A FOCK SPACE MODEL FOR ADDITION AND MULTIPLICATION OF C-FREE RANDOM VARIABLES

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ABSTRACT. The paper presents a Fock space model suitable for construction of c -free algebras. Immediate applications are direct proofs for the properties of the c -free R - and S -transforms.

1. INTRODUCTION

Two important tools in free probability theory are the R - and S -transforms which play a similar role to the Fourier, respectively Mellin, transform. More precisely, besides strong regularity properties, if X and Y are two free non-commutative random variables, then $R_{X+Y}(z) = R_X(z) + R_Y(z)$ and $S_{XY}(z) = S_X(z) \cdot S_Y(z)$ if X, Y have non-zero first order moments.

In literature there are two main techniques to prove the additive, respectively multiplicative, properties of the R - and S -transforms. The proofs given by D.-V. Voiculescu ([19], [20]) and U. Haagerup ([10]) are based on functional analysis techniques, namely on the properties of the annihilation and creation operators on the full Fock space, while the proofs of R. Speicher and A. Nica ([13]) are based on combinatorial techniques on the lattice of non-crossing partitions (also, non-crossing linked partitions appear in the proofs for the multiplicative property of the S -transform in [8], [16]).

In the early 1990's, M. Bożejko, M. Leinert and R. Speicher introduced the notion of c -freeness, which extends the notion of freeness to the framework of an algebra endowed with two (ϕ, ψ) , rather than one, normalized linear functionals (see Section 2 for the exact definitions). (A more general approach to c -freeness, considering pairs of completely positive maps and conditional expectations, has been pursued by F. Boca ([3]), K. Dykema and E. Blanchard ([9]), M. Popa, V. Vinnikov ([14], [17], etc.). The addition of c -free random variables was studied in [7] (and later in [1], [22], [11], etc.), where a c -free version of the R -transform, the cR -transform, with similar additivity and analytic properties (for $\phi = \psi$, the two transforms coincide) was constructed. A multiplication of c -free random variables was studied in [18], where a c -free extension of the S -transform was constructed. In both cases, the proofs of the key properties (addition for the cR - and multiplication for the cS -transform) are combinatorial, much like the proofs from [13], heavily relying on the properties on non-crossing partitions. The present material gives

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a new approach to c -free random variables, in the spirit of the construction from [10]. Particularly, we give a more direct proof of the additive and multiplicative properties of the cR - and cS -transforms, based on the properties of the creation and annihilation operators on a certain type of Fock space.

Besides the Introduction, the paper is organized in 3 sections. Section 2 presents basic definitions, the construction of the space $\mathcal{E}(\mathfrak{H}, \mathcal{T}(\mathcal{K}))$ and an operator algebra model for c -free algebras. Section 3 presents the construction of some operators of prescribed cR and cS -transforms, and Section 4 gives the proof for the additive, respectively multiplicative, properties of cR and cS .

In this paper, rather than the S - or cS -transforms, we use, to simplify the notation, their multiplicative inverses, the so-called T - and cT -transforms, i.e. $T_X(z) \cdot S_X(z) = 1$ (see [8], [16]), respectively ${}^cT(z) \cdot {}^cS(z) = 1$ (see [18]).

2. A CONSTRUCTION OF C -FREE ALGEBRAS

2.1. Suppose $\mathcal{A}$ is a complex unital algebra endowed and $\psi : \mathcal{A} \longrightarrow \mathbb{C}$ is a linear map such that $\psi(1) = 1$. A family of unital subalgebras $\{\mathcal{A}_i\}_{i \in I}$ of $\mathcal{A}$ is said to be *free* (with respect to ψ) if

$$\psi(x_1 \cdots x_n) = 0$$

whenever $x_j \in \mathcal{A}_{\epsilon(j)}$ with $\epsilon(k) \neq \epsilon(k+1)$ and $\psi(x_j) = 0$ for $1 \leq j \leq n$ and $1 \leq k < n$.

If $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ is another linear map with $\phi(1) = 1$, the family $\{\mathcal{A}_i\}_{i \in I}$ of unital subalgebras of $\mathcal{A}$ is said to be *c-free* with respect to (ϕ, ψ) if $\{\mathcal{A}_i\}_{i \in I}$ are free with respect to ψ and

$$\phi(x_1 \cdots x_n) = \phi(x_1) \cdots \phi(x_n)$$

whenever $x_j \in \mathcal{A}_{\epsilon(j)}$ with $\epsilon(k) \neq \epsilon(k+1)$ and $\psi(x_j) = 0$ for $1 \leq j \leq n$ and $1 \leq k < n$.

Take X to be an element from $\mathcal{A}$ and let $m_X(z) = \sum_{k=1}^{\infty} \psi(X^k) z^k$ denote the moment generating series of X with respect to ψ . As formal power series, the transforms $R_X(z)$ and, if $\psi(X) \neq 0$, $T_X(z)$ are defined by the equations

$$\begin{aligned} (1) \quad m_X(z) &= R_X(z[1 + m_X(z)]), \\ \frac{1}{z} m_X(z) &= [T_X(m_X(z))] \cdot (1 + m_X(z)). \end{aligned}$$

We warn the reader that the version of the R -transform that is used in the present material differs from the original definition of D.-V. Voiculescu (that we will here call $\mathcal{R}$) by a multiplication with the variable z : $R_X(z) = z \cdot \mathcal{R}_X(z)$. As also seen in [18], [13], this shift of coefficients simplifies the notation in several recurrence relations from subsection 2.2.

