THERE IS NO STRICTLY SINGULAR CENTRALIZER ON L_p

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ABSTRACT. We prove that if Φ is a centralizer on L_p , where $0 < p < \infty$, then there is a copy of ℓ_2 inside L_p where Φ is bounded. If Φ is symmetric, then it is also bounded on a copy of ℓ_q , provided $0 < p < q < 2$. This shows that for a wide class of exact sequences $0 \rightarrow L_p \rightarrow Z \rightarrow L_p \rightarrow 0$ the quotient map is not strictly singular, which generalizes a recent result of Jesús Suárez.

1. INTRODUCTION

An operator acting between Banach or quasi-Banach spaces is said to be strictly singular if it is not an isomorphism on any infinite dimensional subspace of its domain.

Exact sequences of Banach or quasi-Banach spaces $0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$ in which the quotient map $\pi : Z \rightarrow X$ is strictly singular has spurred moderate interest since the early studies on the ‘three space problem’. Let us call them ‘strictly singular sequences’. In some sense, if one has a strictly singular sequence in which the spaces X and Y are ‘nice’, the middle space Z must be ‘exotic’.

Amongst the most striking examples of this phenomenon one finds that for each $p \in (0, \infty)$ there is a strictly singular sequence

$$(1) \quad 0 \longrightarrow \ell_p \longrightarrow Z_p \longrightarrow \ell_p \longrightarrow 0.$$

These were constructed by Kalton and Peck in [9]; see also [3].

More often than not the construction of strictly singular sequences is achieved by means of a quasilinear map from X to Y , and this is certainly the case for the Kalton-Peck sequences, whose associated quasilinear maps are centralizers (a special type of quasilinear map; see Section 1.2). There is a function space analogue of (1),

$$(2) \quad 0 \longrightarrow L_p \longrightarrow ZF_p \longrightarrow L_p \longrightarrow 0,$$

whose associated quasilinear map is the ‘classical’ centralizer

$$\Omega(f) = f \log \left(\frac{|f|}{\|f\|} \right).$$

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The space ZF_p was introduced in [5], although it arises quite naturally in interpolation theory; see [10, Section 3D].

Very recently Jesús Suárez proved the following remarkable results on the behaviour of Ω on L_p :

- (a) For every $0 < p < \infty$, there is a copy of ℓ_2 in L_p where the restriction of Ω is bounded.
- (b) If $0 < p < q < 2$, then Ω is bounded on a copy of ℓ_q inside L_p .

See [12, Propositions 3.1 and 4.1]. Roughly this means that the sequence (2) is not strictly singular because the quotient map is invertible on an isomorphic copy of ℓ_2 (or ℓ_q) inside L_p .

The purpose of this short note is to prove that (a) holds for all centralizers and (b) holds (at least) for symmetric centralizers. Our approach is based on a result by Kalton that describes centralizers as differentials of interpolation scales of Köthe function spaces from [7]. We also use results from [6] and a recent result of the author on the behaviour of centralizers acting between two different Lebesgue spaces [2].

1.1. Function spaces. Let L_0 denote the space of all real or complex measurable functions on the unit interval $\mathbb{I}$, where we identify two functions if they agree almost everywhere with respect to Lebesgue measure. A function space X is a linear subspace of L_0 , together with a quasinorm $\|\cdot\|$ having the following properties:

- The unit ball $B_X = \{f \in X : \|f\| \leq 1\}$ is closed in L_0 for the topology of convergence in measure.
- If $f \in X, g \in L_0$ and $|g| \leq |f|$, then $g \in X$ and $\|g\| \leq \|f\|$.

Important examples of function spaces are the spaces L_p for $0 < p \leq \infty$.

Given a function space X and $A \subset \mathbb{I}$ we write $X(A)$ for the space of those functions in X vanishing outside A .

We consider Köthe function spaces in the sense of [7]. Thus they are Banach function spaces whose norm satisfies the inequalities $\|hx\|_1 \leq \|x\|_X \leq \|kx\|_\infty$ for some everywhere positive functions h, k and for every $x \in X$.

1.2. Centralizers and extensions. Let X and Y be function spaces. A centralizer from X to Y is a homogeneous mapping $\Phi : X \rightarrow L_0$ satisfying the following condition: there is a constant C such that, for every $a \in L_\infty$ and for every $f \in X$, the difference $\Phi(af) - a\Phi(f)$ belongs to Y and

$$\|\Phi(af) - a\Phi(f)\|_Y \leq C\|a\|_\infty\|f\|_X.$$

When $Y = X$ we say that Φ is a centralizer on X .

