MANIFOLDS WITH A WEIGHTED POINCARÉ INEQUALITY

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ABSTRACT. We study complete manifolds satisfying a weighted Poincaré type property. We establish a splitting and vanishing theorem for L^2 harmonic forms provided that the weight function ρ is of exponential growth of the distance function. Our theory generalizes the results of Li-Wang, Lam and Chen-Sung.

1. INTRODUCTION

It is an interesting theme to understand the relations among the curvature, the topology and the function theorem on a Riemannian manifold. In the work of Li and Wang [L-W1] they proved a splitting type theorem for manifolds whose Ricci curvature is bounded from below by a negative multiple of the lower bound of the spectrum. In particular, their theorem generalized the work of Witten-Yau [W-Y], Cai-Galloway [C-G] and X. Wang [Wang] on conformally compact manifolds to arbitrary complete manifolds. On the other hand, Li and Wang also proved the following vanishing theorem for L^2 integrable harmonic 1-forms on M in [L-W1].

Theorem 1.1 (Li-Wang). *Let M be an m -dimensional complete Riemannian manifold with $\lambda_1(M) > 0$ and*

$$\text{Ric}_M \geq -\frac{m}{m-1}\lambda_1 + \delta,$$

for some $\delta > 0$. Then $H^1(L^2(M)) = 0$, where $H^1(L^2(M))$ denotes the space of L^2 integrable harmonic 1-forms on M .

Later, Lam [Lam] generalized Li-Wang's theory to manifolds satisfying a weighted Poincaré inequality.

Theorem 1.2 (Lam). *Let M be an m -dimensional complete Riemannian manifold satisfying a weighted Poincaré inequality with a nonnegative weight function $\rho(x)$. Assume that the Ricci curvature satisfies*

$$\text{Ric}_M(x) \geq -\frac{m}{m-1}\rho(x) + \delta,$$

for some $\delta > 0$. Let $r(x, p)$ be the distance function from x to some fixed point p , if $\rho(x) = O(r^{2-\alpha}(x, p))$, for some $0 < \alpha < 2$. Then $H^1(L^2(M)) = 0$.

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The main purpose of this article is to study structure theorems for manifolds satisfying a weighted Poincaré type property. In our theorems, a growth assumption is needed for the weighted function. Let us first recall some definitions.

Definition 1.3. Let M^m be an m -dimensional complete Riemannian manifold. We say that M^m satisfies a weighted Poincaré inequality (cf. [L-W2]) with a non-negative weight function $\rho(x)$ if the inequality

$$\int_M \rho(x) \phi^2 \leq \int_M |\nabla \phi|^2 \quad \text{for } \phi \in C_0^\infty(M)$$

is valid for all compactly supported smooth functions $\phi \in C_0^\infty(M)$.

Definition 1.4. Let M^m be an m -dimensional complete Riemannian manifold. We say that M^m has property $(\mathcal{P}_\rho)$ if a weighted Poincaré inequality is valid on M with some nonnegative weight function ρ . Moreover, the ρ -metric, defined by

$$ds_\rho^2 = \rho ds_M^2,$$

is complete.

Let $\lambda_1(M)$ denote the bottom spectrum of the Laplacian on M , namely,

$$\lambda_1(M) = \inf \left\{ \frac{\int_M |\nabla \phi|^2}{\int_M \phi^2} : \phi \in C_0^\infty(M) \right\}.$$

Then the variation principle for $\lambda_1(M)$ asserts the validity of the Poincaré inequality

$$\lambda_1(M) \int_M \phi^2 \leq \int_M |\nabla \phi|^2$$

for all compactly supported functions $\phi \in C_0^\infty(M)$. In particular, if $\lambda_1(M)$ is assumed to be positive, then obviously M possesses property $(\mathcal{P}_\rho)$ with $\rho(x) = \lambda_1(M)$. The notion of property $(\mathcal{P}_\rho)$ may be viewed as a generalization of the assumption $\lambda_1(M) > 0$.

For harmonic p -forms, let $C_0^\infty(\wedge^p M)$ denote the space of smooth p -forms with compact support on M . Then we define property $(\mathcal{P}_{p,\rho})$ as follows.