For $\mathcal{H}$ a complex Hilbert space, we define $\mathcal{T}^0(\mathcal{H}) = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \dots$ and $\mathcal{T}(\mathcal{H}) = \mathbb{C}\omega \oplus \mathcal{T}^0(\mathcal{H})$, where $\|\omega\| = 1$. For $e \in \mathcal{H}$ a non-zero vector, the creation operator over e , $a_e^* \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$, is given by the relations

$$\begin{aligned} a_e^* \omega &= e, \\ a_e^* v_1 \otimes v_2 \otimes \cdots \otimes v_k &= e \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_k, \text{ for } v_1, \dots, v_k \in \mathcal{H}, \end{aligned}$$

while the annihilation operator over e , $a_e \in \mathcal{L}(\mathcal{T}(\mathcal{H}))$, is given by

$$\begin{aligned} a_e \omega &= 0, \\ a_e v_1 \otimes v_2 \otimes \cdots \otimes v_k &= \langle v_1, e \rangle \cdot v_2 \otimes \cdots \otimes v_k. \end{aligned}$$

We recall the following result (see [21] for (1), [10], Theorem 2.2 and Theorem 2.3, for (2) and (3)):

Theorem 2.1. *Let $\mathcal{H}$ be a complex Hilbert space, e_1 and e_2 be two orthogonal vectors from $\mathcal{H}$, and f_1, f_2 be polynomials with complex coefficients. For $e \in \mathcal{H} \setminus \{0\}$ we will denote by $\mathcal{A}(e)$ the algebra generated by the creation and annihilation operators over e .*

- (1) *The algebras $\mathcal{A}(e_1)$ and $\mathcal{A}(e_2)$ are free with respect to the vacuum state $T \mapsto \langle T\omega, \omega \rangle$.*
- (2) *If $\alpha_i = a_{e_i}^* + f(a_{e_i})$ ($i = 1, 2$), then $R_{\alpha_i}(z) = z \cdot f_i(z)$ and $R_{\alpha_1 + \alpha_2}(z) = z \cdot f_1(z) + z \cdot f_2(z)$.*
- (3) *If $f_i(0) \neq 0$ and $\beta_i = [\mathbf{Id}_{\mathcal{T}(\mathcal{H})} + a_{e_i}^*]f(a_{e_i})$ ($i = 1, 2$), then $T_{\beta_i}(z) = f_i(z)$ and $T_{\beta_1\beta_2}(z) = f_1(z) \cdot f_2(z)$.*

2.2. Now consider two complex Hilbert spaces $\mathcal{K}$ and $\mathfrak{H}$ and ω a distinguished unit vector in $\mathfrak{H}$. Take $\mathcal{T}(\mathcal{K}) = \omega_1 \oplus \mathcal{T}^0(\mathcal{K})$, where again $\|\omega_1\| = 1$ and

$$\mathcal{E}(\mathfrak{H}, \mathcal{K}) = \mathfrak{H} \oplus (\mathfrak{H} \otimes \mathcal{T}(\mathcal{K})).$$

Later in this section and in Section 4, we will consider $\mathcal{E}(\mathfrak{H}, \mathcal{K})$ for a particular $\mathfrak{H}$. When there is no possibility of confusion, to simplify the writing, we will use $\mathcal{E}$ for $\mathcal{E}(\mathfrak{H}, \mathcal{K})$. Put $\mathfrak{H}^0 = \mathfrak{H} \ominus \mathbb{C}\omega$, $\Omega = \omega \otimes \omega_1$, $\mathcal{E}^0 = \mathcal{E} \ominus \mathbb{C}\Omega$. We define the following embedding $\pi : \mathcal{L}(\mathcal{K}) \longrightarrow \mathcal{L}(\mathcal{T}(\mathcal{H}))$:

$$\pi(a) = a \oplus a \otimes (\mathbf{Id}_{\mathfrak{H} \otimes \mathcal{T}^0(\mathcal{K})} \oplus 0_{\mathbb{C}\Omega}).$$

Note that $\pi(\mathcal{L}(\mathfrak{H}))$ has unit $\pi(\mathbf{Id}_{\mathfrak{H}}) = \mathbf{Id}_{\mathcal{E}^0} \neq \mathbf{Id}_{\mathcal{E}}$.

For a non-zero vector $\eta \in \mathcal{K}$ we define the operators A_η^* and $\{A_{\eta,n}\}_{n \geq 0}$ from $\mathcal{L}(\mathcal{E})$ as follows:

$$\begin{aligned} A_\eta^* \zeta &= 0, \text{ if } \zeta \in \mathfrak{H} \oplus \mathfrak{H}^0 \otimes \mathcal{T}(\mathcal{K}), \\ A_\eta^* \omega \otimes \omega_1 &= \omega \otimes \eta, \\ A_\eta^* \omega \otimes \zeta &= \omega \otimes (\eta \otimes \zeta) \text{ for all } \zeta \in \mathcal{T}^0(\mathcal{K}). \end{aligned}$$

We put $A_{\eta,0} = \mathbf{Id}_{\mathbb{C}\Omega}$ and, for $n \leq 1$, we define $A_{\eta,n}$ via

$$\begin{aligned} A_{\eta,n} \omega \otimes (\eta^{\otimes n}) &= \omega \otimes \omega_1, \\ A_{\eta,n} \zeta &= 0, \text{ if } \zeta \notin \mathbb{C}\omega_1 \otimes (\eta^{\otimes n}), \text{ where } \eta^{\otimes n} = \underbrace{\eta \otimes \cdots \otimes \eta}_{n \text{ times}}. \end{aligned}$$

We will use the notation $\mathcal{D}(\eta)$ for the algebra generated by A_η^* and $\{A_{\eta,n}\}_{n \geq 0}$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two subalgebras of $\mathcal{L}(\mathcal{E})$, then the notation $\mathcal{A}_1 \vee \mathcal{A}_2$ will stand for the algebra generated by them in $\mathcal{L}(\mathcal{E})$.

