QUANTUM DOUBLE SCHUBERT POLYNOMIALS REPRESENT SCHUBERT CLASSES

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ABSTRACT. The quantum double Schubert polynomials studied by Kirillov and Maeno, and by Ciocan-Fontanine and Fulton, are shown to represent Schubert classes in Kim's presentation of the equivariant quantum cohomology of the flag variety. Parabolic analogues of quantum double Schubert polynomials are introduced and shown to represent Schubert classes in the equivariant quantum cohomology of partial flag varieties. This establishes a new method for computing equivariant Gromov-Witten invariants for partial flag varieties. For complete flags Anderson and Chen have announced a proof with different methods.

1. INTRODUCTION

Let $H^*(\text{Fl})$, $H^T(\text{Fl})$, $QH^*(\text{Fl})$, and $QH^T(\text{Fl})$ be the ordinary, T -equivariant, quantum, and T -equivariant quantum cohomology rings of the variety $\text{Fl} = \text{Fl}_n$ of complete flags in $\mathbb{C}^n$ where T is the maximal torus of GL_n . All cohomologies are with $\mathbb{Z}$ coefficients. The flag variety Fl_n has a stratification by Schubert varieties X_w , labeled by permutations $w \in S_n$, which gives rise to *Schubert bases* for each of these rings. The structure constants of these bases play an important role in classical and modern enumerative geometry. For $QH^T(\text{Fl})$ they are known as the 3-point genus 0 Gromov-Witten invariants.

This paper is concerned with the problem of finding polynomial representatives for the Schubert bases in a presentation of these (quantum) cohomology rings. These ring presentations are due to Borel [Bo] in the classical case, and to Ciocan-Fontanine [Cio], Givental and Kim [GK] and Kim [Kim] in the quantum case.

This problem has been solved in the first three cases: the Schubert polynomials are known to represent Schubert classes in $H^*(\text{Fl}_n)$ by work of Bernstein, Gelfand, and Gelfand [BGG] and Lascoux and Schützenberger [LS]; the double Schubert polynomials, also due to Lascoux and Schützenberger, represent Schubert classes in $H^T(\text{Fl}_n)$ (see for example [Bi]); and the quantum Schubert polynomials of Fomin, Gelfand, and Postnikov [FGP] represent Schubert classes in $QH^*(\text{Fl})$. These polynomials are the subject of much research by combinatorialists and geometers, and we refer the reader to these references for a complete discussion of these ideas. Our first main result (Theorem 3.4) is that the quantum double Schubert polynomials of [KM, CF] represent equivariant quantum Schubert classes in $QH^T(\text{Fl})$. Anderson and Chen [AC] have announced a proof using the geometry of Quot schemes.

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Let SL_n/P be a partial flag variety, where P denotes a parabolic subgroup of SL_n . In nonquantum Schubert calculus, the functoriality of (equivariant) cohomology implies that the (double) Schubert polynomials labeled by minimal length coset representatives again represent Schubert classes in $H^*(SL_n/P)$ or $H^T(SL_n/P)$. This is not the case in quantum cohomology. Ciocan-Fontanine [Cio2] solved the corresponding problem in $QH^*(SL_n/P)$, extending Fomin, Gelfand, and Postnikov's work to the parabolic case. Here we introduce parabolic quantum double Schubert polynomials. Our second main result asserts that these polynomials represent Schubert classes in the torus-equivariant quantum cohomology $QH^T(SL_n/P)$ of a partial flag variety. Earlier, Mihalcea [Mi2] had observed that the double (or factorial) Schur polynomials represent the Schubert basis in the special case of the equivariant quantum cohomology of the Grassmannian.

Our results imply that the equivariant 3-point genus 0 Gromov-Witten invariants for partial flag varieties coincide with the structure constants for the parabolic quantum double Schubert polynomials. Moreover no additional computation is required for reduction modulo the ideal (denoted J_P^{qa} in §6.2): a parabolic quantum double Schubert polynomial either projects to a Schubert basis element or to zero, according as the indexing permutation is in the correct symmetric group or not.

Our proof relies on Mihalcea's characterization [Mi] of the equivariant quantum Schubert classes and ring by his equivariant quantum Chevalley-Monk formula (Theorems 2.1 and 6.11).

2. THE (QUANTUM) COHOMOLOGY RINGS OF FLAG MANIFOLDS

2.1. Presentations. Let $x = (x_1, \dots, x_n)$, $a = (a_1, \dots, a_n)$, and $q = (q_1, \dots, q_{n-1})$ be indeterminates. We work in the graded polynomial ring $\mathbb{Z}[x; q; a]$ with $\deg(x_i) = \deg(a_i) = 1$ and $\deg(q_i) = 2$. Let $S = \mathbb{Z}[a]$ be identified with $H^T(\text{pt})$ and let $e_j(a)$ be the elementary symmetric polynomial. Let C_n be the tridiagonal $n \times n$ matrix with entries x_i on the diagonal, -1 on the superdiagonal, and q_i on the subdiagonal. Define the polynomials $E_j^n \in \mathbb{Z}[x; q]$ by

$$\det(C_n - t \text{Id}) = \sum_{j=0}^n (-t)^{n-j} E_j^n.$$

Let J (resp. J^a , J^q , J^{qa}) be the ideal in $\mathbb{Z}[x]$ (resp. $S[x]$, $\mathbb{Z}[x; q]$, $S[x; q]$) generated by the elements $e_j(x)$ (resp. $e_j(x) - e_j(a)$; E_j^n ; $E_j^n - e_j(a)$) for $1 \leq j \leq n$; in all cases the j -th generator is homogeneous of degree j . We have

$$(2.1) \quad H^*(\text{Fl}) \cong \mathbb{Z}[x]/J,$$

$$(2.2) \quad H^T(\text{Fl}) \cong S[x]/J^a,$$

$$(2.3) \quad QH^*(\text{Fl}) \cong \mathbb{Z}[x; q]/J^q,$$

$$(2.4) \quad QH^T(\text{Fl}) \cong S[x; q]/J^{qa}$$

as algebras over $\mathbb{Z}$, S , $\mathbb{Z}[q]$, and $S[q]$ respectively. The presentation of $H^*(\text{Fl})$ is a classical result due to Borel. The presentations of $QH^*(\text{Fl})$ and $QH^T(\text{Fl})$ are due to Ciocan-Fontanine [Cio], Givental and Kim [GK] and Kim [Kim].

2.2. Schubert bases. Let $X_w = \overline{B_- w B / B} \subset \text{Fl}$ be the opposite Schubert variety, where $w \in W = S_n$ is a permutation, $B \subset SL_n$ is the upper triangular Borel and

B_- the opposite Borel. The ring $H^*(\text{Fl})$ (resp. $H^T(\text{Fl})$) has a basis over $\mathbb{Z}$ (resp. S), denoted $[X_w]$ (resp. $[X_w]_T$), associated with the Schubert varieties.

Given three elements $u, v, w \in W$ and an element of the coroot lattice $\beta \in Q^\vee$ one may define a genus zero Gromov-Witten invariant $c_{uv}^w(\beta) \in \mathbb{Z}_{\geq 0}$ (see [GK, Kim]) and an associative ring $QH^*(\text{Fl})$ with $\mathbb{Z}[q]$ -basis $\{\sigma^w \mid w \in W\}$ (called the quantum Schubert basis) such that

$$\sigma^u \sigma^v = \sum_{w, \beta} q_\beta c_{uv}^w(\beta) \sigma^w,$$

where $q_\beta = \prod_{i=1}^{n-1} q_i^{k_i}$ where $\beta = \sum_{i=1}^{n-1} k_i \alpha_i^\vee$ for $k_i \in \mathbb{Z}$. Similarly there is a basis of a ring $QH^T(\text{Fl})$ with $S[q]$ -basis given by the equivariant quantum Schubert classes σ_T^w , defined using equivariant Gromov-Witten invariants, which are elements of S .

We shall use the following characterization of $QH^T(\text{Fl})$ and its Schubert basis $\{\sigma_T^w \mid w \in W\}$ due to Mihalcea [Mi]. Let Φ^+ be the set of positive roots and ρ the half sum of positive roots. For $w \in W$ define

$$\begin{aligned} A_w &= \{\alpha \in \Phi^+ \mid ws_\alpha \succ w\}, \\ B_w &= \{\alpha \in \Phi^+ \mid \ell(ws_\alpha) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho \rangle\}. \end{aligned}$$

Let $\omega_i^a = a_1 + \cdots + a_i \in S$ be the fundamental weight. We write A_w^n and B_w^n to emphasize that the computation pertains to $\text{Fl} = SL_n/B$.