Definition 1.5. Let M^m be an m -dimensional complete Riemannian manifold. We say that M^m has property $(\mathcal{P}_{p,\rho})$ if a weighted p -Poincaré inequality is valid on M with some non-negative weight function ρ , i.e.,

$$(1) \quad \int_M \rho(x) |\phi|^2 \leq \int_M |d\phi|^2 + |\delta\phi|^2 \quad \text{for } \phi \in C_0^\infty(\wedge^p M).$$

Moreover, the ρ -metric, defined by $ds_\rho^2 = \rho ds_M^2$, is complete.

Note that if the greatest lower bound $\lambda_{1,p}$ of the p spectrum satisfies $\lambda_{1,p}(M) > 0$, then M has property $(\mathcal{P}_{p,\rho})$ with the weight function $\rho(x) = \lambda_{1,p}(M)$. Hence the notion of property $(\mathcal{P}_{p,\rho})$ can also be viewed as a generalization of the assumption that $\lambda_{1,p}(M) > 0$.

Throughout this article, we use $H^p(L^2(M))$ to denote the space of L^2 harmonic p -forms, and $r_\rho(x)$ the geodesic distance from some fixed point to x with respect to the metric ds_ρ^2 .

It is interesting to see that a similar theorem holds by relaxing the Ricci curvature bound condition with $\delta = 0$. In this article, we show a result of [L-W1] and [Lam] to weighted p -Poincaré type property and establish a splitting and vanishing result for

L^2 harmonic 1-forms provided that the weight function ρ is of exponential growth of the distance function. The following theorem has been established.

Theorem 1.6. *Let M be an m -dimensional complete Riemannian manifold satisfying a $\mathcal{P}_{1,\rho}$ property with a nonnegative weight function $\rho(x)$. Assume that the Ricci curvature satisfies*

$$\text{Ric}_M(x) \geq -\frac{m}{m-1}\rho(x)$$

and $\rho \leq O(e^{2r_\rho})$, Then either

- (1) $H^1(L^2(M)) = 0$ or
- (2) $\widetilde{M} = \mathbb{R} \times N$, where $\widetilde{M}$ is the universal cover of M and N is a manifold of dimension $(m-1)$.

On the other hand, we prove a similar result for L^2 harmonic p -forms on manifolds with property $(\mathcal{P}_{p,\rho})$. The curvature operator K_p on M is assumed to satisfy the inequality

$$K_p \geq -\frac{m-p+1}{m-p}\rho(x)$$

for all $x \in M$. In this theorem, a growth assumption is also needed for the weight function ρ in terms of the ρ -distance.

Theorem 1.7. *Let M be a complete noncompact manifold with property $(\mathcal{P}_{p,\rho})$ for some nonnegative weight function ρ . Assume that the curvature operator K_p on M has the lower bound*

$$K_p \geq -\frac{m-p+1}{m-p}\rho(x)$$

and $\rho \leq O(e^{2r_\rho})$. Suppose that ω is a harmonic p -form in $L^2(M)$ ($p \geq 2$). Then either

- (1) $H^p(L^2(M)) = 0$ or
- (2) there exists a 1-form α with $\alpha \wedge \omega = 0$ such that

$$\nabla \omega = \alpha \otimes \omega - \frac{1}{m-p+1} \sum_{j=1}^m \theta^j \otimes (\theta^j \wedge i_{\alpha^\#} \omega),$$

where $\{\theta^1, \dots, \theta^m\}$ is an orthonormal basis for the cotangent bundle and $\alpha^\#$ is the dual vector of α .