Although we will not use it, we remark that every centralizer is quasilinear; that is, there is a constant Q such that for every $f, g \in X$ the difference $\Phi(f+g) - \Phi f - \Phi g$ falls in Y and one has $\|\Phi(f+g) - \Phi f - \Phi g\|_Y \leq Q(\|f\|_X + \|g\|_X)$.

A centralizer from X to Y gives rise to an exact sequence

$$0 \longrightarrow Y \xrightarrow{\iota} Y \oplus_\Phi X \xrightarrow{\pi} X \longrightarrow 0$$

as follows:

- The middle space is $Y \oplus_\Phi X = \{(g, f) \in L_0 \times X : g - \Phi(f) \in Y\}$ with the quasinorm given by $\|(g, f)\|_\Phi = \|g - \Phi f\|_Y + \|f\|_X$.
- $\iota(g) = (g, 0)$ and $\pi(g, f) = f$.

Actually only quasilinearity of Φ is required here.

We say that two centralizers Φ and Ψ are equivalent, and we write $\Phi \approx \Psi$ if the difference takes values in Y and is bounded in the sense that $\|\Phi(f) - \Psi(f)\|_Y \leq B\|f\|_X$ for some B and every $f \in X$.

Let U be a subspace of X and suppose Φ is bounded on U in the sense that Φ maps U into Y (not L_0) and $\|\Phi(u)\|_Y \leq B\|u\|_X$ for some constant B and every $u \in U$. Then the map $s : U \rightarrow Y \oplus_\Phi X$ defined by $s(u) = (0, u)$ is a bounded linear operator and $\pi \circ s = I_U$. Thus π cannot be strictly singular if Φ is bounded on some infinite dimensional subspace of X .

Important examples of centralizers are the following (see [6], Section 3 and especially Theorem 3.1). Let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a Lipschitz function vanishing at the origin. Then the map $L_p \rightarrow L_0$ given by

$$f \mapsto f\varphi\left(\log \frac{|f|}{\|f\|_p}, \log \frac{|r_f|}{\|f\|_p}\right)$$

is a (real, symmetric) centralizer on L_p . Here r_g is the so-called rank-function of $g \in L_0$ defined by

$$r_g(t) = \lambda\{s \in \mathbb{R}^+ : |g(s)| > |g(t)| \text{ or } s \leq t \text{ and } |g(s)| = |g(t)|\},$$

which arises in real interpolation.

1.3. Real centralizers. Let X be a complex function space and let $X^{\mathbb{R}} = \Re(X)$ be a corresponding real function space. A centralizer on X is said to be real if it sends real functions into real functions. Clearly, every real centralizer on X induces a centralizer on $X^{\mathbb{R}}$ by restriction. On the other hand each centralizer Φ on $X^{\mathbb{R}}$ extends to a real centralizer on X by the formula $\Phi^{\mathbb{C}}f = \Phi(u) + i\Phi(v)$, where $u = \Re f$ and $v = \Im f$. These processes are each inverse of the other, up to equivalence.

Moreover, if Φ is any centralizer on X , there are real centralizers Φ_1 and Φ_2 such that $\Phi \approx \Phi_1 + i\Phi_2$; see [7, Lemma 7.1].

2. RESULTS

Let A be a Borel subset of $\mathbb{I}$. A Rademacher sequence in A is a sequence (r_n) in $L_0(A)$ such that $\lambda(\{t \in A : r_n(t) = 1\}) = \lambda(\{t \in A : r_n(t) = -1\}) = \frac{1}{2}\lambda(A)$ for all n and $\mathbb{E}[r_n r_m | A] = 0$ for $n \neq m$.

Khintchine's inequality states that if (r_n) is a Rademacher sequence and (t_n) is in ℓ_2 , then $f = \sum_n t_n r_n$ belongs to L_s for every $s \in (0, \infty)$ and, moreover, there is a constant M depending only on s and $\lambda(A)$ such that

$$M^{-1}\|(t_n)\|_{\ell_2} \leq \|f\|_s \leq M\|(t_n)\|_{\ell_2}.$$

Thus a Rademacher sequence spans a subspace isomorphic to ℓ_2 in L_s for any $s \in (0, \infty)$.

