sellers [9] used some elementary generating function manipulations to prove (1.3) and then employed them to derive an explicit formula for $a_3(n)$ in terms of the factorization of $3n+1$. As corollaries, they derived several infinite families of arithmetic results involving $a_3(n)$, including the generalizations of (1.4) and (1.5).

The main purpose of this paper is to use Ramanujan's theta function identities to prove that $u(12n+4) = 6a_3(n)$, where $u(n)$ denotes the number of representations of a nonnegative integer n in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$. With the help of a classical result by L. Lorenz in 1871, we then deduce (1.3).

2. PRELIMINARY RESULTS

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined by

$$(2.1) \quad f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Jacobi's famous triple product identity [3, p. 35, Entry 19] takes the form

$$(2.2) \quad f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}.$$

The three most important special cases of $f(a, b)$ are

$$(2.3) \quad \varphi(q) := f(q, q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty},$$

$$(2.4) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

and

$$(2.5) \quad f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty},$$

where the product representations in (2.3)–(2.5) arise from (2.2).

We note that, by (2.5), the formula (1.1) reduces to

$$(2.6) \quad \sum_{n=0}^{\infty} a_3(n) q^n = \frac{f^3(-q^3)}{f(-q)}.$$

In the following lemmas, we state some properties satisfied by $f(a, b)$.

Lemma 2.1 ([3, Entry 29, p. 45]). *If $ab = cd$, then*

$$(2.7) \quad f(a, b) f(c, d) = f(ac, bd) f(ad, bc) + af(b/c, ac^2d) f(b/d, acd^2).$$

Lemma 2.2 ([3, Entry 31, p. 48]). *If $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for each integer n , then*

$$(2.8) \quad f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

Lemma 2.3 ([3, Entry 25 (i) and (ii), p. 40]).

$$(2.9) \quad \varphi(q) = \varphi(q^4) + 2q\psi(q^8).$$

Lemma 2.4 ([3, Corollary, p. 49]).

$$(2.10) \quad \varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}),$$

$$(2.11) \quad \psi(q) = f(q^3, q^6) + q\psi(q^9).$$

3. MAIN THEOREMS

Theorem 3.1. *If $u(n)$ denotes the number of representations of a nonnegative integer n in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, and $a_3(n)$ is the number of 3-cores of n , then*

$$(3.1) \quad u(12n + 4) = 6a_3(n).$$

In the following process for proving (3.1), we also find some other results involving $u(n)$.

Proof. We have

$$(3.2) \quad \sum_{n=0}^{\infty} u(n)q^n = \varphi(q)\varphi(q^3),$$

which we rewrite with the help of (2.9) as

$$(3.3) \quad \begin{aligned} \sum_{n=0}^{\infty} u(n)q^n &= (\varphi(q^4) + 2q\psi(q^8)) (\varphi(q^{12}) + 2q^3\psi(q^{24})) \\ &= (\varphi(q^4)\varphi(q^{12}) + 4q^4\psi(q^8)\psi(q^{24})) + 2q (\psi(q^8)\varphi(q^{12}) + q^2\varphi(q^4)\psi(q^{24})). \end{aligned}$$

Extracting the terms involving q^{2n} and q^{2n+1} , respectively, in (3.3), we find that

$$(3.4) \quad \sum_{n=0}^{\infty} u(2n)q^n = \varphi(q^2)\varphi(q^6) + 4q^2\psi(q^4)\psi(q^{12})$$

and

$$(3.5) \quad \sum_{n=0}^{\infty} u(2n+1)q^n = 2\psi(q^4)\varphi(q^4) + 2q\varphi(q^2)\psi(q^{12}) = 2\psi(q)\psi(q^3),$$

where the last equality of (3.5) is proved by several authors, for examples in [4, p. 356], [2], [10], and [5].