2. SPLITTING AND VANISHING THEOREMS

Let $\{e_1, \dots, e_m\}$ be an orthonormal basis and harmonic 1-form $\omega = \sum_{i=1}^m \omega_i e_i$. It is well known that if ω is an L^2 harmonic 1-form, then it is both closed and co-closed:

$$(2) \quad |\nabla \omega|^2 = \sum_{i,j=1}^m w_{ij}^2 \geq \sum_{j=1}^m w_{1j}^2 + \sum_{\alpha=2}^m w_{\alpha 1}^2 + \sum_{\alpha=2}^m w_{\alpha \alpha}^2 \geq \frac{m}{m-1} \sum_{j=1}^m w_{1j}^2.$$

We also know that $|\omega|^2 |\nabla \omega|^2 = w_1^2 \sum_{j=1}^m w_{1j}^2$, hence

$$|\nabla \omega|^2 \geq \frac{m}{m-1} |\nabla \omega|^2.$$

Therefore, ω satisfies the following Bochner type formula:

$$(3) \quad |\omega| \Delta |\omega| \geq \text{Ric}(\omega, \omega) + \frac{|\nabla |\omega||^2}{m-1}.$$

By the lemma in [Lam], we have an estimate for functions which satisfy the above Bochner type formula. For the sake of completeness, we restate the lemma in [Lam] as follows.

Lemma 2.1 ([Lam]). *Let $b > -1$. Assume that h is a nonnegative function and satisfies the differential inequality $h\Delta h \geq -ah^2 + b|\nabla h|^2$ in the weak sense for some constant $a > 0$. For any $\varepsilon > 0$, we have the estimate*

$$(b(1-\varepsilon) + 1) \int_M |\nabla(\phi h)|^2 \leq \left\{ b \left(\frac{1}{\varepsilon} - 1 \right) + 1 \right\} \int_M h^2 |\nabla \phi|^2 + a \int_M \phi^2 h^2$$

for any compactly supported smooth function $\phi \in \mathcal{C}_0^\infty(M)$. In addition, if

$$\int_{B_p(R)} h^2 = o(R^2),$$

then

$$\int_M |\nabla h|^2 \leq \frac{a}{b+1} \int_M h^2.$$

In particular, h has a finite Dirichlet integral if $h \in L^2(M)$.

Note that the results of Lemma 3.1 in [Lam] are still valid if $a = a(x)$ is a function of x . Hence, we have

$$(4) \quad (b(1-\varepsilon) + 1) \int_M |\nabla(\varphi h)|^2 \leq \left(b \left(\frac{1}{\varepsilon} - 1 \right) + 1 \right) \int_M h^2 |\nabla \varphi|^2 + \int_M a(x) \varphi^2 h^2$$

and

$$(5) \quad \int_M |\nabla h|^2 \leq \frac{1}{b+1} \int_M a(x) h^2.$$

In this section, the structure theorems are established for complete manifolds that satisfy a weighted Poincaré type inequality with weight function $\rho(x)$. The following theorem is given on a manifold with positive spectrum $\lambda_1(M)$; i.e. M possesses property $(\mathcal{P}_\rho)$ with weight function $\rho = \lambda_1(M)$.

Theorem 2.2. *Let M be a complete Riemannian manifold of dimension $m \geq 3$. Suppose that $\lambda_1(M) > 0$ and*

$$\text{Ric}_M \geq -\frac{m}{m-1} \lambda_1(M).$$

Then, either

- (1) $H^1(L^2(M)) = 0$ or
- (2) $\widetilde{M} = \mathbb{R} \times N$, where $\widetilde{M}$ is the universal cover of M and N is a manifold of dimension $(m-1)$.

Proof. Let ω be an L^2 harmonic 1-form and $h = |\omega|$ the length of ω . By the Bochner type formula (3) and the curvature assumption, we have

$$(6) \quad h\Delta h \geq -\frac{m\lambda_1(M)}{m-1} h^2 + \frac{1}{m-1} |\nabla h|^2.$$

By Lemma 2.1, h has a finite Dirichlet integral. Let ϕ be a cut-off function such that

$$\phi = \begin{cases} 1 & \text{on } B(R), \\ 0 & \text{on } M \setminus B(2R) \end{cases}$$

and $|\nabla\phi|^2 \leq \frac{C}{R^2}$ on $B(2R) \setminus B(R)$.

Now we consider the following cases.