Theorem 2.1 ([Mi, Corollary 8.2]). *For $w \in S_n$ and $1 \leq i \leq n-1$ a Dynkin node, the equivariant quantum Schubert classes σ_T^w satisfy the equivariant quantum Chevalley-Monk formula*

$$(2.5) \quad \sigma_T^{s_i} \sigma_T^w = (-\omega_i^a + w \cdot \omega_i^a) \sigma_T^w + \sum_{\alpha \in A_w^n} \langle \alpha^\vee, \omega_i \rangle \sigma_T^{ws_\alpha} + \sum_{\alpha \in B_w^n} q_{\alpha^\vee} \langle \alpha^\vee, \omega_i \rangle \sigma_T^{ws_\alpha}.$$

Moreover these structure constants determine the Schubert basis $\{\sigma_T^w \mid w \in S_n\}$ and the ring $QH^T(SL_n/B)$ up to isomorphism as $S[q]$ -algebras.

3. QUANTUM DOUBLE SCHUBERT POLYNOMIALS

Now we work with infinite sets of variables $x = (x_1, x_2, \dots)$, $q = (q_1, q_2, \dots)$, and $a = (a_1, a_2, \dots)$.

3.1. Various Schubert polynomials. Let $\partial_i^a = \alpha_i^{-1}(1 - s_i^a)$ be the divided difference operator, where $\alpha_i = a_i - a_{i+1}$ and s_i^a is the operator that exchanges a_i and a_{i+1} . Since the operators $\partial_i = \partial_i^a$ satisfy the braid relations one may define $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$, where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition. For $w \in S_n$ define the double Schubert polynomial $\mathfrak{S}_w(x; a)$ [LS] and the quantum double Schubert polynomial $\tilde{\mathfrak{S}}_w(x; a) \in S[x; q]$ [KM, CF] by

$$(3.1) \quad \mathfrak{S}_w(x; a) = (-1)^{\ell(w)} \partial_{w_0^{(n)}}^a \prod_{i=1}^{n-1} \prod_{j=1}^i (x_j - a_{n-i}),$$

$$(3.2) \quad \tilde{\mathfrak{S}}_w(x; a) = (-1)^{\ell(w)} \partial_{w_0^{(n)}}^a \prod_{i=1}^{n-1} \det(C_i - a_{n-i} \text{Id}),$$

where $w_0^{(n)} \in S_n$ is the longest element.¹ Note that it is equivalent to define $\mathfrak{S}_w(x; a)$ by setting the q_i variables to zero in $\tilde{\mathfrak{S}}_w(x; a)$.

Let $S_\infty = \bigcup_{n \geq 1} S_n$ be the infinite symmetric group under the embeddings $i_n : S_n \rightarrow S_{n+1}$ that add a fixed point at the end of a permutation. Due to the stability property [KM] $\tilde{\mathfrak{S}}_{i_n(w)}(x; a) = \tilde{\mathfrak{S}}_w(x; a)$ for $w \in S_n$, the quantum double Schubert polynomials $\tilde{\mathfrak{S}}_w(x; a)$ are well-defined for $w \in S_\infty$. Similarly, $\mathfrak{S}_w(x; a)$ is well-defined for $w \in S_\infty$.

For $w \in S_\infty$, define the (resp. quantum) Schubert polynomial $\mathfrak{S}_w(x) = \mathfrak{S}_w(x; 0)$ (resp. $\tilde{\mathfrak{S}}_w(x) = \tilde{\mathfrak{S}}_w(x; 0)$) by setting the a_i variables to zero in the (resp. quantum) double Schubert polynomial. Note that $\mathfrak{S}_w(x)$, $\mathfrak{S}_w(x; a)$, $\tilde{\mathfrak{S}}_w(x)$, and $\tilde{\mathfrak{S}}_w(x; a)$ are all homogeneous of degree $\ell(w)$. The original definition of a quantum Schubert polynomial in [FGP] is different. However their definition and the one used here are easily seen to be equivalent [KM] due to the commutation of the divided differences in the a variables and the quantization map θ of [FGP], which we review in §3.3.

Lemma 3.1 ([Mac]). *For $w \in S_\infty$, the term that is of highest degree in the x variables and then is the reverse lex leading such term, in any of $\mathfrak{S}_w(x)$, $\mathfrak{S}_w(x; a)$, $\tilde{\mathfrak{S}}_w(x)$, and $\tilde{\mathfrak{S}}_w(x; a)$, is the monomial $x^{\text{code}(w)}$, where $\text{code}(w) = (c_1, c_2, \dots)$ and*

$$c_i = |\{j \in \mathbb{Z}_{>0} \mid i < j \text{ and } w(j) < w(i)\}| \quad \text{for } i \in \mathbb{Z}_{>0}.$$

Lemma 3.2 ([Mac]). *There is a bijection from S_∞ to the set of tuples $(c_1, c_2, \dots)$ of nonnegative integers, almost all zero, given by $w \mapsto \text{code}(w)$. Moreover it restricts to a bijection from S_n to the set of tuples $(c_1, \dots, c_n)$ such that $0 \leq c_i \leq n - i$ for all $0 \leq i \leq n$.*

Lemma 3.3. *Let $x = (x_1, \dots, x_n)$, $a = (a_1, \dots, a_n)$, and $q = (q_1, \dots, q_{n-1})$. Then:*

- (1) $\{\mathfrak{S}_w(x) \mid w \in S_n\}$ is a $\mathbb{Z}$ -basis of $\mathbb{Z}[x]/J_n$.
- (2) $\{\mathfrak{S}_w(x; a) \mid w \in S_n\}$ is a $\mathbb{Z}[a]$ -basis of $\mathbb{Z}[x; a]/J_n^a$.
- (3) $\{\tilde{\mathfrak{S}}_w(x) \mid w \in S_n\}$ is a $\mathbb{Z}[q]$ -basis of $\mathbb{Z}[x; q]/J_n^q$.
- (4) $\{\tilde{\mathfrak{S}}_w(x; a) \mid w \in S_n\}$ is a $\mathbb{Z}[q, a]$ -basis of $\mathbb{Z}[x; q; a]/J_n^{q, a}$.

Proof. Since in each case the highest degree part of the j -th ideal generator in the x variables is $e_j(x_1, \dots, x_n)$, any polynomial may be reduced modulo the ideal until its leading term in the x variables is x^γ , where $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_{\geq 0}^n$ with $\gamma_i \leq n - i$ for $1 \leq i \leq n$. But these are the leading terms of the various kinds of Schubert polynomials. $\square$

3.2. Geometric bases. Under the isomorphism (2.1) (resp. (2.2), (2.3)) the Schubert basis $[X_w]$ (resp. $[X_w]_T$, σ^w) corresponds to $\mathfrak{S}_w(x)$ (resp. $\mathfrak{S}_w(x; a)$, $\tilde{\mathfrak{S}}_w(x)$) by [LS, BGG] for $H^*(\text{Fl})$, [Bi] for $H^T(\text{Fl})$, and [FGP] for $QH^*(\text{Fl})$. Our first main result is:

Theorem 3.4. *Under the $S[q]$ -algebra isomorphism (2.4) the quantum equivariant Schubert basis element σ_T^w corresponds to the quantum double Schubert polynomial $\tilde{\mathfrak{S}}_w(x; a)$.*

Theorem 3.4 is proved in Section 5.

¹This is not the standard definition of a double Schubert polynomial. However it is easily seen to be equivalent using, say, the identity $\mathfrak{S}_{w^{-1}}(x; a) = \mathfrak{S}_w(-a; -x)$ [Mac].