Remark 2.2. Fix $\eta, \eta_0 \in \mathcal{K}$ unit vectors. From the definitions of $\pi, A_\eta^*, A_{\eta,n}$, trivial verifications give that

$$A_{\eta,n} (A_\eta^*)^p = \begin{cases} A_{\eta,n-p} & \text{if } n \geq p, \\ 0 & \text{if } n < p. \end{cases}$$

If $x \in \pi(\mathcal{L}(\mathfrak{H}))$ and $n \geq 0$, then $xA_{\eta,n} = 0$. Also, if $m, n > 0$, then

$$\begin{aligned} A_{\eta,n} A_{\eta_0,m} &= 0, \\ \mathbf{Id}_{\mathbb{C}\Omega} A_{\eta,n} &= A_{\eta,n}; \quad A_{\eta,n} \mathbf{Id}_{\mathbb{C}\Omega} = 0, \\ A_{\eta,n} \mathbf{Id}_{\mathcal{E}^0} &= 0; \quad \mathbf{Id}_{\mathcal{E}^0} A_\eta^* = A_\eta^* \mathbf{Id}_{\mathcal{E}^0} = A_\eta^*. \end{aligned}$$

Remark 2.3. For $\eta_1, \eta_2 \in \mathcal{K}$, we have that

$$\text{Range}\left((A_{\eta_1}^*)^p A_{\eta_2}^*\right) = \text{Span}\{\omega \otimes \eta_2 \otimes \eta_1^{\otimes p}, \omega \otimes \zeta \otimes \eta_2 \otimes \eta_1^{\otimes p} : \zeta \in \mathcal{T}^0(\mathcal{K})\};$$

therefore, if $\eta_1 \perp \eta_2$, then

$$A_{\eta_1, n}(A_{\eta_1}^*)^p A_{\eta_2}^* = 0.$$

On $\mathcal{L}(\mathcal{E})$ we consider the functionals $\phi(\cdot) = \langle \cdot, \Omega, \Omega \rangle$ and $\psi(\cdot) = \langle \cdot, \Omega_1, \Omega_1 \rangle$.

Lemma 2.4. *Suppose $\eta_1, \eta_2 \in \mathcal{K}$ and $x \in \pi(\mathcal{L}(\mathfrak{H}))$. Then*

$$\begin{aligned} A_{\eta_1}^* x A_{\eta_2}^* &= A_{\eta_1}^* \psi(x) A_{\eta_2}^*, \\ A_{\eta_1, n} x A_{\eta_2}^* &= A_{\eta_1, n} \psi(x) A_{\eta_2}^*. \end{aligned}$$

Proof. Since $A_{\eta_2}(\mathcal{E}) = \omega_1 \otimes \mathcal{T}^0(\mathcal{K})$, for any $\zeta \in \mathcal{E}$ we have that

$$\begin{aligned} A_{\eta_2} \zeta &= \omega_1 \otimes \zeta' \text{ for some } \zeta' \in \mathcal{T}^0(\mathcal{K}), \\ x A_{\eta_2} \zeta &= \psi(x) \omega_1 \otimes \zeta' + v \otimes \zeta' \text{ for some } v \in \mathcal{T}^0(\mathcal{H}). \end{aligned}$$

But $\mathcal{T}^0(\mathcal{H}) \otimes \mathcal{T}^0(\mathcal{K}) \subset \ker(A_{\eta_1}^*), \ker(A_{\eta_1, n})$; hence the conclusion. $\square$

In the proof of the main result of this section, Theorem 2.6, we will use the following lemma:

Lemma 2.5. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two free independent subalgebras of an algebra $\mathcal{A}$ with respect to some linear map φ and that $a_0, a_1, \dots, a_{n+1} \in \mathcal{A}_1$, $b_1, \dots, b_n \in \mathcal{A}_2$ are such that $\varphi(a_k) = \varphi(b_k) = 0$ for all $1 \leq k \leq n$. Then*

$$\varphi(a_0 b_1 \cdots b_n a_{n+1}) = 0.$$

Proof. Take $d_j = a_j - \varphi(a_j)$, $j \in \{0, n+1\}$. Then $\varphi(d_j) = 0$ and $a_j = \varphi(a_j) + d_j$; therefore

$$\begin{aligned} \varphi(a_0 b_1 \cdots b_n a_{n+1}) &= \varphi(a_0) \varphi(a_0 b_1 \cdots b_n d_{n+1}) + \varphi(a_0) \varphi(a_0 b_1 \cdots b_n) \varphi(a_{n+1}) \\ &\quad + \varphi(d_0 b_1 \cdots b_n d_{n+1}) + \varphi(d_0 b_1 \cdots b_n) \varphi(a_{n+1}), \end{aligned}$$

and all the above four terms cancel from the definition of free independence. $\square$

Theorem 2.6. *Let $\mathcal{A}_1, \mathcal{A}_2$ be two subalgebras of $\mathcal{L}(\mathfrak{H})$ which are free independent with respect to ψ and let η_1, η_2 be two orthogonal unit vectors $\mathcal{K}$. Then the algebras $\mathfrak{A}_1 = \pi(\mathcal{A}_1) \vee \mathcal{D}(\eta_1)$ and $\mathfrak{A}_2 = \pi(\mathcal{A}_2) \vee \mathcal{D}(\eta_2)$ are c -free with respect to (ϕ, ψ) .*