Case 1 ($\lambda_1(M) \int_M h^2 < \int_M |\nabla h|^2$). Since h has finite Dirichlet integral, we have

$$\lim_{R \rightarrow \infty} \int_{B(2R)} \phi^2 h^2 = \int_M h^2 \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_{B(2R)} |\nabla(\phi h)|^2 = \int_M |\nabla h|^2.$$

Hence, there exists a positive number η such that

$$(\lambda_1(M) + \eta) \int_M (\phi h)^2 \leq \int_M |\nabla(\phi h)|^2 \quad \text{if } R \text{ is large enough.}$$

By Lemma 2.1, we have

$$\begin{aligned} \left(\frac{1}{m-1} (1-\varepsilon) + 1 \right) (\lambda_1(M) + \eta) \int_M \phi^2 h^2 &\leq \left(\frac{1}{m-1} (1-\varepsilon) + 1 \right) \int_M |\nabla(\phi h)|^2 \\ &\leq \frac{m\lambda_1(M)}{m-1} \int_M \phi^2 h^2 + \left\{ \frac{1}{m-1} \left(\frac{1}{\varepsilon} - 1 \right) + 1 \right\} \int_M h^2 |\nabla\phi|^2 \end{aligned}$$

for any $\varepsilon > 0$ and any compactly supported smooth function $\phi \in \mathcal{C}_0^\infty(M)$ when R is large. The above inequality yields

$$\frac{\eta(1-\varepsilon)}{m-1} \int_{B(R)} h^2 \leq \frac{\varepsilon\lambda_1(M)}{m-1} \int_{B(R)} h^2 + \left\{ \frac{1}{m-1} \left(\frac{1}{\varepsilon} - 1 \right) + 1 \right\} \frac{C}{R^2} \int_{B(2R) \setminus B(R)} h^2.$$

Since $h \in L^2(M)$, first let $R \rightarrow \infty$, then take $\varepsilon \rightarrow 0$. We conclude that $\int_M h^2 \leq 0$.

Thus, $h \equiv 0$, equivalently, $\omega \equiv 0$. This yields a contradiction, since $\lambda_1(M) \int_M h^2 < \int_M |\nabla h|^2$.

Case 2 ($\lambda_1(M) \int_M h^2 = \int_M |\nabla h|^2$).

Multiplying both sides of inequality (6) by ϕ and integrating it over M , the divergence theorem implies that

$$-\int_M \langle \nabla(\phi h), \nabla h \rangle \geq -\frac{m\lambda_1(M)}{m-1} \int_M \phi h^2 + \frac{1}{m-1} \int_M \phi |\nabla h|^2.$$

Let $R \rightarrow \infty$ and use the fact that h and $|\nabla h|$ are in $L^2(M)$:

$$-\int_M |\nabla h|^2 \geq -\frac{m\lambda_1(M)}{m-1} \int_M h^2 + \frac{1}{m-1} \int_M |\nabla h|^2.$$

We have

$$-\lambda_1(M) \int_M h^2 \geq -\frac{m\lambda_1(M)}{m-1} \int_M h^2 + \frac{1}{m-1} \int_M |\nabla h|^2.$$

This implies

$$\lambda_1(M) \int_M h^2 \geq \int_M |\nabla h|^2.$$

Hence equality (3) holds. In particular, the gradient of ω must be

$$(w_{ij}) = \begin{pmatrix} -(n-1)\mu & 0 & 0 & \dots & 0 \\ 0 & \mu & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}$$

if ω is harmonic.

Since the equality (6) holds,

$$h\Delta h = -\frac{m\lambda_1(M)}{m-1}h^2 + \frac{1}{m-1}|\nabla h|^2.$$

Let $g = h^{\frac{m-2}{m-1}}$; we have

$$(7) \quad -\frac{m(m-2)\lambda_1(M)}{(m-1)^2}g = \Delta g.$$

Now, suppose that the space of an L^2 harmonic 1-form is nontrivial, i.e., $H^1(L^2(M)) \neq 0$. Then by equation (7),

$$H^1(L^2(M)) \subset \left\{ \omega : \omega \text{ is a harmonic 1-form: } \Delta|\omega|^{\frac{m-2}{m-1}} = -\frac{m(m-2)\lambda_1(M)}{(m-1)^2}|\omega|^{\frac{m-2}{m-1}} \right\}.$$

Next, we show that $\widetilde{M}$ is equal to $\mathbb{R} \times N$ if $\pi : \widetilde{M} \rightarrow M$ is a universal cover of M .