Our first result is based on certain ideas from complex interpolation. Let us indicate the minimal background one needs to understand the proof.

Let X_0 and X_1 be (complex) Köthe function spaces on the unit interval. Consider the closed strip $\mathbb{S} = \{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$ and let $\mathcal{F}(X_0, X_1)$ denote the space of bounded, continuous functions $F : \mathbb{S} \rightarrow X_0 + X_1$ having the following properties:

- F is analytic on the interior of $\mathbb{S}$.
- $F(k + it) \in X_k$ for each $k = 0, 1$ and all $t \in \mathbb{R}$.
- For $k = 0, 1$, the map $t \in \mathbb{R} \mapsto F(k + it) \in X_k$ is bounded and continuous.

Then $\mathcal{F} = \mathcal{F}(X_0, X_1)$ is a Banach space under the norm

$$\|F\|_{\mathcal{F}} = \sup\{\|F(k + it)\|_{X_k} : t \in \mathbb{R}, k = 0, 1\}.$$

For $\theta \in [0, 1]$ we define the interpolation space

$$X_{\theta} = [X_0, X_1]_{\theta} = \{f \in L_0 : f = F(\theta) \text{ for some } F \in \mathcal{F}\}$$

with the (quotient) norm $\|f\|_{X_{\theta}} = \inf\{\|F\|_{\mathcal{F}} : f = F(\theta)\}$.

The equation $[X_0, X_1]_{\theta} = X$ induces a ‘derivation’ on X as follows. We fix a small $\epsilon > 0$ and for each $f \in X$ we choose $F \in \mathcal{F}(X_0, X_1)$ such that $F(\theta) = f$, with $\|F\|_{\mathcal{F}} \leq (1 + \epsilon)\|f\|_X$. Then we put $\Omega(f) = F'(\theta)$. The map $\Omega : X \rightarrow L_0$ is a centralizer on X , and two centralizers obtained with different choices of F are equivalent.

An important result of Kalton [7, Theorem 7.6] states that if Φ is a real centralizer on L_p , with $p > 1$, then there is a constant $c > 0$ and a couple of Köthe functions spaces such that $L_p = [X_0, X_1]_{\theta=1/2}$ with equivalent norms, in the sense that both spaces contain the same functions and there is M such that

$$M^{-1}\|f\|_p \leq \inf_{f=F(\frac{1}{2})} \|F\|_{\mathcal{F}} \leq M\|f\|_p$$

for all $f \in L_p$, and $\Phi \approx c\Omega$, where Ω is the corresponding derivation on $X_{1/2} = L_p$.

Proposition 1. *Let Φ be a centralizer on L_p , where $0 < p < \infty$. Then for each $\delta > 0$ there is a set $B \subset \mathbb{I}$ with $\lambda(B) \geq 1 - \delta$ such that, for each $A \subset B$ of positive measure, Φ is bounded on the closed subspace spanned by any Rademacher sequence in A . In particular, the sequence $0 \rightarrow L_p \rightarrow L_p \oplus_{\Phi} L_p \rightarrow L_p \rightarrow 0$ is not strictly singular.*

Proof. It should be clear from the remarks in Section 1.3 that it suffices to prove the proposition assuming that Φ is a real centralizer on the complex L_p .

First suppose $p > 1$. Then by the result of Kalton quoted above, we know that there are a couple of Köthe spaces (X_0, X_1) and $c > 0$ such that $L_p = [X_0, X_1]_{1/2}$ and $\Phi \approx c\Omega$.

Let us take a look at Ω . First, by iteration, we have $L_p = [X_{1/4}, X_{3/4}]_{1/2}$ where $X_{k/4} = [X_0, X_1]_{k/4}$ for $k = 1, 3$ and both $X_{1/4}$ and $X_{3/4}$ are super-reflexive by [8, Theorem 5.8]. On the other hand, if $F \in \mathcal{F}(X_0, X_1)$, then the function G defined by $G(z) = F(\frac{1}{2}(z + \frac{1}{2}))$ belongs to $\mathcal{F}(X_{1/4}, X_{3/4})$ and one has $\|G\|_{\mathcal{F}} \leq \|F\|_{\mathcal{F}}$, $G(\frac{1}{2}) = F(\frac{1}{2})$ and $G'(\frac{1}{2}) = \frac{1}{2}F'(\frac{1}{2})$.