Proof. It suffices to prove that for $x_1, \dots, x_m$ such that $x_j \in \mathcal{A}(e_{\epsilon(j)}, \eta_{\epsilon(j)})$ with $\epsilon(i) \neq \epsilon(i+1)$ and $\psi(x_k) = 0$, we have

$$(2) \quad \psi(x_m \cdots x_2 x_1) = 0,$$

$$(3) \quad \phi(x_m \cdots x_2 x_1) = \phi(x_m) \cdots \phi(x_2) \phi(x_1).$$

Note that $\mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K}) \perp \Omega_1$ and

$$\begin{aligned} \mathcal{D}(\eta_i)(\mathcal{E}) &\subseteq \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K}), \\ \pi(\mathcal{A}_i)(\mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K})) &\subseteq \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K}). \end{aligned}$$

Hence ψ cancels on all reduced products from $\mathfrak{A}_1 \cup \mathfrak{A}_2$ that contain factors from $\mathcal{D}(\eta_1)$ or $\mathcal{D}(\eta_2)$. It follows that we only need to prove the relation (2) for $x_1, \dots, x_m \in \pi(\mathcal{A}_1) \cup \pi(\mathcal{A}_2)$, the statement which is equivalent to the free independence of $\mathcal{A}_1$ and $\mathcal{A}_2$.

We will prove (3) by induction on n . For $n = 1$, the assertion is trivial. For the induction step, it suffices to prove that

$$(4) \quad \phi(x_n \cdots x_1) = \phi(x_n)\phi(x_{n-1} \cdots x_1).$$

Taking $x'_n = x_n - \phi(x_n)\mathbf{Id}_{\mathbb{C}\Omega}$, we have that $\phi(x'_n) = 0$; hence (4) is equivalent to $\phi(x_n \cdots x_1) = 0$ whenever $\phi(x_n) = 0$.

Suppose $x_n \in \mathcal{A}_1 \vee \mathcal{D}(\eta_1)$. Then x_n is a linear combination of monomials in elements from $\mathcal{A}_1$ and $\mathcal{D}(\eta_1)$. From Lemma 2.4, we can suppose that all factors from $\mathcal{D}(\eta_1)$ are consecutive, so x_n is a sum of elements from $\mathcal{A}_1$ and monomials of the types $y'_1(A_{\eta_1}^*)^p A_{\eta_1, m} y_1$ or $y'_1(A_{\eta_1}^*)^p y_1$, with $y'_1, y_1 \in \mathcal{A}_1 \cup \mathbf{Id}$ and $p \geq 0$. If $y'_1 \neq \mathbf{Id}$ or $p \neq 0$, then $x_n(\mathcal{E}) \perp \Omega$; hence $\phi(x_n \cdots x_1) = 0$. Also, if $m = 0$, then either $y_1 = \mathbf{Id}$ and $\phi(x_n) \neq 0$ or $y_1 \in \mathcal{A}_1$ and $x_n = 0$. Therefore we can suppose that $x_n = A_{\eta_1, m} y_1$ for some $m > 0$ and $y_1 \in \mathcal{A}_1$ and that all other x_j are either elements of $\mathcal{A}_{\epsilon(j)}$ or monomials as above.

Let $k = \max\{j : x_j \text{ contains } A_{\epsilon(j)}^*\}$ and $p = \max\{j : x_j \text{ contains } A_{\epsilon(j)}\}$. If $p > k$, then $x_n \cdots x_p = A_{\eta_1, m} y_1 A_{\eta_{\epsilon(p)}} y'$, for some $y \in \mathcal{A}_1 \vee \mathcal{A}_2$ and $y' \in \mathcal{A}_{\epsilon(p)} \vee \mathcal{D}(\eta_{\epsilon(j)})$. From Lemma 2.4 and Remark 2.2, $A_{\eta_1, m} y_1 A_{\eta_{\epsilon(p)}} = A_{\eta_1, m} \psi(y) A_{\eta_{\epsilon(p)}} = 0$.

Suppose that $p \leq k$. If $\epsilon(k) = 2$, then $x_n \cdots x_k = A_{\eta_1, m} y_1 A_{\eta_2}^* y'$, for some $y \in \mathcal{A}_1 \vee \mathcal{A}_2$ and $y' \in \mathcal{A}_2 \vee \mathcal{D}(\eta_2)$. Applying Lemma 2.4 and Remark 2.3, we have

$$x_n \cdots x_k = A_{\eta_1, m} \psi(y) A_{\eta_2}^* y' = 0.$$

If $\epsilon(j) = 1$, then, from Remark 2.3,

$$\begin{aligned} x_n \cdots x_k &= A_{\eta_1, m} y_1 x_{n-1} \cdots x_{k+1} y_1 A_{\eta_1}^* y' \\ &= A_{\eta_1, m} \psi(y_1 x_{n-1} \cdots x_{k+1} y_1) A_{\eta_1}^* y', \end{aligned}$$

but $\psi(y_1 x_{n-1} \cdots x_{k+1} y_1) = 0$ from Lemma 2.5, so the proof is complete. $\square$

Corollary 2.7. *With the notation from subsection 2.1, take $\mathfrak{H} = \mathbb{C}\omega \oplus \mathcal{T}^0(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space of dimension at least 2.*

Let e_1, e_2 , respectively η_1, η_2 , be two pairs of orthogonal unit vectors from $\mathcal{H}$, respectively $\mathcal{K}$. Then the algebras $\mathcal{D}(\eta_1) \vee \pi(\mathcal{A}(e_1))$ and $\mathcal{D}(\eta_2) \vee \pi(\mathcal{A}(e_2))$ are c-free with respect to the maps ϕ and ψ considered above.