If ω is an L^2 harmonic 1-form, then ω is closed. Let $\tilde{\omega}$ be the lifting of ω on $\widetilde{M}$ with respect to covering map π . Note that $\tilde{\omega}$ is closed. Because $\widetilde{M}$ is simply connected, there exists a function f on $\widetilde{M}$ such that $df = \tilde{\omega}$ and the Hessian matrix of f is a diagonal matrix. In particular, the level set of f and $|\nabla f|$ coincide. Let N be a fixed level set of f in $\widetilde{M}$. Since $|\nabla f| \neq 0$, $\widetilde{M}$ is equal to $\mathbb{R} \times N$.

Let $(x_2, \dots, x_m)$ be a fixed point in N . The image of $\mathbb{R} \times \{x_2, \dots, x_m\}$ via the covering map π becomes a totally geodesic hypersurface of M .

This completes the proof of Theorem 2.2. $\square$

Remark 2.3.

- (1) Note that by equation (7) and Li-Wang's argument, we have that M has only one end. Recall that an end of M is an unbounded component of $M \setminus K$, where K is a compact subset of M .
- (2) If the Ricci curvature of M satisfies $\text{Ric}_M \geq -\frac{m}{m-1}\lambda_1(M) + \delta$ for some positive constant $\delta > 0$, then equality (7) is no longer true and $H^1(L^2(M)) = 0$. Hence Theorem 2.2 is a generalization of the work of Li-Wang [L-W1] and Lam [Lam].

Next, we generalize the above result to show that manifold M satisfies a weighted Poincaré type inequality with a nonconstant weight function.

Theorem 2.4. *Let M be an m -dimensional complete Riemannian manifold satisfying a $(P_{1,\rho})$ property with a nonnegative weight function $\rho(x)$. Assume that the Ricci curvature satisfies*

$$\text{Ric}_M(x) \geq -\frac{m}{m-1}\rho(x)$$

and $\rho \leq O(e^{2r_\rho})$, Then either

- (1) $H^1(L^2(M)) = 0$ or
- (2) $\widetilde{M} = \mathbb{R} \times N$, where $\widetilde{M}$ is the universal cover of M and N is a manifold of dimension $(m-1)$.

Proof. Suppose that $H^1(L^2(M)) \neq 0$. Let ω be an $L^2(M)$ harmonic 1-form and $h = |\omega|$. We will first prove that

$$\int_M \rho |\omega|^2 < \infty.$$

Indeed, $\rho \leq O(e^{2r_\rho})$ and by $|\omega| \in L^2(M)$, for any $R > 0$, we obtain

$$\int_{B_\rho(2R) \setminus B_\rho(R)} \rho e^{-2r_\rho} |\omega|^2 \leq C$$

for some constant C . By Lemma 2.1 in [C-S],

$$\int_{B_\rho(R+1) \setminus B_\rho(R)} \rho |\omega|^2 \leq C e^{-2R}.$$

Hence,

$$\begin{aligned} \int_{B_\rho(2R) \setminus B_\rho(R)} \rho |\omega|^2 &\leq \sum_{r=[R]}^{2[R]+1} \int_{B_\rho(r+1) \setminus B_\rho(r)} \rho |\omega|^2 \\ &\leq \sum_{r=[R]}^{2[R]+1} C e^{-2r} \leq 2C[R] e^{-2[R]}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} [R] 2^n e^{-2^n [R]}$ converges, we have

$$\begin{aligned} \int_M \rho |\omega|^2 &= \int_{B(R)} \rho |\omega|^2 + \sum_{n=1}^{\infty} \int_{B_\rho(2^n R) \setminus B_\rho(2^{n-1} R)} \rho |\omega|^2 \\ (8) \quad &\leq \int_{B(R)} \rho |\omega|^2 + C[R] \sum_{n=1}^{\infty} 2^n e^{-2^n [R]} < \infty. \end{aligned}$$

Now, by the Bochner type formula (3) and the curvature assumption,

$$(9) \quad h \Delta h \geq -\frac{m\rho(x)}{m-1} h^2 + \frac{1}{m-1} |\nabla h|^2.$$

Note that in this formula, we have $a(x) = \frac{m\rho(x)}{m-1}$, $b = \frac{1}{m-1}$. By (5) and (8), h has a finite Dirichlet integral. Let ϕ be a cut-off function such that

$$\phi = \begin{cases} 1, & \text{on } B(R), \\ 0, & \text{on } M \setminus B(2R) \end{cases}$$

and $|\nabla \phi|^2 \leq \frac{C}{R^2}$ on $B(2R) \setminus B(R)$.