3.3. Stable quantization. This section follows [FGP]. Let $e_i^r = e_i(x_1, x_2, \dots, x_r)$ be the elementary symmetric polynomial for integers $0 \leq i \leq r$. By [FGP, Prop. 3.3], $\mathbb{Z}[x]$ has a $\mathbb{Z}$ -basis of *standard monomials* $e_I = \prod_{r \geq 1} e_{i_r}^r$, where $I = (i_1, i_2, \dots)$ is a sequence of nonnegative integers, almost all zero, with $0 \leq i_r \leq r$ for all $r \geq 1$. The stable quantization map is the $\mathbb{Z}[q]$ -module automorphism θ of $\mathbb{Z}[x; q]$ given by

$$e_I \mapsto E_I := \prod_{j \geq 1} E_{i_j}^j.$$

By [FGP, CF, KM] we have²

$$(3.3) \quad \theta(\mathfrak{S}_w(x)) = \tilde{\mathfrak{S}}_w(x) \quad \text{for all } w \in S_\infty.$$

The map θ is extended by $\mathbb{Z}[a]$ -linearity to a $\mathbb{Z}[q, a]$ -module automorphism of $\mathbb{Z}[x, q, a]$.

3.4. Cauchy formulae. The double Schubert polynomials satisfy [Mac]

$$(3.4) \quad \mathfrak{S}_w(x; a) = \sum_{v \leq^L w} \mathfrak{S}_{vw^{-1}}(-a) \mathfrak{S}_v(x),$$

where $v \leq^L w$ denotes the left weak order, defined by $\ell(wv^{-1}) + \ell(v) = \ell(w)$. For a geometric explanation of this identity see [And]. We have [KM, CF]

$$(3.5) \quad \theta(\mathfrak{S}_w(x; a)) = \tilde{\mathfrak{S}}_w(x; a) = \sum_{v \leq^L w} \mathfrak{S}_{vw^{-1}}(-a) \tilde{\mathfrak{S}}_v(x).$$

The first equality follows from the divided difference definitions of $\tilde{\mathfrak{S}}_w(x; a)$ and $\mathfrak{S}_w(x; a)$ and the commutation of the divided differences in the a -variables with quantization. The second equality follows from quantizing (3.4).

We require the explicit formulae for Schubert polynomials indexed by simple reflections.

Lemma 3.5. *We have*

$$(3.6) \quad \mathfrak{S}_{s_i}(x) = \omega_i^x = x_1 + x_2 + \dots + x_i,$$

$$(3.7) \quad \tilde{\mathfrak{S}}_{s_i}(x) = \mathfrak{S}_{s_i}(x),$$

$$(3.8) \quad \tilde{\mathfrak{S}}_{s_i}(x; a) = \omega_i^x - \omega_i^a.$$

Proof. Since s_i is an i -Grassmannian permutation with associated partition consisting of a single box, its Schubert polynomial is the Schur polynomial [Mac] $\mathfrak{S}_{s_i}(x) = S_1[x_1, \dots, x_i] = \omega_i^x$, proving (3.6). For (3.7) we have $\tilde{\mathfrak{S}}_{s_i}(x) = \theta(\mathfrak{S}_{s_i}(x)) = \theta(e_{1,i}) = E_{1,i} = \omega_i^x$. For (3.8), by (3.5) we have $\tilde{\mathfrak{S}}_{s_i}(x; a) = \tilde{\mathfrak{S}}_{s_i}(x) - \mathfrak{S}_{s_i}(a) = \omega_i^x - \omega_i^a$, as required. $\square$

4. CHEVALLEY-MONK RULES FOR SCHUBERT POLYNOMIALS

The Chevalley-Monk formula describes the product of a divisor class and an arbitrary Schubert class in the cohomology ring $H^*(\text{Fl})$. The goal of this section is to establish the Chevalley-Monk rule for quantum double Schubert polynomials. The Chevalley-Monk rules for (double) Schubert polynomials should be viewed as

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²This is the definition of a quantum Schubert polynomial in [FGP].

product rules for the (equivariant) cohomology of an infinite-dimensional flag ind-variety Fl_∞ of type A_∞ with Dynkin node set $\mathbb{Z}_{>0}$ and simple bonds between i and $i+1$ for all $i \in \mathbb{Z}_{>0}$.

Let $\Phi^+ = \{\alpha_{ij} = a_i - a_j \mid 1 \leq i < j\}$ be the set of positive roots. Let $\alpha_{ij}^\vee = a_i - a_j$ and $\alpha_i^\vee = \alpha_{i,i+1}^\vee$ by abuse of notation. Let $s_{ij} = s_{\alpha_{ij}}$ for $1 \leq i < j$. For $w \in S_\infty$ let A_w and B_w be defined as before Theorem 2.1 but using the infinite set of positive roots Φ^+ and letting $\rho = (0, -1, -2, \dots)$. To distinguish between the finite and limiting infinite cases, we denote by A_w^n the set A_w for $w \in S_n$ which uses the positive roots of SL_n .

4.1. Schubert polynomials.

Proposition 4.1 ([Che, Mo]). *For $w \in S_n$ and $1 \leq i \leq n-1$, in $H^*(\mathrm{Fl})$ we have*

$$[X_{s_i}][X_w] = \sum_{\alpha \in A_w^n} \langle \alpha^\vee, \omega_i \rangle [X_{ws_\alpha}].$$

Proposition 4.2 ([Mac]). *For $w \in S_\infty$ and $i \in \mathbb{Z}_{>0}$ the Schubert polynomials satisfy the identity in $\mathbb{Z}[x]$ given by*

$$(4.1) \quad \mathfrak{S}_{s_i}(x) \mathfrak{S}_w(x) = \sum_{\alpha \in A_w} \langle \alpha^\vee, \omega_i \rangle \mathfrak{S}_{ws_\alpha}(x).$$

Example 4.3. It is necessary to take a large-rank limit ($n \gg 0$) to compare Propositions 4.1 and 4.2. Let $n = 2$. We have $[X_{s_1}]^2 = 0$ in $H^*(\mathrm{Fl}_2)$ since $A_{s_1} = \emptyset$ for SL_2 . Lifting to polynomials we have $\mathfrak{S}_{s_1}^2 = x_1^2 = \mathfrak{S}_{s_2s_1}$ since $A_{s_1} = \{\alpha_{13}\}$, which is not a positive root for SL_2 . Note that $\mathfrak{S}_{s_2s_1} \in J_2$ and $s_2s_1 \in S_3 \setminus S_2$. In $H^*(\mathrm{Fl}_n)$ for $n \geq 3$ we have $[X_{s_1}]^2 = [X_{s_2s_1}]$.

4.2. Quantum Schubert polynomials.

Proposition 4.4 ([FGP]). *For $w \in S_n$ and $1 \leq i \leq n-1$, in $QH^*(\mathrm{Fl}_n)$ we have*

$$\sigma^{s_i} \sigma^w = \sum_{\alpha \in A_w^n} \langle \alpha^\vee, \omega_i \rangle \sigma^{ws_\alpha} + \sum_{\alpha \in B_w^n} q_{\alpha^\vee} \langle \alpha^\vee, \omega_i \rangle \sigma^{ws_\alpha}.$$

Proposition 4.5 ([FGP]). *For $i \in \mathbb{Z}_{>0}$ and $w \in S_\infty$ the quantum Schubert polynomials satisfy the identity in $\mathbb{Z}[x; q]$ given by*

$$(4.2) \quad \tilde{\mathfrak{S}}_{s_i}(x) \tilde{\mathfrak{S}}_w(x) = \sum_{\alpha \in A_w} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{ws_\alpha}(x) + \sum_{\alpha \in B_w} q_{\alpha^\vee} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{ws_\alpha}(x).$$

4.3. Double Schubert polynomials.

Proposition 4.6 ([KK], [Rob]). *For $w \in S_n$ and $1 \leq i \leq n-1$, in $H^T(\mathrm{Fl}_n)$ we have*

$$(4.3) \quad [X_{s_i}]_T [X_w]_T = (-\omega_i^a + w \cdot \omega_i^a) [X_w]_T + \sum_{\alpha \in A_w^n} \langle \alpha^\vee, \omega_i \rangle [X_{ws_\alpha}]_T.$$

The following result is a consequence of [Man, Ex. 2.7.3].