Proof. From Theorem 2.1(1), the algebras $\mathcal{A}(e_1)$ and $\mathcal{A}(e_2)$ are free in $\mathcal{L}(\mathcal{T}(\mathcal{H}))$ with respect to $\langle \cdot, \omega, \omega \rangle$, and the conclusion follows from Theorem 2.6. $\square$

3. THE cR - AND cT -TRANSFORMS

Consider an algebra $\mathcal{A}$ with two states $\phi, \psi : \mathcal{A} \rightarrow \mathbb{C}$ and $X \in \mathcal{A}$. Let $m_X(z) = \sum_{k=1}^{\infty} \psi(X^k)$, respectively $M_X(z) = \sum_{k=1}^{\infty} \phi(X^k)$, be the moment-generating series of X with respect to ψ , respectively ϕ . We define the cR - and, if $\psi(X) \neq 0$, the cT -transforms of X by the following equations:

$$(5) \quad {}^cR_X(z[1 + m_X(z)]) \cdot (1 + M_X(z)) = M_X(z)[1 + m_X(z)],$$

$$(6) \quad [{}^cT_X(m_X(z))] \cdot (1 + M_X(z)) = \frac{M_X(z)}{z}.$$

With notation from Section 2, for $\eta \in \mathcal{K}$ a non-zero vector and $f = \sum_{k=0}^N f_k X^k$ a polynomial with complex coefficients, we define $A_{\eta, f \otimes}$ via

$$A_{\eta, f \otimes} = \sum_{k=0}^N f_k \cdot A_{\eta, k}.$$

3.1. The ${}^c R$ -transform.

Theorem 3.1. *Let η be a unit vector from $\mathcal{K}$, $b \in \pi(\mathcal{L}(\mathfrak{H}))$ and $f = \sum_{p=0}^M g_p \cdot z^p$ be a polynomial with complex coefficients. Consider $\alpha \in \mathcal{L}(\mathcal{E})$ given by*

$$\alpha = b + A_{\eta}^* + A_{\eta, f \otimes}.$$

Then ${}^c R_{\alpha}(z) = zf(z)$.

Proof. It suffices to show that $zf(z)$ satisfies equation (5), which is equivalent to the following recurrence:

$$(7) \quad \phi(\alpha^n) = \sum_{0 \leq p \leq n} \sum_{\substack{q_1, \dots, q_p \geq 0 \\ n \geq 1 + p + q_1 + \dots + q_p}} \phi\left(\alpha^{n-1-(p+q_1+\dots+q_p)}\right) \cdot f_p \cdot \psi(\alpha^{q_1}) \cdots \psi(\alpha^{q_p})$$

for all $n > 0$.

Let us denote $A = A_{\eta}^*$ and $B = A_{\eta, f \otimes}$. The triple (b, A, B) satisfies the following relations:

$$(8) \quad b\Omega = 0, B(\mathcal{E}) = \mathbb{C}\Omega,$$

$$(9) \quad Ab^q A = A\psi(b^q)A, Bb^q A = B\psi(b^q)A \text{ for all } q > 0,$$

$$(10) \quad \phi(BA^n) = f_n, \text{ for all } n \geq 0$$

(equations (8) and (9) are consequences of the relations from Remark 2.2, and (10) follows from Lemma 2.4).

Let $I = \{b, A, B\}$. Since $\alpha = \sum_{x \in I} x$, we have that

$$(11) \quad \phi(\alpha^n) = \sum_{(x_1, \dots, x_n) \in I^n} \phi(x_n x_{n-1} \cdots x_1).$$

To further simplify the writing, we introduce the following notation:

$$I[n, j] = \left\{ (x_1, \dots, x_n) \in I^n, \min\{k : x_k = B\} = j \right\}.$$

Since $b\Omega = 0$ and $A(\mathcal{E}) \perp \Omega$, we have that $\phi(x_n \cdots x_1) = 0$ unless $x_n = B$; hence $(x_n, \dots, x_1) \in I[n, j]$ for some j . Also, for $(x_n, \dots, x_1) \in I[n, j]$, since $B(\mathcal{E}) = \mathbb{C}\Omega$, we have that $x_j \cdots x_1 \Omega = \phi(x_j \cdots x_1) \Omega$, so $\phi(x_n \cdots x_1) = \phi(x_n \cdots x_{j+1}) \phi(x_j \cdots x_1)$. Therefore (11) becomes

$$\begin{aligned} \phi(\alpha^n) &= \sum_{j=1}^n \sum_{(x_1, \dots, x_n) \in I[n, j]} \phi(x_n \cdots x_1) \\ &= \sum_{j=1}^n \sum_{(x_1, \dots, x_n) \in I[n, j]} \phi(x_n \cdots x_{j+1}) \phi(x_j \cdots x_1) \\ (12) \quad &= \sum_{j=1}^n \sum_{(x_1, \dots, x_j) \in I[j, j]} \phi(\alpha^{n-j}) \phi(x_j \cdots x_1). \end{aligned}$$