By a similar proof in Theorem 2.2 and inequality (5), we have

$$\int_M |\nabla h|^2 \leq \frac{1}{\frac{1}{m-1} + 1} \int_M \frac{m\rho(x)}{m-1} h^2 = \int_M \rho(x) h^2.$$

However, if M satisfies the property $(\mathcal{P}_{1,\rho})$, then the weighted Poincaré inequality implies $\int_M \rho(x) h^2 = \int_M |\nabla h|^2$. On the other hand, if we multiply both sides of (9) by ϕ and integrate over M , the divergence theorem implies that

$$-\int_M \langle \nabla(\phi h), \nabla h \rangle \geq -\frac{m}{m-1} \int_M \phi \rho(x) h^2 + \frac{1}{m-1} \int_M \phi |\nabla h|^2.$$

Now, letting $R \rightarrow \infty$ and using the fact that $h \in L^2(M)$, $|\nabla h| \in L^2(M)$, we have

$$-\frac{m}{m-1} \int_M \rho(x) h^2 + \frac{1}{m-1} \int_M |\nabla h|^2 \leq -\int_M |\nabla h|^2$$

and

$$-\frac{m}{m-1} \int_M \rho(x) h^2 + \frac{1}{m-1} \int_M |\nabla h|^2 \leq -\int_M \rho(x) h^2.$$

This implies

$$\int_M |\nabla h|^2 \leq \int_M \rho(x) h^2.$$

So all preceding inequalities becomes equalities. In particular, the inequality (9) is an equality, namely,

$$h\Delta h = -\frac{m\rho(x)}{m-1} h^2 + \frac{1}{m-1} |\nabla h|^2.$$

By a similar proof in Theorem 2.2, we conclude that either $H^1(L^2(M)) = 0$ or $\widetilde{M} = \mathbb{R} \times N$, where $\widetilde{M}$ is the universal cover of M and N is a manifold of dimension $m-1$. The proof is complete. $\square$

Remark 2.5. If $\rho = O(r^{2-\alpha})$ where $\alpha > 0$ is given, then $ds_\rho^2 = \rho ds_M^2 = O(r^{2-\alpha} ds_M^2)$. We have

$$\rho e^{-2r_\rho} = O(r^{2-\alpha}) e^{-2O(r^{2-\alpha})} \leq C,$$

namely, $\rho \leq O(e^{2r_\rho})$. This shows that our result is more general than Lam's results [Lam].

3. WEIGHTED p POINCARÉ INEQUALITY

In this section, we study the space of harmonic p -forms on manifolds with property $\mathcal{P}_{p,\rho}$. The curvature operator K_p on M is assumed to satisfy the inequality

$$K_p \geq -\frac{m-p+1}{m-p} \rho(x)$$

for all $x \in M$. In this theorem, a growth assumption is needed for the weighted function of the Poincaré type inequality. For the sake of completeness, we restate the following decay estimate in [C-S] (cf. [D] and [L-W2]).

Lemma 3.1. *Let M be a complete noncompact manifold. If E is an end of M with the property $(\mathcal{P}_{p,\rho})$ for some nonnegative weight function ρ , then for any smooth harmonic p form w satisfying*

$$(10) \quad \int_{E_\rho(2R) \setminus E_\rho(R)} \rho(x) \exp(-2r_\rho) |w|^2 dv = o(R),$$

we have

$$\int_{E_\rho(R)} \rho(x) \exp(2r_\rho) |w|^2 dv \leq CR$$

and

$$\int_{E_\rho(R+1) \setminus E_\rho(R)} \rho(x) \exp(2r_\rho) |w|^2 dv \leq C,$$

for all R sufficiently large and for some constant C depending on w and ρ .