Thus replacing the couple (X_0, X_1) by $(X_{1/4}, X_{3/4})$ preserves the induced centralizer, up to a constant factor, and so we may assume X_0 and X_1 are super-reflexive Köthe spaces.

Now, for $i = 0, 1$, take everywhere positive functions h_i and k_i so that $\|h_i f\|_1 \leq \|f\|_{X_i} \leq \|k_i f\|_{\infty}$ for all $f \in X_i$ and observe that for fixed $\delta > 0$ there is M large enough and a subset $B \subset \mathbb{I}$ with $\lambda(B) > 1 - \delta$ where $k_i \leq M$ and $h_i \geq 1/M$ for $i = 0, 1$.

It follows that $L_{\infty}(B) \subset X_i(B) \subset L_1(B)$, with continuous inclusions, and since X_i is super-reflexive it is also s_i -concave for some finite s_i , and so we have a continuous inclusion $L_{s_i}(B) \subset X_i(B)$ (see [4, p. 14]). Now taking $s = \max s_i$ we conclude that $L_s(B)$ embeds continuously into X_i , and so there is a constant M such that $\|f\|_{X_i} \leq M\|f\|_s$ for every $f \in L_s(B)$ and $i = 0, 1$.

Now, let (r_n) be a Rademacher sequence in $L_s(A)$, where $A \subset B$, and let R be the closed linear span of (r_n) in $L_s(A)$. Then, for $(\lambda_n) \in \ell_2$ the sum $\sum_n \lambda_n r_n$ is in

$L_s(A)$, hence in $X_0(A) \cap X_1(A)$ and $\|f\|_{X_i} \leq M\|f\|_s \leq M'\|f\|_p$, by Khintchine's inequality. Actually the restriction of the norm of the spaces X_0, X_1, L_p and L_s to R is equivalent to the norm of (λ_n) in ℓ_2 .

Hence for $f \in R$ we may take $F(z) = f$ for all $z \in \mathbb{S}$ since $\|F\|_{\mathcal{F}} \leq M'\|f\|_p$, and so $\Omega(f) = F'(\frac{1}{2}) = 0$. As $\Phi \approx c\Omega$ we see that Φ is bounded on R .

Now suppose $p \leq 1$ and let Φ be a centralizer on L_p . We define r by the identity $p^{-1} = r^{-1} + 2^{-1}$. Then there is a centralizer Ψ on L_2 and a constant M such that

$$\|\Phi(gf) - g\Psi(f)\|_p \leq M\|g\|_r\|f\|_2 \quad (g \in L_r, f \in L_2)$$

(see [6, Theorem 8.1] for the case $p = 1$ and [2, Corollary 3] for $p < 1$). We know from the first part of the proof that there is a set B with measure arbitrarily close to 1 such that if $A \subset B$ and R is a subspace of $L_2(A)$ spanned by a Rademacher sequence in A , then Ψ is bounded on R : $\|\Psi(f)\|_2 \leq M'\|f\|_2$ for $f \in R$. Now taking $g = 1$ and $f \in R$ we have

$$\|\Phi(f) - \Psi(f)\|_p \leq M\|1\|_r\|f\|_2 \leq M''\|f\|_p,$$

and so Φ is also bounded on R . □

A centralizer Φ on L_p is said to be symmetric if there is a constant S such that

$$\|\Phi(f \circ \sigma) - \Phi(f) \circ \sigma\|_p \leq S\|f\|_p$$

for every $f \in L_p$ and every measure preserving Borel automorphism σ of $\mathbb{I}$.

The decreasing rearrangement of a real-valued $f \in L_0$ is defined by the formula

$$f^*(t) = \inf_{\lambda(B)=t} \sup_{s \in A \setminus B} f(s) \quad (0 \leq t \leq 1)$$

where B runs over the Borel subsets of $\mathbb{I}$. That is, f^* is the only decreasing, right-continuous function having the same distribution as f . It is a basic fact from measure theory that for each $f \in L_0$, there is a measure preserving Borel automorphism σ of $\mathbb{I}$ (depending on f) such that $f^* = f \circ \sigma$ (almost everywhere), and so f^* is a true rearrangement of f ; see [11, Lemma 2].