Now, with the help of (3.2) and (3.5) we can rewrite the identity (3.4) in the form

$$(3.6) \quad \sum_{n=0}^{\infty} u(2n)q^n = \sum_{n=0}^{\infty} u(n)q^{2n} + 2q^2 \sum_{n=0}^{\infty} u(2n+1)q^{4n}.$$

Equating the coefficients of q^{2n+1} , q^{4n} , and q^{4n+2} , respectively, from both sides of (3.6), we obtain

$$(3.7) \quad u(4n+2) = 0,$$

$$(3.8) \quad u(8n) = u(2n),$$

$$(3.9) \quad u(8n+4) = 3u(2n+1).$$

Again, employing (2.10) in (3.2), we have

$$(3.10) \quad \sum_{n=0}^{\infty} u(n)q^n = \varphi(q^3) (\varphi(q^9) + 2qf(q^3, q^{15})).$$

Either extracting the terms or equating the coefficients of the terms involving q^{3n} , q^{3n+2} , and q^{3n+1} , respectively, from both sides of (3.10), we find that

$$(3.11) \quad u(3n) = u(n),$$

$$(3.12) \quad u(3n+2) = 0,$$

$$(3.13) \quad \sum_{n=0}^{\infty} u(3n+1)q^n = 2\varphi(q)f(q, q^5).$$

Now, we find a 2-dissection of $\varphi(q)f(q, q^5)$. To this end, setting $a = -q\omega$ and $b = -q\omega^2$, where ω is a nonreal cube root of unity, in Jacobi triple product identity (2.2), we find that

$$(3.14) \quad \begin{aligned} \frac{f(-q\omega, -q\omega^2)}{(q^2\omega, q^2\omega^2; q^4)_{\infty}} &= \frac{(q\omega; q^2)_{\infty}(q\omega^2; q^2)_{\infty}(q^2; q^2)_{\infty}}{(q^2\omega, q^2\omega^2; q^4)_{\infty}} \\ &= \frac{(q\omega^2; q^2)_{\infty}(q^2; q^2)_{\infty}}{(q^2\omega; q^4)_{\infty}(-q\omega; q^2)_{\infty}} \\ &= \frac{(q^2; q^2)_{\infty}}{(-q\omega^2; q^2)_{\infty}(-q\omega; q^2)_{\infty}}. \end{aligned}$$

Changing the base of the q -products $(-q\omega^2; q^2)_{\infty}$ and $(-q\omega; q^2)_{\infty}$ in (3.14), we deduce that

$$\begin{aligned} \frac{f(-q\omega, -q\omega^2)}{(q^2\omega, q^2\omega^2; q^4)_{\infty}} &= (q^2; q^2)_{\infty}(-q; q^6)_{\infty}(-q^5; q^6)_{\infty} \\ &= \frac{(q^2; q^2)_{\infty}f(q, q^5)}{(q^6; q^6)_{\infty}}, \end{aligned}$$

where we also used (2.2).

Thus,

$$(3.15) \quad \begin{aligned} \varphi(q)f(q, q^5) &= \frac{\varphi(q)f(-q\omega, -q\omega^2)(q^6; q^6)_{\infty}}{(q^2\omega, q^2\omega^2; q^4)_{\infty}(q^2; q^2)_{\infty}} \\ &= \frac{\varphi(q)f(-q\omega, -q\omega^2)(q^{12}; q^{12})_{\infty}}{(q^4; q^4)_{\infty}} \\ &= \frac{f(q, q)f(-q\omega, -q\omega^2)f(-q^{12})}{f(-q^4)}, \end{aligned}$$

where we have used (2.3) and (2.5).

Now, setting $a = b = q$, $c = -q\omega$, and $d = -q\omega^2$ in (2.7), we find that

$$(3.16) \quad f(q, q)f(-q\omega, -q\omega^2) = f^2(-q^2\omega, -q^2\omega^2) + qf(-\omega^2, -\omega q^4)f(-\omega, -q^4\omega^2).$$

Employing (2.2) in the expressions on the right side of (3.16), manipulating the q -products, and then using the resulting identity in (3.15), we obtain

$$(3.17) \quad \varphi(q)f(q, q^5) = \psi(q^2)f(q^2, q^4) + 3q \frac{f^3(-q^{12})}{f(-q^4)}.$$

Now, employing (3.17) in (3.13), we find that

$$(3.18) \quad \sum_{n=0}^{\infty} u(3n+1)q^n = 2\psi(q^2)f(q^2, q^4) + 6q \frac{f^3(-q^{12})}{f(-q^4)}.$$

Extracting the terms involving q^{4n+1} from both sides of (3.18) and then using (2.6), we obtain

$$\sum_{n=0}^{\infty} u(12n+4)q^n = 6 \frac{f^3(-q^3)}{f(-q)} = 6 \sum_{n=0}^{\infty} a_3(n)q^n,$$

from which we readily deduce (3.1) to complete the proof. $\square$

Now the identity (1.5) can easily be deduced.