Theorem 3.2. *Let M be a complete noncompact manifold with property $(\mathcal{P}_{p,\rho})$ for some nonnegative weight function ρ . Assume that the curvature operator K_p on M has the lower bound*

$$K_p \geq -\frac{m-p+1}{m-p} \rho(x)$$

and $\rho \leq O(e^{2r_\rho})$. Suppose that ω is a harmonic p -form in L^2 ($p \geq 2$). Then either

- (1) $H^p(L^2(M)) = 0$ or
- (2) *there exists a 1-form α with $\alpha \wedge \omega = 0$ such that*

$$\nabla \omega = \alpha \otimes \omega - \frac{1}{m-p+1} \sum_{j=1}^m \theta^j \otimes (\theta^j \wedge i_{\alpha^\#} \omega),$$

where $\{\theta^1, \dots, \theta^m\}$ is an orthonormal basis for the cotangent bundle and $\alpha^\#$ is the dual vector of α .

Proof. The proof is similar to the proof of Theorem 1.6. Suppose that $\omega \neq 0$. First, we claim that

$$\int_M \rho |\omega|^2 < \infty.$$

Indeed, since $\rho \leq O(e^{2r_\rho})$ and $|\omega| \in L^2(M)$, for any $R > 1$, we have

$$\int_{B_\rho(2R) \setminus B_\rho(R)} \rho e^{-2r_\rho} |\omega|^2 \leq C$$

for some constant C . By Lemma 3.1, we have

$$\int_{B_\rho(R+1) \setminus B_\rho(R)} \rho |\omega|^2 \leq C e^{-2R}.$$

Hence,

$$\begin{aligned} \int_{B_\rho(2R) \setminus B_\rho(R)} \rho |\omega|^2 &\leq \sum_{r=[R]}^{2[R]+1} \int_{B_\rho(r+1) \setminus B_\rho(r)} \rho |\omega|^2 \\ &\leq \sum_{r=[R]}^{2[R]+1} C e^{-2r} \leq 2C[R] e^{-2[R]}. \end{aligned}$$

Since $\sum_{n=1}^{\infty} 2^n [R] e^{-2^n [R]}$ converges,

$$(11) \quad \int_M \rho |\omega|^2 = \int_{B(R)} \rho |\omega|^2 + \sum_{n=1}^{\infty} \int_{B_\rho(2^n R) \setminus B_\rho(2^{n-1} R)} \rho |\omega|^2 < \infty.$$

The claim is proved. Now, by the Bochner type formula for harmonic p -forms (see [C-S]) and the curvature assumption, we have

$$(12) \quad h \Delta h \geq -\frac{(m-p+1)\rho(x)}{m-p} h^2 + \frac{1}{m-p} |\nabla h|^2.$$

Note that in this formula, we have $a(x) = \frac{(m-p+1)\rho(x)}{m-p}$, $b = \frac{1}{m-p}$. By (5) and (11), h has a finite Dirichlet integral. Let ϕ be a cut-off function such that

$$\phi = \begin{cases} 1, & \text{on } B(R), \\ 0, & \text{on } M \setminus B(2R) \end{cases}$$

and $|\nabla \phi|^2 \leq \frac{C}{R^2}$ on $B(2R) \setminus B(R)$.

By (5),

$$\int_M |\nabla h|^2 \leq \frac{1}{\frac{1}{m-p} + 1} \int_M \frac{(m-p+1)\rho(x)}{m-p} h^2 = \int_M \rho(x) h^2.$$

Hence $\int_M \rho(x) h^2 = \int_M |\nabla h|^2$ for M has property $(\mathcal{P}_{p,\rho})$.