Note that if Φ is a symmetric centralizer on L_p and $f^* = f \circ \sigma$, then $\|\Phi(f) - (\Phi(f^*)) \circ \sigma^{-1}\|_p \leq S\|f\|_p$ and so the map $\Phi_s(f) = (\Phi(f^*)) \circ \sigma^{-1}$ is a symmetric centralizer equivalent to Φ with the additional property that the distribution of $\Phi_s(f)$ depends only on the distribution of f .

We emphasize that, in general, centralizers take values in L_0 . For symmetric centralizers we have, however, the following.

Lemma 1. *Suppose $0 < p < r < \infty$ and let Φ be a symmetric centralizer on L_p . If $f \in L_r$, then $\Phi f \in L_p$.*

Proof. It suffices to prove the lemma for real spaces. Let $\Phi_r : L_r \rightarrow L_0$ be the restriction of Φ to L_r . This is a centralizer from L_r to L_p , so by the main result in [2] Φ_r must be trivial and there is $\phi \in L_0$ and a constant M such that

$$\|\Phi_r(f) - \phi f\|_p \leq M\|f\|_r \quad (f \in L_r).$$

We claim that $\phi \in L_s$, where $s^{-1} + r^{-1} = p^{-1}$. By the Hölder inequality this implies that $\phi f \in L_p$, and the same occurs for $\Phi(f) = \Phi_r(f)$. To see this, observe that since $f \mapsto \phi f$ is equivalent to Φ_r it is a symmetric centralizer from L_r to L_p , and so there is a constant S such that

$$\|(\phi \circ \sigma)(f \circ \sigma) - \phi(f \circ \sigma)\|_p \leq S\|f\|_r \quad (f \in L_r)$$

whenever σ is a measure preserving automorphism of the unit interval. Now, since for every $g \in L_s$ one has $\|g\|_s = \sup_{\|f\|_r \leq 1} \|gf\|_p$, we see that $\|\phi \circ \sigma - \phi\|_s \leq M'$ for some M' independent on σ . By symmetry one also has $\|\phi^* \circ \sigma - \phi^*\|_s \leq M'$, where ϕ^* is the decreasing arrangement of ϕ and σ is as before. In particular $\|\phi^* \circ \sigma - \phi^*\|_s$ is finite when $\sigma(t) = 1 - t$. Let $m = \phi^*(\frac{1}{2})$ be the median of ϕ . Now, since ϕ^* is decreasing, $\phi^* \circ \sigma$ is increasing and both agree with m at $t = \frac{1}{2}$, we see that

$$\|\phi - m1\|_s = \|\phi^* - m1\|_s \leq \|\phi^* \circ \sigma - \phi^*\|_s$$

is finite and $\phi \in L_s$. □

We are now ready to prove the following.

Proposition 2. *Let $0 < p < q < 2$. There is a subspace U of L_p isomorphic to ℓ_q where the restriction of any symmetric centralizer is bounded.*

Proof. It suffices to prove the result for real spaces. Moreover, we may and do assume that the distribution of $\Phi(f)$ depends only on that of f .

We proceed as in [12, Proof of Proposition 4.1]. For fixed $q \in (p, 2)$ we consider a q -stable random variable $\vartheta \in L_p$ and a sequence of independent copies (ϑ_n) . We recall that a random variable is said to be q -stable if its characteristic function (= Fourier transform) is $e^{-|t|^q/q}$. We refer the reader to [1, Chapter 6, Section 4] for basic information on stable variables. Here we use the following facts:

- If ϑ is q -stable, then $\mathbb{E}[|\vartheta|^r] < \infty$ for $p < r < q$.
- If (ϑ_n) is a sequence on independent copies of a q -stable random variable ϑ and (λ_n) is a sequence normalized in ℓ_q , then $\sum_n \lambda_n \vartheta_n$ has the same distribution as ϑ .

Therefore the map $(\lambda_n) \in \ell_q \mapsto \sum_n \lambda_n \vartheta_n \in L_p$ is well defined and it is an isometric embedding whose image we denote by U . Moreover, by the lemma, $\Phi(\vartheta)$ belongs to L_p , and then for (λ_n) normalized in ℓ_q we have $\|\Phi(\sum \lambda_n \vartheta_n)\|_p = \|\Phi(\vartheta)\|_p$, and so Φ is bounded on U . □

Problem. Is there a strictly singular sequence $0 \rightarrow L_p \rightarrow Z \rightarrow L_p \rightarrow 0$ for $0 < p < 2$? (See [3, Theorem 2(c)] for the case $2 \leq p < \infty$.)

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