Corollary 3.2. *The identity (1.5) holds well.*

Proof. Replacing n by $3n+2$ in (3.7), we have

$$u(12n+10) = 0.$$

By (3.8), the above identity is equivalent to

$$u(48n+40) = 0,$$

which, with the help of (3.1), can easily be reduced to (1.5). $\square$

With the aid of a result of Lorenz [11], the identity (1.3) can also be deduced easily.

Corollary 3.3. *The identity (1.3) holds well.*

Proof. A classical result of Lorenz [11] states that

$$(3.19) \quad u(n) = 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)),$$

where $d_{r,3}(n)$ is the number of divisors of n congruent to $r \pmod{3}$.

Hirschhorn [8] proved the equivalent form of (3.19) as

$$\varphi(q)\varphi(q^3) = 1 + 2 \sum_{n=1}^{\infty} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n=1}^{\infty} (d_{4,12}(n) - d_{8,12}(n))q^n,$$

which can be rewritten in the form

$$(3.20) \quad \sum_{n=0}^{\infty} u(n)q^n = 1 + 2 \sum_{n=1}^{\infty} (d_{1,3}(n) - d_{2,3}(n))q^n + 4 \sum_{n=1}^{\infty} (d_{1,3}(n) - d_{2,3}(n))q^{4n}.$$

Equating the coefficients of $12n+4$ from both sides of (3.20), we find that

$$u(12n+4) = 2(d_{1,3}(12n+4) - d_{2,3}(12n+4)) + 4(d_{1,3}(3n+1) - d_{2,3}(3n+1)).$$

Noting that $d_{1,3}(12n+4) - d_{2,3}(12n+4) = d_{1,3}(3n+1) - d_{2,3}(3n+1)$, we immediately arrive at (1.3). $\square$

Employing (1.3) and a standard counting argument, Hirschhorn and Sellers [9] found an explicit formula for $a_3(n)$ in terms of the factorization of $3n + 1$ and deduced several infinite families of arithmetic results involving $a_3(n)$. Now we can also find different proofs of their results without considering the factorization of $3n + 1$. We demonstrate this by proving one of their results in Theorem 3.5. First we prove the following lemma.

Lemma 3.4. *If $u(n)$ denotes the number of representations of a nonnegative integer n in the form $x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and if $p \equiv 2 \pmod{3}$ is an odd prime, then*

$$(3.21) \quad u(p^2 n) = u(n).$$

Proof. Let $p \equiv 2 \pmod{3}$ be an odd prime. Setting $n = p$ and $a = b = q$ in (2.8), we obtain

$$(3.22) \quad \varphi(q) = \varphi(q^{p^2}) + \sum_{r=1}^{p-1} q^{r^2} f(q^{p(p-2r)}, q^{p(p+2r)}).$$

With successive use of the trivial identity $f(a, b) = af(a^{-1}, a^2 b)$, we can rewrite the above identity in the form

$$(3.23) \quad \begin{aligned} \varphi(q) &= \varphi(q^{p^2}) + 2qf(q^{p(p-2)}, q^{p(p+2)}) + 2q^{2^2}f(q^{p(p-2 \cdot 2)}, q^{p(p+2 \cdot 2)}) \\ &\quad + 2q^{3^2}f(q^{p(p-2 \cdot 3)}, q^{p(p+2 \cdot 3)}) + \dots + 2q^{\left(\frac{p-1}{2}\right)^2}f(q^p, q^{p(2p-1)}). \end{aligned}$$