If we multiply both sides of (12) by ϕ and integrate it over M , the divergence theorem gives

$$-\int_M \langle \nabla(\phi h), \nabla h \rangle \geq -\frac{m-p+1}{m-p} \int_M \phi \rho(x) h^2 + \frac{1}{m-p} \int_M \phi |\nabla h|^2.$$

Now, letting $R \rightarrow \infty$ and using the fact that $h \in L^2(M)$, $|\nabla h| \in L^2(M)$,

$$-\int_M |\nabla h|^2 \geq -\frac{m-p+1}{m-p} \int_M \rho(x) h^2 + \frac{1}{m-p} \int_M |\nabla h|^2;$$

consequently,

$$-\int_M \rho(x) h^2 \geq -\frac{m-p+1}{m-p} \int_M \rho(x) h^2 + \frac{1}{m-p} \int_M |\nabla h|^2.$$

This implies

$$\int_M \rho(x) h^2 \geq \int_M |\nabla h|^2.$$

Hence all of the preceding inequalities become equalities. In particular, inequality (12) is an equality, i.e.,

$$h \Delta h = -\frac{(m-p+1)\rho(x)}{m-p} h^2 + \frac{1}{m-p} |\nabla h|^2.$$

By a similar argument in [C-S], the proof is complete. $\square$

Now we focus on a complete manifold that satisfies both a weighted p -Poincaré inequality and the curvature operator lower bound. A similar vanishing result for harmonic forms (see [L-W1], [Lam]) is given as follows.

Theorem 3.3. *Let M^m be a complete noncompact Riemannian manifold with property $(\mathcal{P}_{p,\rho})$ for some nonnegative weight function ρ . Assume that $\rho \leq O(e^{2r_\rho})$ and the curvature operator K_p of M has the lower bound*

$$K_p \geq -\left(\frac{m-p+1}{m-p}\rho - \epsilon\right)$$

for some given $\epsilon > 0$. Then $\dim H^p(L^2(M)) = 0$.

Proof. Let ω be a harmonic p -form in $L^2(M)$ and $h = |\omega|$. By the argument given in the proof of Theorem 1.7, we have that

$$\int_M \rho |\omega|^2 < \infty.$$

Moreover, we have the Bochner formula

$$(13) \quad h\Delta h \geq -\left(\frac{m-p+1}{m-p}\rho - \epsilon\right)h^2 + \frac{1}{m-p}|\nabla h|^2.$$

By (5),

$$\int_M |\nabla h|^2 \leq \frac{1}{\frac{1}{m-p} + 1} \int_M \left(\frac{m-p+1}{m-p}\rho - \epsilon\right)h^2 = \int_M \rho(x)h^2 - \epsilon \int_M h^2.$$

Since $h \in L^2$ and $\int_M \rho h^2 < \infty$, it implies $|\nabla h| \in L^2(M)$. Note that M has $(\mathcal{P}_{p,\rho})$ property; hence M also has a weighted Poincaré inequality with the nonnegative weight function ρ . Combining this with the L^2 property as above, we infer that

$$\int_M \rho h^2 \leq \int_M |\nabla h|^2.$$

Thus,

$$\int_M |\nabla h|^2 \leq \int_M \rho(x)h^2 - \frac{m-p}{m-p+1}\epsilon \int_M h^2 \leq \int_M |\nabla h|^2 - \frac{m-p}{m-p+1}\epsilon \int_M h^2.$$

This yields $\int_M h^2 = 0$; consequently $h = 0$ or equivalently $\omega = 0$. The proof is complete. $\square$

In the following theorem, we study the space of an L^2 harmonic function with finite energy of M which satisfies a weighted Poincaré inequality with a nonnegative weight function $\rho(x)$. By Theorem 2.1 in [D-S], we have

$$\int_{B_\rho(R+1) \setminus B_\rho(R)} \rho u^2 \leq C e^{-2R}.$$

By a similar argument in Theorem 2.2, the splitting and vanishing properties are established on M .

Theorem 3.4. *Let M^m be a complete noncompact Riemannian manifold with property $(\mathcal{P}_{p,\rho})$ for some nonnegative weight function ρ . Assume that the Ricci curvature satisfies*

$$\text{Ric}_M(x) \geq -\frac{m}{m-1}\rho(x)$$

and $\rho \leq O(e^{2r_\rho})$. Then either

- (1) $H(L^2(M)) = \mathbb{R}$ or
- (2) $M = \mathbb{R} \times N$ with warped metric product, where N is a manifold of dimension $(m-1)$.

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