Proposition 4.7. *For $w \in S_\infty$ and $i \in \mathbb{Z}_{>0}$, the double Schubert polynomials satisfy the identity in $\mathbb{Z}[x, a]$ given by*

$$(4.4) \quad \mathfrak{S}_{s_i}(x; a) \mathfrak{S}_w(x; a) = (-\omega_i^a + w \cdot \omega_i^a) \mathfrak{S}_w(x; a) + \sum_{\alpha \in A_w} \langle \alpha^\vee, \omega_i \rangle \mathfrak{S}_{ws_\alpha}(x; a).$$

Proof. Fix $w \in S_\infty$. We observe that the set A_w is finite. Let N be large enough so that all appearing terms make sense for S_N . By [Bi] under the isomorphism (2.2), $[X_w]_T \mapsto \mathfrak{S}_w(x; a) + J_N^a$ for $w \in S_N$. By Proposition 4.6 for $H^T(\mathrm{Fl}_N)$, equation (4.4) holds modulo an element $f \in J_N^a$. We may write $f = \sum_{v \in S_\infty} b_v \mathfrak{S}_v(x; a)$ where $b_v \in \mathbb{Z}[a]$ and only finitely many are nonzero. Choose $n \geq N$ large enough so that $v \in S_n$ and $b_v \in \mathbb{Z}[a_1, \dots, a_n]$ for all v with $b_v \neq 0$. Applying Proposition 4.6 again for $H^T(\mathrm{Fl}_n)$ we deduce that $f \in J_n^a$. By Lemma 3.3 it follows that $f = 0$, as required. $\square$

4.4. Quantum double Schubert polynomials. Theorem 2.1 gives the equivariant quantum Chevalley-Monk rule for $QH^T(\mathrm{Fl}_n)$. We cannot use the multiplication rule in Theorem 2.1 directly because we are trying to prove that the quantum double Schubert polynomials represent Schubert classes. We deduce the following product formula by cancelling down to the equivariant case which was proven above.

Proposition 4.8. *The quantum double Schubert polynomials satisfy the equivariant quantum Chevalley-Monk rule in $\mathbb{Z}[x, q, a]$: for all $w \in S_\infty$ and $i \geq 1$ we have*

$$(4.5) \quad \tilde{\mathfrak{S}}_{s_i}(x; a) \tilde{\mathfrak{S}}_w(x; a) = (-\omega_i^a + w \cdot \omega_i^a) \tilde{\mathfrak{S}}_w(x; a) + \sum_{\alpha \in A_w} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{ws_\alpha}(x; a) \\ + \sum_{\alpha \in B_w} q_{\alpha^\vee} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{ws_\alpha}(x; a).$$

Proof. Let $C = \sum_{\alpha \in A_w} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{ws_\alpha}(x; a)$. Starting with (4.4) and using Lemma 3.5 and (3.4) we have

$$0 = -\mathfrak{S}_{s_i}(x; a) \mathfrak{S}_w(x; a) + (-\omega_i^a + w \cdot \omega_i^a) \mathfrak{S}_w(x; a) + C \\ = (\omega_i^a - \mathfrak{S}_{s_i}(x)) \mathfrak{S}_w(x; a) + (-\omega_i^a + w \cdot \omega_i^a) \mathfrak{S}_w(x; a) + C \\ = -\mathfrak{S}_{s_i}(x) \sum_{v \leq L_w} \mathfrak{S}_{vw^{-1}}(-a) \mathfrak{S}_v(x) + (w \cdot \omega_i^a) \mathfrak{S}_w(x; a) + C \\ = - \sum_{v \leq L_w} \mathfrak{S}_{vw^{-1}}(-a) \sum_{\alpha \in A_v} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{vs_\alpha}(x) + (w \cdot \omega_i^a) \mathfrak{S}_w(x; a) + C.$$

Quantizing and rearranging, we have

$$(w \cdot \omega_i^a) \tilde{\mathfrak{S}}_w(x; a) + \sum_{\alpha \in A_w} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{ws_\alpha}(x; a) \\ = \sum_{v \leq L_w} \mathfrak{S}_{vw^{-1}}(-a) \sum_{\alpha \in A_v} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{vs_\alpha}(x) \\ = \sum_{v \leq L_w} \mathfrak{S}_{vw^{-1}}(-a) \left(\tilde{\mathfrak{S}}_{s_i}(x) \tilde{\mathfrak{S}}_v(x) - \sum_{\alpha \in B_v} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{vs_\alpha}(x) \right) \\ = \tilde{\mathfrak{S}}_{s_i}(x) \tilde{\mathfrak{S}}_w(x; a) - \sum_{v \leq L_w} \mathfrak{S}_{vw^{-1}}(-a) \sum_{\alpha \in B_v} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{vs_\alpha}(x).$$

Therefore to prove (4.5) it suffices to show that

$$(4.6) \quad \sum_{v \leq L_w} \mathfrak{S}_{vw^{-1}}(-a) \sum_{\alpha \in B_v} q_{\alpha^\vee} \langle \alpha^\vee, \omega_i \rangle \tilde{\mathfrak{S}}_{vs_\alpha}(x) \\ = \sum_{\alpha \in B_w} \langle \alpha^\vee, \omega_i \rangle q_{\alpha^\vee} \sum_{v \leq L_{ws_\alpha}} \mathfrak{S}_{vs_\alpha w^{-1}}(-a) \tilde{\mathfrak{S}}_v(x).$$

To prove (4.6) it suffices to show that there is a bijection $A \rightarrow B$ given by $(v, \alpha) \mapsto (vs_\alpha, \alpha)$, where

$$\begin{aligned} A &= \{(v, \alpha) \in W \times \Phi^+ \mid v \leq^L w \text{ and } \alpha \in B_v\}, \\ B &= \{(u, \alpha) \in W \times \Phi^+ \mid u \leq^L ws_\alpha \text{ and } \alpha \in B_w\}. \end{aligned}$$

Let $(v, \alpha) \in A$. Then $w = (wv^{-1})(v)$ is length-additive since $v \leq^L w$ and $v = (vs_\alpha)(s_\alpha)$ is length-additive because $\ell(s_\alpha) = \langle \alpha^\vee, 2\rho \rangle - 1$ and $\alpha \in B_v$. Therefore $w = (wv^{-1})(vs_\alpha)(s_\alpha)$ is length-additive. It follows that $ws_\alpha = (wv^{-1})(vs_\alpha)$ is length-additive and that $vs_\alpha \leq^L ws_\alpha$. Moreover we have $\ell(ws_\alpha) = \ell(wv^{-1}) + \ell(vs_\alpha) = \ell(wv^{-1}) + \ell(v) + 1 - \langle \alpha^\vee, 2\rho \rangle = \ell(w) + 1 - \langle \alpha^\vee, 2\rho \rangle$. Therefore $(vs_\alpha, \alpha) \in B$.

Conversely suppose $(u, \alpha) \in B$. Let $v = us_\alpha$. Arguing as before, we have that $w = (ws_\alpha u^{-1})(u)(s_\alpha) = (wv^{-1})(v)$ is length-additive. We deduce that $v \leq^L w$ and that $\ell(v) = \ell(w) - \ell(wv^{-1}) = \ell(ws_\alpha) + 1 - \langle \alpha^\vee, 2\rho \rangle - \ell(wv^{-1}) = \ell(vs_\alpha) + 1 - \langle \alpha^\vee, 2\rho \rangle$ so that $\alpha \in B_v$, as required. $\square$

5. PROOF OF THEOREM 3.4

Let I^a be the ideal in $\mathbb{Z}[x, a]$ generated by $e_i^p(x) - e_i^p(a)$ for $p \geq n$ and $i \geq 1$, and a_i for $i > n$. Let $J^a \subset \mathbb{Z}[x, a]$ be the $\mathbb{Z}[a]$ -submodule spanned by $\mathfrak{S}_w(x; a)$ for $w \in S_\infty \setminus S_n$ and $a_i \mathfrak{S}_u(x; a)$ for $i > n$ and any $u \in S_\infty$. We shall show that $I^a = J^a$. Let $c_{i,p} = s_{p+1-i} \cdots s_{p-2} s_{p-1} s_p \in S_\infty \setminus S_p$ be the cycle of length i . We note that the family $\{e_i^p(x) - e_i^p(a) \mid 1 \leq i \leq p\}$ is unitriangular over $\mathbb{Z}[a]$ with the family $\{\mathfrak{S}_{c_{i,p}}(x; a) \mid 1 \leq i \leq p\}$. Since $\mathbb{Z}[x, a] = \bigoplus_{u \in S_\infty} \mathbb{Z}[a] \mathfrak{S}_u(x; a)$, to show that $I^a \subset J^a$ it suffices to show that $\mathfrak{S}_u(x; a) \mathfrak{S}_{c_{i,p}}(x; a) \in J^a$ for all $p \geq n$, $i \geq 1$, and $u \in S_\infty$. But this follows from the fact that the product of $\mathfrak{S}_u(x; a) \mathfrak{S}_v(x; a)$ is a $\mathbb{Z}[a]$ -linear combination of $\mathfrak{S}_w(x; a)$ where $w \geq u$ and $w \geq v$.