Consider $(x_1, \dots, x_n) \in I[n, n]$. If $n = 1$, then $\phi(x_n \cdots x_1) = \phi(B) = f_0$. If $n > 1$, then $x_1 \Omega = 0$ unless $x_1 = A$. Let $1 = k_1 < \dots < k_p < n$ be the set of all indices k such that $x_k = A$. Letting $q_j = k_{j+1} - k_j - 1$, $q_p = n - k_p - 1$ and applying property (10), we obtain

$$\begin{aligned}
 \sum_{(x_1, \dots, x_n) \in I[n, n]} \phi(x_n \cdots x_1) &= \sum_{p=1}^{n-1} \sum_{\substack{0 \leq q_1, \dots, q_p \\ q_1 + \dots + q_p < n-p}} \phi(B \cdot \psi(b^{q_p}) \cdot A \cdots \psi(b^{q_1})A) \\
 (13) \qquad \qquad \qquad &= \sum_{\substack{0 \leq q_1, \dots, q_p \\ q_1 + \dots + q_p < n-p}} f_p \cdot \psi(b^{q_p}) \cdots \psi(b^{q_1}).
 \end{aligned}$$

Finally, the equality $\psi(b^q) = \psi(\alpha^q)$ and equations (12), (13) imply (7), so the proof is complete. $\square$

3.2. The cT -transform.

Theorem 3.2. *Let η be a unit vector from $\mathcal{K}$, $d \in \pi(\mathcal{L}(\mathfrak{H}))$ and $f(z) = \sum_{k=0}^M f_k \cdot z^k$ be a polynomial with complex coefficients such that $\psi(b), f_0 \neq 0$. Consider $\beta \in \mathcal{L}(\mathcal{E})$ given by*

$$\beta = d + dA_\eta^* + A_{\eta, f \otimes}.$$

Then ${}^cT_\beta(z) = f(z)$.

Proof. The proof is similar to the one of Theorem 3.1. Again denote $A = A_\eta^*$, $B = A_{\eta, f \otimes}$ and consider the sets

$$J = \{d, dA, B\},$$

$$J[n, l] = \{(x_1, \dots, x_n) \in J^n, \min\{k : x_k = B\} = l\}.$$

For $(x_1, \dots, x_n) \in J^n$, we have that $\phi(x_n \cdots x_1) = 0$ unless $x_n = B$; hence $(x_1, \dots, x_n) \in J[n, l]$ for some $1 \leq l \leq n$. Also, note that equation (12) holds true if we replace I with J ; therefore

$$(14) \qquad \phi(\beta^n) = \sum_{l=1}^n \phi(\beta^{n-l}) \sum_{(x_1, \dots, x_l) \in J[l, l]} \phi(x_l \cdots x_1).$$

Fix $n > 0$ and let $(x_1, \dots, x_1) \in J[n, n]$. If $n = 1$, then $\phi(x_n, \dots, x_1) = \phi(B) = f_0$. If $n = 1$, then $\phi(x_n \cdots x_1)$ cancels unless $x_1 = dA$. Let $1 = k_1 < k_2 < \dots < k_p \leq n - 1$ be the set of indices k such that $x_k = dA$. Taking $q_j = k_{j+1} - k_j - 1$ for $1 < j < n - 1$ and $q_p = n - k_p - 1$, and applying property (10), we obtain

$$\begin{aligned}
 (15) \qquad \sum_{(x_1, \dots, x_n) \in J[n, n]} \phi(x_n \cdots x_1) &= \sum_{p=1}^{n-1} \sum_{\substack{0 \leq q_1, \dots, q_p \\ q_1 + \dots + q_p < n}} \phi(B \cdot (b^{q_p-1}) \cdot dA \cdot b^{q_{p-1}-1} \cdots b^{q_1-1} \cdot dA) \\
 &= \sum_{p=1}^{n-1} \sum_{\substack{0 \leq q_1, \dots, q_p \\ q_1 + \dots + q_p < n}} \left(B \cdot \psi(d^{q_p}) \cdot A \cdot \psi(b^{q_{p-1}}) \cdots \psi(d^{q_1})A \right) \\
 &= \sum_{p=1}^{n-1} \sum_{\substack{0 \leq q_1, \dots, q_p \\ q_1 + \dots + q_p < n}} f_p \cdot \psi(d^{q_p}) \cdots \psi(d^{q_1}).
 \end{aligned}$$

The conclusion follows, since (15), (14) and the identity $\psi(\beta^q) = \psi(d^q)$ imply that the $f(z)$ satisfies (6). $\square$

4. ADDITION AND MULTIPLICATION OF C-FREE RANDOM VARIABLES

Theorem 4.1. *Let η_1, η_2 be orthogonal unit vectors from $\mathcal{K}$; let b_1, b_2, d_1, d_2 be some elements from $\pi(\mathcal{L}(\mathfrak{H}))$; and let f_1, f_2, F_1, F_2 be polynomials with complex coefficients, such that $\psi(d_i \neq 0 \neq F_i(0))$. Define ($i = 1, 2$) :*

$$\begin{aligned}\alpha_i &= b_i + A_{\eta_i}^* + A_{\eta_i, f_i \otimes}, \\ \beta_i &= d_i + d_i \cdot A_{\eta_i}^* + A_{\eta_i, F_i \otimes}.\end{aligned}$$

Then ${}^cR_{\alpha_1 + \alpha_2}(z) = {}^cR_{\alpha_1}(z) + {}^cR_{\alpha_2}(z)$ and ${}^cT_{\beta_1 \cdot \beta_2}(z) = {}^cT_{\beta_1}(z) \cdot {}^cT_{\beta_2}(z)$.

Proof. Suppose that $F_1(z) = \sum_{k=0}^M h_k \cdot z^k$ and $F_2(z) = \sum_{k=0}^M l_k \cdot z^k$ (eventually h_M or l_M are zero).