Replacing q by q^3 in (3.23), we have

$$(3.24) \quad \begin{aligned} \varphi(q^3) &= \varphi(q^{3p^2}) + 2q^3f(q^{3p(p-2)}, q^{3p(p+2)}) + 2q^{3 \cdot 2^2}f(q^{3p(p-2 \cdot 2)}, q^{3p(p+2 \cdot 2)}) \\ &\quad + 2q^{3 \cdot 3^2}f(q^{3p(p-2 \cdot 3)}, q^{3p(p+2 \cdot 3)}) + \dots + 2q^{3\left(\frac{p-1}{2}\right)^2}f(q^{3p}, q^{3p(2p-1)}). \end{aligned}$$

Employing (3.23) and (3.24) in (3.2) and then extracting the terms involving q^{pn} from both sides of the resulting identity by noting that prime $p \equiv 2 \pmod{3}$ and squares are always congruent to 0 or 1 modulo 3, we find that

$$(3.25) \quad \sum_{n=0}^{\infty} u(pn)q^n = \varphi(q^p)\varphi(q^{3p}) = \sum_{n=0}^{\infty} u(n)q^{pn}.$$

Equating the coefficients of q^{pn} from both sides of (3.25), we readily arrive at (3.21) to complete the proof. $\square$

Theorem 3.5 ([9, Corollary 8]). *Let $p \equiv 2 \pmod{3}$ be prime and k be a positive even integer. If $a_3(n)$ denotes the number of 3-cores of n , then for any positive integer n , we have*

$$(3.26) \quad a_3(n) = a_3\left(p^k n + \left(\frac{p^k - 1}{3}\right)\right).$$

Proof. First we prove the theorem for $p = 2$.

Replacing n by $6n + 2$ in (3.8), we obtain

$$(3.27) \quad u(12n + 4) = u(12(4n + 1) + 4).$$

Employing (3.1) in (3.27), we arrive at (1.4), from which (3.26) for $p = 2$ can be readily deduced by induction.

Next, we prove the theorem for an odd prime $p \equiv 2 \pmod{3}$.

Replacing n by $12n + 4$ in (3.21), we have

$$(3.28) \quad u(12n + 4) = u(p^2(12n + 4)) = u\left(12\left(p^2n + \frac{p^2 - 1}{3}\right) + 4\right).$$

Employing (3.1) in (3.28), we arrive at

$$a_3(n) = a_3\left(p^2n + \frac{p^2 - 1}{3}\right),$$

which implies (3.26) by induction. $\square$

If $asc_3(n)$ denotes the number of 3-cores of n that are self-conjugates, then Baruah and Berndt [1] proved that $asc_3(4n + 1) = asc_3(n)$. In the following theorem, we generalize this result by proving a theorem analogous to Theorem 3.5, where $a_3(n)$ is replaced by $asc_3(n)$.

Theorem 3.6. *Let $p \equiv 2 \pmod{3}$ be prime and k be a positive even integer. If $asc_3(n)$ denotes the number of 3-cores of n that are self-conjugates, then for any positive integer n , we have*

$$(3.29) \quad asc_3(n) = asc_3\left(p^kn + \left(\frac{p^k - 1}{3}\right)\right).$$

Proof. Let us define $s(n)$ by

$$(3.30) \quad \varphi(q) = \sum_{n \geq 0} s(n)q^n.$$

Employing (2.10) in (3.30) and extracting the terms involving q^{3n+1} , we find that

$$(3.31) \quad \sum_{n \geq 0} s(3n + 1)q^n = 2f(q, q^5).$$

Again, by [3, p. 51, Example (v)], (2.4) and (1.2) with $t = 3$, we have

$$(3.32) \quad f(q, q^5) = \sum_{n \geq 0} asc_3(n)q^n.$$

From (3.31) and (3.32), we arrive at

$$(3.33) \quad s(3n + 1) = 2asc_3(n).$$

Again, employing (3.30) in (3.23) and then proceeding as in the proof of Theorem 3.5, we obtain

$$(3.34) \quad s(p^2n) = s(n).$$

From (3.33) and (3.34), we find that

$$\begin{aligned} 2asc_3(n) &= s(3n + 1) = s(p^2(3n + 1)) = s\left(3\left(p^2n + \frac{p^2 - 1}{3}\right) + 1\right) \\ &= 2asc_3\left(p^2n + \frac{p^2 - 1}{3}\right), \end{aligned}$$

from which we readily arrive at (3.29) by induction. $\square$

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