Let K be the ideal in $\mathbb{Z}[x, a]$ generated by a_i for $i > n$. Then I^a/K has a $\mathbb{Z}[a_1, \dots, a_n]$ -basis given by standard monomials e_I with $i_r > 0$ for some $r \geq n$, while J^a/K has a $\mathbb{Z}[a_1, \dots, a_n]$ -basis given by $\mathfrak{S}_w(x; a)$ for $w \in S_\infty \setminus S_n$. The quotient ring $\mathbb{Z}[x, a]/K$ has a $\mathbb{Z}[a_1, \dots, a_n]$ -basis given by all standard monomials e_I for $I = (i_1, i_2, \dots)$ with $0 \leq i_p \leq p$ for all $p \geq 1$ and almost all i_p zero, and also by all double Schubert polynomials $\mathfrak{S}_w(x; a)$ for $w \in S_\infty$. But the standard monomials e_I with $i_n = i_{n+1} = \cdots = 0$ are in graded bijection with the $\mathfrak{S}_w(x; a)$ for $w \in S_n$. It follows that $I^a = J^a$ by graded dimension counting.

Let J_∞^{qa} be the ideal of $\mathbb{Z}[x, q, a]$ generated by $E_i^p - e_i^p(a)$ for all $i \geq 1$ and $p \geq n$, together with q_i for $i \geq n$ and a_i for $i > n$. We wish to show that

$$(5.1) \quad \tilde{\mathfrak{S}}_w(x; a) \in J_\infty^{qa} \quad \text{for all } w \in S_\infty \setminus S_n.$$

For this it suffices to show that

$$(5.2) \quad \theta(J_\infty^a) \subset J_\infty^{qa}.$$

To prove (5.2) it suffices to show that

$$(5.3) \quad \theta(e_I(e_i^p(x) - e_i^p(a))) \in J_\infty^{qa} \quad \text{for standard monomials } e_I, i \geq 1 \text{ and } p \geq n.$$

To apply θ to this element we must express $e_I e_i^p$ in standard monomials. The only nonstandardness that can occur is if $i_p > 0$. In that case one may use [FGP, (3.2)]:

$$e_i^p e_j^p = e_{i-1}^p e_{j+1}^p + e_j^p e_i^{p+1} - e_{i-1}^p e_{j+1}^{p+1}.$$

Ultimately, the straightening of $e_I e_i^p$ into standard monomials only changes factors of the form e_k^q for $k \geq 1$ and $q \geq p$.

Let $E_I = \prod_{r \geq 1} E_{i_r}^r$ for $I = (i_1, i_2, \dots)$. If we consider $E_I(E_i^p - e_i^p(a))$ and use [FGP, (3.6)]

$$E_i^p E_j^p = E_{i-1}^p E_{j+1}^p + E_j^p E_i^{p+1} - E_{i-1}^p E_{j+1}^{p+1} + q_p(E_{j-1}^{p-1} E_{i-1}^p - E_{i-2}^{p-1} E_j^p)$$

to rewrite it into quantized standard monomials, we see that the two straightening processes differ only by multiples of q_p, q_{p+1} , etc. Therefore

$$\theta(e_I(E_i^p(x) - e_i^p(a))) - E_I(E_i^p - e_i^p(a)) \in J_\infty^{qa}.$$

But $E_I(E_i^p - e_i^p(a)) \in J_\infty^{qa}$, so (5.3) holds and (5.1) follows.

The ring $\mathbb{Z}[x, q, a]/J_\infty^{qa}$ has a $\mathbb{Z}[q_1, \dots, q_{n-1}; a_1, \dots, a_n]$ -basis given by $\tilde{\mathfrak{S}}_w(x; a)$ for $w \in S_n$. This follows from Lemmata 3.1 and 3.2. Moreover this basis satisfies the equivariant quantum Chevalley-Monk rule for SL_n by Proposition 4.8. By Theorem 2.1 there is an isomorphism of $\mathbb{Z}[q_1, \dots, q_{n-1}; a_1, \dots, a_n]$ -algebras $QH^T(SL_n/B) \rightarrow \mathbb{Z}[x, q, a]/J_\infty^{qa}$. Moreover, σ_T^w and $\tilde{\mathfrak{S}}_w(x; a)$ (or, rather, its pre-image in $QH^T(\text{Fl})$) are related by an automorphism of $QH^T(\text{Fl})$. But the Schubert divisor class $\sigma_T^{s_i}$ is (by definition) represented by a usual double Schubert polynomial $\mathfrak{S}_{s_i}(x; a) = \tilde{\mathfrak{S}}_{s_i}(x; a)$ (Lemma 3.5) in Kim's presentation, and these divisor classes generate $QH^T(\text{Fl})$ over $\mathbb{Z}[q_1, \dots, q_{n-1}; a_1, \dots, a_n]$. Thus the automorphism must be the identity, completing the proof.

6. PARABOLIC CASE

6.1. Notation. Fix a composition $(n_1, n_2, \dots, n_k) \in \mathbb{Z}_{>0}^k$ with $n_1 + n_2 + \dots + n_k = n$. Let $P \subset SL_n(\mathbb{C})$ be the parabolic subgroup consisting of block upper triangular matrices with block sizes $n_1, n_2, \dots, n_k$. Then SL_n/P is isomorphic to the variety of partial flags in $\mathbb{C}^n$ with subspaces of dimensions $N_j := n_1 + n_2 + \dots + n_j$ for $0 \leq j \leq k$. Denote by W_P the Weyl group for the Levi factor of P and by W^P the set of minimum length coset representatives in W/W_P . For every $w \in W$ there exist unique elements $w^P \in W^P$ and $w_P \in W_P$ such that $w = w^P w_P$; moreover this factorization is length-additive. Let $w_0 \in W$ be the longest element and let $w_0 = w_0^P w_{0,P}$ so that $w_{0,P} \in W_P$ is the longest element.

Example 6.1. Let $(n_1, n_2, n_3) = (2, 1, 3)$. Then $(N_1, N_2, N_3) = (2, 3, 6)$, $w_0^P = 564123$ (that is, $w \in S_6$ is the permutation with $w(1) = 5$, $w(2) = 6$, etc.), and $w_{0,P} = 213654$.

6.2. Parabolic quantum double Schubert polynomials. In this section let $x = (x_1, \dots, x_n)$ and $q = (q_1, \dots, q_{k-1})$. Let $\mathbb{Z}[x; q]$ be the graded polynomial ring with $\deg(x_i) = 1$ and $\deg(q_j) = n_j + n_{j+1}$ for $1 \leq j \leq k-1$. Following [AS], [Cio2] let $D = D^P$ be the $n \times n$ matrix with entries x_i on the diagonal, -1 on the superdiagonal, and entry $(N_{j+1}, N_{j-1} + 1)$ given by $-(-1)^{n_{j+1}} q_j$ for $1 \leq j \leq k-1$. For $1 \leq j \leq k$ let D_j be the upper left $N_j \times N_j$ submatrix of D and for $1 \leq i \leq N_j$ define the elements $G_i^j \in \mathbb{Z}[x; q]$ by

$$\det(D_j - t \text{Id}) = \sum_{i=0}^{N_j} (-t)^{N_j-i} G_i^j.$$

For $w \in W^P$, we introduce the parabolic quantum double Schubert polynomial

$$(6.1) \quad \tilde{\mathfrak{S}}_w^P(x; a) = (-1)^{\ell(w(w_0^P)^{-1})} \partial_{w(w_0^P)^{-1}}^a \prod_{j=1}^{k-1} \prod_{i=n-N_{j+1}+1}^{n-N_j} \det(D_j - a_i \text{Id}).$$