To prove the first equality, we introduce the notation

$$\begin{aligned}\tilde{b} &= b_1 + b_2, \\ \tilde{A} &= A_{\eta_1}^* + A_{\eta_2}^*, \\ \tilde{B} &= A_{\eta_1, F_1 \otimes} + A_{\eta_2, F_2 \otimes}.\end{aligned}$$

Trivial verifications show that the triple $(\tilde{b}, \tilde{A}, \tilde{B})$ verifies conditions (8)–(10). Therefore recurrence (7) holds true for $\alpha = \tilde{b} + \tilde{A} + \tilde{B} = \alpha_1 + \alpha_2$ and $\{g_k\}_{k=1}^M$ the coefficients of $f_1(z) + f_2(z)$, so ${}^cR_{\alpha_1 + \alpha_2}(z) = z[f_1(z) + f_2(z)]$, completing the proof. $\square$

For the second equality, we need to prove that $F_1(z) \cdot F_2(z)$ satisfies (6) for $\beta = \beta_1 \cdot \beta_2$; that is, the recurrence formula

$$\begin{aligned}(16) \quad \phi(\beta^n) &= \sum_{p=0}^{n-1} \sum_{\substack{q_1, \dots, q_p > 0 \\ q_1 + \dots + q_p \leq n}} \phi(\beta^{n-1-(q_1 + \dots + q_p)}) \cdot [g_p \cdot \psi(\beta^{q_p}) \dots \psi(\beta^{q_1})] \\ &= \sum_{m=1}^n \phi(\beta^{n-m}) \cdot \left[\sum_{p=1}^{m-1} \sum_{\substack{q_1, \dots, q_p > 0 \\ q_1 + \dots + q_p < m}} g_p \cdot \psi(\beta^{q_p}) \dots \psi(\beta^{q_1}) \right]\end{aligned}$$

is verified for $\beta = \beta_1 \cdot \beta_2$ and g_m the coefficient of z^m in $F_1(z) \cdot F_2(z)$.

We introduce the notation

$$\begin{aligned}b &= d_1; \quad d = d_2, \\ A_1 &= A_{\eta_1}^*; \quad A_2 = A_{\eta_2}^*, \\ B_1 &= A_{\eta_1, F_1 \otimes}; \quad B_2 = A_{\eta_2, F_2 \otimes}.\end{aligned}$$

Then $B_i A_j = 0$ whenever $i \neq j$ and $\beta_1 = b + bA_1 + B_1$, $\beta_2 = d + dA_2 + B_2$. For $\beta = \beta_1 \cdot \beta_2$, we have that

$$\begin{aligned}\beta &= (b + bA_1 + B_1)(d + dA_2 + B_2) \\ &= bd + bdA_2 + bB_2 + bA_1d + bA_1dA_2 + bA_1B_2 + B_1d + B_1dA_2 + B_1B_2.\end{aligned}$$

Consider the sets

$$\begin{aligned}\mathcal{J} &= \{bd, bdA_2, bA_1d, bA_1dA_2, bA_1B_2, B_1d, B_1dA_2, B_1B_2\}, \\ \mathcal{J}[n, m] &= \left\{ (x_1, \dots, x_n) \in \mathcal{J}^n, \min_k \{x_k \in \{B_1d, B_1B_2\}\} = m \right\}, \\ \overline{\mathcal{J}} &= \mathcal{J} \cup \{bA_1dA_2\}.\end{aligned}$$

Note first that $bB_2 = 0$. Also, since Lemma 2.4 implies $B_1dA_2 = B_1\psi(d)A_2 = 0$, we have that $\beta = \sum_{x \in \overline{\mathcal{J}}} x$; hence

$$\phi(\beta^n) = \sum_{(x_1, \dots, x_n) \in \overline{\mathcal{J}}} \phi(x_n \cdots x_1).$$

If some x_k is bA_1dA_2 , then $\phi(x_n \cdots x_1)$ cancels, since the vectors from $\mathcal{E}$ with the $\mathcal{T}(\mathcal{K})$ component containing mixed tensors in η_1 and η_2 are cancelled by any B_i ($i = 1, 2$) and are also orthogonal to Ω . On the other hand, $\phi(x_n \cdots x_1)$ also cancels if $x_n \cdots x_1 \perp \Omega$, that is, if x_n does not start with some B_i . It follows that only terms having $x_n \in \{B_1d, B_1B_2\}$ contribute to the sum; that is, the sum can be taken only for $(x_1, \dots, x_n) \in \mathcal{J}[n, m]$ ($1 \leq m \leq n$).

Now consider $(x_1, \dots, x_n) \in \mathcal{J}[n, m]$. Then $x_m \cdots x_1 \Omega = \phi(x_m \cdots x_1) \Omega$; hence $\phi(x_n \cdots x_1) = \phi(x_n \cdots x_m) \phi(x_m \cdots x_1)$, and

$$\begin{aligned}(17) \quad \sum_{(x_1, \dots, x_n) \in \mathcal{J}[n, m]} \phi(x_n \cdots x_1) &= \sum_{(x_1, \dots, x_n) \in \mathcal{J}[n, m]} \phi(x_n \cdots x_{m+1}) \cdot \phi(x_m \cdots x_1) \\ &= \phi(\beta^{n-m}) \cdot \left[\sum_{(x_1, \dots, x_m) \in \mathcal{J}[m, m]} \phi(x_m \cdots x_1) \right].\end{aligned}$$

Therefore it suffices to prove that the second factors from the right hand sides of (17) coincide to the second factor of the m -th summand in (16); that is (we use that $g_m = \sum_{p+k=m} l_p \cdot h_k$),

$$\begin{aligned}(18) \quad \sum_{(x_1, \dots, x_n) \in \mathcal{J}[n, n]} \phi(x_n \cdots x_1) &= \sum_{m=0}^{n-1} g_m \cdot \sum_{\substack{q_1, \dots, q_m > 0 \\ q_1 + \dots + q_m < n}} \psi(\beta^{q_m}) \cdots \psi(\beta^{q_1}) \\ &= \sum_{\substack{p, k \geq 0 \\ p+q < n}} E(p, k)\end{aligned}$$

for $E(p, k) = [l_p \psi(\beta^{q_m}) \cdots \psi(\beta^{q_1})] \cdot [h_k \psi(\beta^{s_k}) \cdots \psi(\beta^{s_1})]$, where the summation is done over all $p, q \geq 0$ such that $p + q < n$ and all $q_1, \dots, q_p, s_1, \dots, s_k > 0$ with $q_1 + \dots + q_p + s_1 + \dots + s_k = n - 1$.