Example 6.2. Continuing Example 6.1 we have

$$\begin{aligned}\tilde{\mathfrak{S}}_{w_0^P}^P(x; a) &= \det(D_1 - a_4 \text{Id}) \det(D_2 - a_1 \text{Id}) \det(D_2 - a_2 \text{Id}) \det(D_2 - a_3 \text{Id}) \\ &= (x_1 - a_4)(x_2 - a_4) \prod_{i=1}^3 ((x_1 - a_i)(x_2 - a_i)(x_3 - a_i) + q_1).\end{aligned}$$

Example 6.3. For the Grassmannian $\text{Gr}(r, n)$ of r -dimensional subspaces of $\mathbb{C}^n$ we have $k = 2$, $(n_1, n_2) = (r, n - r)$, $(N_0, N_1, N_2) = (0, r, n)$, and

$$\tilde{\mathfrak{S}}_w^P(x; a) = (-1)^{\ell(w(w_0^P)^{-1})} \partial_{w(w_0^P)^{-1}}^a \prod_{i=1}^{n-r} \det(D_1 - a_i \text{Id}).$$

Here D_1 is the $r \times r$ matrix with diagonal entries x_i for $1 \leq i \leq r$ and super-diagonal entries -1 ; the variable q_1 does not appear. One may verify that in this case the parabolic quantum double Schubert polynomial is just a double Schubert polynomial $\mathfrak{S}_w(x; a)$ for $w \in W^P$; these are also known as double or factorial Schur polynomials. This agrees with [Mi2].

The parabolic quantum double Schubert polynomials $\tilde{\mathfrak{S}}_w^P(x; a)$ have specializations similar to the quantum double Schubert polynomials. Let $w \in W^P$.

- (1) We define the parabolic quantum Schubert polynomials by the specialization $\tilde{\mathfrak{S}}_w^P(x) = \tilde{\mathfrak{S}}_w^P(x; 0)$ which sets $a_i = 0$ for all i . In Lemma 6.9 it is shown that these polynomials coincide with those of Ciocan-Fontanine [Cio2], whose definition uses a parabolic analogue of the quantization map of [FGP].
- (2) Setting $q_i = 0$ for all i , one obtains the double Schubert polynomial $\mathfrak{S}_w(x; a)$.
- (3) Setting both a_i and q_i to zero, one obtains the Schubert polynomial $\mathfrak{S}_w(x)$.

Let J_P be the ideal in $\mathbb{Z}[x]^{W_P}$ (resp. $J_P^a \subset S[x]^{W_P}$, $J_P^q \subset \mathbb{Z}[x]^{W_P}[q]$, $J_P^{qa} \subset S[x]^{W_P}[q]$) generated by the elements $e_i^n(x)$ (resp. $e_i^n(x) - e_i^n(a)$, G_i^k , $G_i^k - e_i^n(a)$) for $1 \leq i \leq n$. The aim of this section is to establish (4) of the following theorem.

Theorem 6.4.

- (1) *There is an isomorphism of $\mathbb{Z}$ -algebras [BGG, LS]:*

$$\begin{aligned}H^*(SL_n/P) &\cong \mathbb{Z}[x]^{W_P}/J_P \\ [X_w] &\mapsto \mathfrak{S}_w(x) + J_P.\end{aligned}$$

- (2) *There is an isomorphism of S -algebras [Bi]:*

$$\begin{aligned}H^T(SL_n/P) &\cong S[x]^{W_P}/J_P^a \\ [X_w]_T &\mapsto \mathfrak{S}_w(x; a) + J_P^a.\end{aligned}$$

- (3) *There is an isomorphism of $\mathbb{Z}[q]$ -algebras [AS], [Kim2], [Kim3]:*

$$\begin{aligned}QH^*(SL_n/P) &\cong \mathbb{Z}[x]^{W_P}[q]/J_P^q \\ \sigma^{P,w} &\mapsto \tilde{\mathfrak{S}}_w^P(x) + J_P^q.\end{aligned}$$

- (4) *There is an isomorphism of $S[q]$ -algebras:*

$$(6.2) \quad QH^T(SL_n/P) \cong S[x]^{W_P}[q]/J_P^{qa}$$

$$(6.3) \quad \sigma_T^{P,w} \mapsto \tilde{\mathfrak{S}}_w^P(x; a) + J_P^{qa}.$$

Here $[X_w]$, $[X_w]_T$, $\sigma^{P,w}$, and $\sigma_T^{P,w}$ denote the Schubert bases for their respective cohomology rings for $w \in W^P$.

The isomorphism (6.2) is due to [Kim3]. We shall establish (6.3), namely, that under this isomorphism, the cosets of parabolic quantum double Schubert polynomials are the images of parabolic equivariant quantum Schubert classes.

Example 6.5. Continuing Example 6.3, the relations for $QH^T(\mathrm{Gr}(r, n))$ equate $e_i(a)$ with the coefficient of $(-t)^{n-i}$ in $\det(D_2 - t\mathrm{Id})$. The $n \times n$ matrix D_2 , aside from the usual diagonal entries x_i and superdiagonal entries -1 , has only one other nonzero entry $-(-1)^{n-r}q_1$, which appears in position $(n, 1)$. Therefore we have relations $e_i(x) - e_i(a)$ for $1 \leq i \leq n-1$ and $e_n(x) - e_n(a) - (-1)^{n-r}q_1$. This agrees with [AS, §5.2].

6.3. Stability of parabolic quantum double Schubert polynomials. The following lemma can be verified by direct computation and induction.

Lemma 6.6. *Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^n$ be such that $\beta_i \leq n-i$ for $2 \leq i \leq n$. Then $\partial_{n-1}^a \cdots \partial_2^a \partial_1^a \cdot a^\beta$ equals $a_1^{\beta_2} a_2^{\beta_3} \cdots a_{n-1}^{\beta_n}$ if $\beta_1 = n-1$ and 0 if $\beta_1 < n-1$.*

Suppose $w \in W^P$ is such that $w(r) = r$ for $N_{k-1} < r \leq n$. Let $w_0^{(p,q)}$ be the minimum length coset representative of the longest element in $S_{p+q}/(S_p \times S_q)$ and let w_0^{P-} be the minimum length coset representative of the longest element in $S_{N_{k-1}}/(S_{n_1} \times \cdots \times S_{n_{k-1}})$. We have the length-additive factorization $w_0^P = w_0^{(N_{k-1}, n_k)} w_0^{P-}$. Also $\ell(w(w_0^P)^{-1}) = \ell(w(w_0^{P-})^{-1}) + \ell(w_0^{(n_k, N_{k-1})})$. Using the above lemma repeatedly and using the notation $Z = \prod_{j=1}^{k-2} \prod_{i=n-N_{j+1}+1}^{n-N_j} \det(D_j - a_i \mathrm{Id})$ we have

$$\begin{aligned} (-1)^{\ell(w w_0^P)} \tilde{\mathfrak{S}}_w^P(x; a) &= \partial_{w(w_0^P)^{-1}}^a \prod_{j=1}^{k-1} \prod_{i=n-N_{j+1}+1}^{n-N_j} \det(D_j - a_i \mathrm{Id}) \\ &= \partial_{w w_0^{P-}}^a \partial_{w_0^{(n_k, N_{k-1})}}^a \cdot \left(\prod_{i=1}^{n_k} \det(D_{k-1} - a_i \mathrm{Id}) \right) Z \\ &= (-1)^{\ell(w_0^{(n_k, N_{k-1})})} \partial_{w w_0^{P-}}^a w_0^{(n_k, N_{k-1})} \cdot Z \\ &= (-1)^{\ell(w_0^{(n_k, N_{k-1})})} \partial_{w w_0^{P-}}^a \cdot Z \\ &= (-1)^{\ell(w_0^{(n_k, N_{k-1})})} (-1)^{\ell(w(w_0^{P-})^{-1})} \tilde{\mathfrak{S}}_w^{P-}(x; a). \end{aligned}$$

The final outcome is

$$\tilde{\mathfrak{S}}_w^P(x; a) = \tilde{\mathfrak{S}}_w^{P-}(x; a).$$

This means that if we append a block of size m to $n_\bullet$ and append m fixed points to $w \in W^P$, the parabolic quantum double Schubert polynomial remains the same.

6.4. Stable parabolic quantization. This section follows [Cio2]. Consider an infinite sequence of positive integers $n_\bullet = (n_1, n_2, \dots)$. Let $N_j = n_1 + \cdots + n_j$ for $j \geq 1$. Let $W = S_\infty = \bigcup_{n \geq 1} S_n$ be the infinite symmetric group, W_P the subgroup of W generated by s_i for $i \notin \{N_1, N_2, \dots\}$, W^P the set of minimum length coset representatives in W/W_P , etc. Let $\mathbb{Y}^P$ be the set of tuples of partitions

$\lambda^\bullet = (\lambda^{(1)}, \lambda^{(2)}, \dots)$ such that $\lambda^{(j)}$ is contained in the rectangle with n_{j+1} rows and N_j columns and almost all $\lambda^{(j)}$ are empty. Define

$$g_{\lambda^\bullet} = \prod_{j \geq 1} \prod_{i=1}^{n_{j+1}} g_{\lambda_i^{(j)}}^j, \quad G_{\lambda^\bullet} = \prod_{j \geq 1} \prod_{i=1}^{n_{j+1}} G_{\lambda_i^{(j)}}^j,$$

where $g_r^j = e_r^{N_j}(x)$. The following is a consequence of [Cio2] by taking a limit.

Proposition 6.7 ([Cio2]). $\{g_{\lambda^\bullet} \mid \lambda^\bullet \in \mathbb{Y}^P\}$ is a $\mathbb{Z}$ -basis of $\mathbb{Z}[x]^{W_P}$.

We also observe that

Lemma 6.8. $\{\mathfrak{S}_w(x) \mid w \in W^P\}$ is a $\mathbb{Z}$ -basis of $\mathbb{Z}[x]^{W_P}$.

Define the $\mathbb{Z}[q]$ -module automorphism θ^P of $\mathbb{Z}[x, q]$ by $\theta(g_{\lambda^\bullet}) = G_{\lambda^\bullet}$. By $\mathbb{Z}[a]$ -linearity it defines a $\mathbb{Z}[q, a]$ -module automorphism of $\mathbb{Z}[x, q, a]$, also denoted θ^P .

Lemma 6.9. For all $w \in W^P$,

$$(6.4) \quad \tilde{\mathfrak{S}}_w^P(x) = \theta^P(\mathfrak{S}_w(x)),$$

$$(6.5) \quad \tilde{\mathfrak{S}}_w^P(x; a) = \theta^P(\mathfrak{S}_w(x; a)).$$

Proof. Let $w \in W^P$. We may compute $\tilde{\mathfrak{S}}_w^P(x; a)$ by working with respect to $(n_1, n_2, \dots, n_k)$ for some finite k ; the result is independent of k by the previous subsection. Then (6.5) is an immediate consequence of the commutation of θ^P and the divided difference operators in the a variables. Equation (6.4) follows from (6.5) by setting the a variables to zero. $\square$

6.5. Cauchy formula. Keeping the notation of the previous subsection, we have

Proposition 6.10. For all $w \in W^P$ we have

$$(6.6) \quad \tilde{\mathfrak{S}}_w^P(x; a) = \sum_{v \leq^L w} \mathfrak{S}_{vw^{-1}}(-a) \tilde{\mathfrak{S}}_v^P(x).$$

Proof. Observe that if $v \leq^L w$, then $v \in W^P$ so that $\tilde{\mathfrak{S}}_v^P(x)$ makes sense. We have

$$\begin{aligned} \tilde{\mathfrak{S}}_w^P(x; a) &= \theta^P(\mathfrak{S}_w(x; a)) \\ &= \theta^P\left(\sum_{v \leq^L w} \mathfrak{S}_{vw^{-1}}(-a) \mathfrak{S}_v(x)\right) \\ &= \sum_{v \leq^L w} \mathfrak{S}_{vw^{-1}}(-a) \tilde{\mathfrak{S}}_v^P(x). \end{aligned}$$

$\square$

6.6. Parabolic Chevalley-Monk rules. Fix $n_\bullet = (n_1, n_2, \dots, n_k)$ such that $\sum_{j=1}^k n_j = n$ and let P be the parabolic defined by $n_\bullet$, and so on. Let $Q_P^\vee$, Φ_P^+ , and ρ_P be respectively the coroot lattice, the positive roots, and the half sum of positive roots for the Levi factor of P . Let $\eta_P : Q^\vee \rightarrow Q^\vee / Q_P^\vee$ be the natural projection. Let $\pi_P : W \rightarrow W^P$ be the map $w \mapsto w^P$ where $w = w^P w_P$ and $w^P \in W^P$ and $w_P \in W_P$. Define the sets of roots

$$(6.7) \quad A_{P,w}^n = \{\alpha \in \Phi^+ \setminus \Phi_P^+ \mid ws_\alpha \geq w \text{ and } ws_\alpha \in W^P\},$$

$$(6.8) \quad B_{P,w}^n = \{\alpha \in \Phi^+ \setminus \Phi_P^+ \mid \ell(\pi_P(ws_\alpha)) = \ell(w) + 1 - \langle \alpha^\vee, 2(\rho - \rho_P) \rangle\}.$$

For $\lambda = \sum_{i=1}^{k-1} b_i \alpha_{N_i}^\vee \in Q^\vee / Q_P^\vee$ with $b_i \in \mathbb{Z}_{\geq 0}$ we let $q_\lambda = \prod_i q_i^{b_i}$. The following is Mihalcea's characterization of $QH^T(SL_n/P)$ which extends Theorem 2.1.

Theorem 6.11 ([Mi]). *For $w \in W^P$ and $i \in \{N_1, N_2, \dots, N_{k-1}\}$ (that is, $s_i \in W^P$), we have*

$$\begin{aligned} \sigma_T^{P, s_i} \sigma_T^{P, w} &= (-\omega_i^a + w \cdot \omega_i^a) \sigma_T^{P, w} \\ &+ \sum_{\alpha \in A_{P, w}^n} \langle \alpha^\vee, \omega_i \rangle \sigma_T^{P, ws_\alpha} + \sum_{\alpha \in B_{P, w}^n} \langle \alpha^\vee, \omega_i \rangle q_{\eta_P(\alpha^\vee)} \sigma_T^{P, \pi_P(ws_\alpha)}. \end{aligned}$$

Moreover these structure constants determine the Schubert basis $\{\sigma_T^{P, w} \mid w \in W^P\}$ and the ring $QH^T(SL_n/P)$ up to isomorphism as $\mathbb{Z}[q_1, \dots, q_{k-1}; a_1, \dots, a_n]$ -algebras.

Now let $(n_1, n_2, \dots)$ be an infinite sequence of positive integers. Let $A_{P, w}$ and $B_{P, w}$ be the analogues of the sets of roots defined in (6.7) and (6.8) where $\rho = (0, -1, -2, \dots)$ and ρ_P is the juxtaposition of $(0, -1, \dots, 1 - n_j)$ for $j \geq 1$.

Proposition 6.12. *For $w \in W^P$ and i such that $i \in \{N_1, N_2, \dots\}$ (that is, $s_i \in W^P$), the parabolic quantum double Schubert polynomials satisfy the identity*

$$\begin{aligned} \tilde{\sigma}_{s_i}^P(x; a) \tilde{\sigma}_w^P(x; a) &= (-\omega_i^a + w \cdot \omega_i^a) \tilde{\sigma}_w^P(x; a) + \sum_{\alpha \in A_{P, w}} \langle \alpha^\vee, \omega_i \rangle \tilde{\sigma}_{ws_\alpha}^P(x; a) \\ &+ \sum_{\alpha \in B_{P, w}} \langle \alpha^\vee, \omega_i \rangle q_{\eta_P(\alpha^\vee)} \tilde{\sigma}_{\pi_P(ws_\alpha)}^P(x; a). \end{aligned}$$

Sketch of the proof. The proof proceeds entirely analogously to that of Proposition 4.8, starting with the Chevalley-Monk formula for double Schubert polynomials (applied in the special case that $w \in W^P$ and $s_i \in W^P$) and reducing to the following identity:

$$\begin{aligned} &\sum_{v \leq^L w} \mathfrak{S}_{vw^{-1}}(-a) \sum_{\alpha \in B_{P, v}} q_{\eta_P(\alpha^\vee)} \langle \alpha^\vee, \omega_i \rangle \tilde{\sigma}_{\pi_P(vs_\alpha)}^P(x) \\ &= \sum_{\alpha \in B_{P, w}} \langle \alpha^\vee, \omega_i \rangle q_{\eta_P(\alpha^\vee)} \sum_{v \leq^L \pi_P(ws_\alpha)} \mathfrak{S}_{v(\pi_P(ws_\alpha))^{-1}}(-a) \tilde{\sigma}_v^P(x). \end{aligned}$$

For this it suffices to show that there is a bijection $(v, \alpha) \mapsto (\pi_P(vs_\alpha), \alpha)$ and inverse bijection $(u, \alpha) \mapsto (\pi_P(us_\alpha), \alpha)$ between the sets

$$\begin{aligned} A &= \{(v, \alpha) \in W^P \times (\Phi^+ \setminus \Phi_P^+) \mid v \leq^L w \text{ and } \alpha \in B_{P, v}\}, \\ B &= \{(u, \alpha) \in W^P \times (\Phi^+ \setminus \Phi_P^+) \mid u \leq^L \pi_P(ws_\alpha) \text{ and } \alpha \in B_{P, w}\} \end{aligned}$$

such that for $(v, \alpha) \in A$ we have

$$(6.9) \quad vw^{-1} = \pi_P(vs_\alpha)(\pi_P(ws_\alpha))^{-1}.$$

To establish this bijection we use [LamSh, Lemma 10.14], which asserts that elements $\alpha \in B_{P, w}$ automatically satisfy the additional condition $\ell(ws_\alpha) = \ell(w) + 1 - \langle \alpha^\vee, 2\rho \rangle$. Let $(v, \alpha) \in A$. Then we have length-additive factorizations $w = (wv^{-1}) \cdot v$, $v = (vs_\alpha) \cdot s_\alpha$, and $vs_\alpha = \pi_P(vs_\alpha) \cdot x$ for some $x \in W_P$. Therefore

$w = (wv^{-1}) \cdot \pi_P(vs_\alpha) \cdot x \cdot s_\alpha$ is length-additive. This implies that $ws_\alpha = (wv^{-1}) \cdot \pi_P(vs_\alpha) \cdot x$ is length-additive. One may deduce from this that $\pi_P(ws_\alpha) = (wv^{-1})\pi_P(vs_\alpha)$ and therefore that (6.9) holds. The conditions that $(\pi_P(vs_\alpha), \alpha) \in B$ are readily verified. Thus the map $A \rightarrow B$ is well-defined. The rest of the proof is similar. $\square$

6.7. Proof of Theorem 6.4(4). The proof is analogous to that of Theorem 3.4.

Given the finite sequence $(n_1, \dots, n_k)$ and parabolic subgroup P , consider the extension $(n_1, \dots, n_k, 1, 1, 1, \dots)$ by an infinite sequence of 1s. We use the notation P_∞ to label the corresponding objects. That is, $W_P \cong W_{P_\infty}$ is the subgroup of S_∞ , W^{P_∞} is the set of minimum-length coset representatives in S_∞/W_{P_∞} , and so on.

Let $R_P = \mathbb{Z}[x, a]^{W_{P_\infty}}$ and $R_P^q = \mathbb{Z}[x, q, a]^{W_{P_\infty}}$, where $x = (x_1, x_2, \dots)$, $q = (q_1, q_2, \dots)$, and $q_1, q_2, \dots, q_{k-1}$ are identified with the quantum parameters in Theorem 6.4(4). Define $J_{P,\infty}^q$ to be the ideal of R_P generated by $g_i^p(x) - e_i^{N_p}(a)$ for $i \geq 1$ and $p \geq k$ and by a_i for $i > n$. Note that for $p = k + j$ with $j \geq 0$ we have $g_i^p(x) - e_i^{N_p}(a) = e_i^{n+j}(x) - e_i^{n+j}(a)$.

Let $I_{P,\infty}^a$ be the $\mathbb{Z}[a]$ -submodule of R_P spanned by $\mathfrak{S}_w(x; a)$ for $w \in W^{P_\infty} \setminus S_n$, and by $a_i \mathfrak{S}_u(x; a)$ for $i > n$ and any $u \in W^{P_\infty}$. We claim that $I_{P,\infty}^a$ is an ideal. This follows easily from the fact that the only double Schubert polynomials occurring in the expansion of a product $\mathfrak{S}_w(x; a) \mathfrak{S}_v(x; a)$ lie above w and v in Bruhat order. But then it follows from Theorem 6.4(2) and dimension counting that $I_{P,\infty}^a = J_{P,\infty}^a$.

Define $J_{P,\infty}^{qa}$ to be the ideal of R_P^q generated by $G_i^p - e_i^{N_p}(a)$ for $p \geq k$ and $i \geq 1$, and a_i for $i > n$, and q_i for $i \geq k$. We claim that $\tilde{\mathfrak{S}}_w^P(x; a) \in J_{P,\infty}^{qa}$ for $w \in W^{P_\infty} \setminus S_n$. This would follow from the definitions if we could establish that $\theta^{P_\infty}(J_{P,\infty}^{qa}) \subset J_{P,\infty}^{qa}$. Since θ^{P_∞} is trivial on the equivariant variables $a_1, a_2, \dots$, it suffices to show that $\theta^{P_\infty}(g_{\lambda^\bullet}^P \cdot (g_i^p(x) - e_i^{N_p}(a))) \in J_{P,\infty}^{qa}$, for each $i \geq 1$, $p \geq k$, and each standard monomial $g_{\lambda^\bullet}^P$. To apply θ^{P_∞} to $g_{\lambda^\bullet}^P \cdot g_i^p$ we must first standardize this monomial. This can be achieved by using a parabolic version of the straightening algorithm of [FGP, Proposition 3.3]. The only nonstandard part of $g_{\lambda^\bullet}^P \cdot g_i^p$ is the possible presence of a factor $g_j^p g_i^p$. This can be standardised using only the (non-parabolic) relation [FGP, (3.2)] – any factors $g_m^{k'}$ for $k' < k$ in $g_{\lambda^\bullet}^P \cdot g_i^p$ are not modified. Similarly, the product $G_{\lambda^\bullet}^P \cdot G_i^p$ can be standardized using the following variant of [FGP, Lemma 3.5] which can be deduced from [Cio2, (3.5)]:

$$G_i^p G_{j+1}^{p+1} + G_{i+1}^p G_j^p \pm q_p G_{i-n_{p+1}-n_p}^{p-1} G_j^p = G_j^p G_{i+1}^{p+1} + G_{k+1}^p G_i^p \pm q_p G_{j-n_{p+1}-n_p}^{p-1} G_i^p.$$

We note that when $p \geq k$, this relation only involves quantum variables $q_k, q_{k+1}, \dots$. Thus modulo $q_k, q_{k+1}, \dots$, the straightening relation for $g_{\lambda^\bullet}^P \cdot g_i^p$ and for $G_{\lambda^\bullet}^P \cdot G_i^p$ coincide. It follows that $\theta^{P_\infty}(g_{\lambda^\bullet}^P \cdot (g_i^p(x) - e_i^{N_p}(a))) = G_{\lambda^\bullet}^P \cdot (G_i^p - e_i^{N_p}(a)) \pmod{J_{P,\infty}^{qa}}$, and thus $\tilde{\mathfrak{S}}_w^P(x; a) \in J_{P,\infty}^{qa}$ for $w \in W^{P_\infty} \setminus S_n$. But $R_{P,\infty}^{qa}/J_{P,\infty}^{qa}$ has rank $|W^P|$ over $\mathbb{Z}[q_1, \dots, q_{k-1}, a_1, \dots, a_n]$, and so it follows that $J_{P,\infty}^{qa}$ is spanned by $\tilde{\mathfrak{S}}_w^P(x; a) \in J_{P,\infty}^{qa}$ for $w \in W^{P_\infty} \setminus S_n$ together with $q_i \tilde{\mathfrak{S}}_u^P(x; a)$ for $i \geq k$, $a_i \tilde{\mathfrak{S}}_u^P(x; a)$ for $i > n$ and any $u \in W^{P_\infty}$.

Theorem 6.4(4) now follows from Proposition 6.12 and the determination of $QH^T(SL_n/P)$ in Theorem 6.11.

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