Consider $(x_1, \dots, x_n) \in \mathcal{J}[n, n]$ such that $\phi(x_n \cdots x_1) \neq 0$; then $x_1 \Omega \neq 0$, so $x_1 \in \{B_1B_2, bA_1B_2, bdA_2\}$.

Case I: $x_1 = B_2B_2$. This implies that $n = 1$ (since x_1 already starts with a B_i) and $\phi(x_1) = \langle B_1B_2\Omega, \Omega \rangle = l_0h_0$, that is, (18) for $n = 1$.

Case II: $x_1 = bA_1B_2$. In this case $x_1\Omega = bA_1B_2\Omega = bA_1\Omega \cdot l_0$.

Let $j = \min\{k : x_k \text{ contains } B_1, \text{ i.e. } x_k \in \{B_1B_2, B_1d\}\}$ (since $x_n \in \{B_1B_2, B_1d\}$, the set is not void). Also, $j \neq n$ will contradict the definition of $\mathcal{J}[n, n]$, so $j = n$.

Tensors containing η_1 are canceled by B_2 and, as seen earlier, summands with A_2 and A_1 not separated by B_1 do not contribute to the sum. Therefore it follows

that $x_2, \dots, x_{n-1}$ do not contain A_2 or B_2 , so they can be only of the types bd and bA_1d .

Let $1 < k_2 < \dots < k_p < n$ be the indices of the factors of type bA_1d and put $k_1 = 1$ and $k_{p+1} = n$. For $q_i = k_{i+1} - k_i$, applying Lemma 2.4, we have

$$\begin{aligned}
 x_n \cdots x_2 bA_1\Omega &= B_1 d(bd)^{q_p-1} bA_1 d(bd)^{q_{p-1}-1} \cdots (bd)^{q_1-1} bA_1\Omega \\
 &= B_1 \psi((db)^{q_p}) A_1 \psi((db)^{q_{p-1}}) \cdots \psi((db)^{q_1}) A_1\Omega \\
 &= (B_1 A_1^p \Omega) \psi((db)^{q_p}) \cdots \psi((db)^{q_1}) \\
 (19) \quad &= h_p \Omega \cdot \psi(\beta^{q_p}) \cdots \psi(\beta^{q_1})
 \end{aligned}$$

since $B_1 A_1^p \Omega = h_p \Omega$ and $\psi(db) = \psi(bd) = \psi(\beta)$ due to the traciality of the vector states. Multiplying with l_0 and summing, we obtain

$$(20) \quad \sum_{\substack{(x_1, \dots, x_n) \in \mathcal{J}[n, n] \\ x_1 = dA_1 B_2}} \phi(x_n \cdots x_1) = \sum_{p=1}^{n-1} E(p, 0).$$

Case III: $x_1 = bdA_2$.

Let x_j be the factor of the smallest index that contains B_2 . Since $x_j \in \mathcal{J}$, we have that $x_j = y \cdot B_2$, with $y \in \{bA_1, B_1\}$.

None of the factors $x_2, \dots, x_{j-1}$ contain B_1 (otherwise it will contradict the definition of $J[n, n]$). If some of them will contain A_1 , then $yB_2x_{j-1} \cdots x_2 = 0$, since B_2 cancels all the tensors mixing η_1 and η_2 . Hence $x_2, \dots, x_{j-1} \in \{bd, bdA_2\}$. Again, let $1 = j_1 < \dots < j_k < n$ be the indices of the factors of type bdA_2 and put $j_{k+1} = j$. For $s_i = k_{i+1} - k_i$, applying Lemma 2.4, we have

$$\begin{aligned}
 x_j \cdots x_1 \Omega &= y \cdot B_2 (bd)^{s_k} A_2 (bd)^{s_{k-1}} A_2 \cdots (bd)^{s_1} A_2 \Omega \\
 (21) \quad &= y \Omega \cdot l_k \psi(\beta)^{s_k} \cdots \psi(\beta^{s_1})
 \end{aligned}$$

(we used that $B_2 A_2^k \Omega = l_k \Omega$ and that $\psi(\beta^s) = \psi((bd)^s)$).

If $y = B_1$, then $y\Omega = h_0$; also, the minimality of n implies $j = n$.

If $y = bA_1$, since $x_n \in \{B_1 B_2, B_1 d\}$, the minimality of n implies that $x_{j+1}, \dots, x_{n-1}$ do not contain B_1 . If they do contain A_2 or B_2 , then $\phi(x_n \cdots x_1) = 0$ as seen earlier, so they must be of the types bd or bA_1d . Also, if $x_n = B_1 B_2$, then again $\phi(x_n \cdots x_1) = 0$, so we can suppose $x_n = B_1 d$. In this case, $x_n \cdots x_{j+1} y \Omega$ is in the setting of formula (19), so it is computed accordingly.

Summing, we obtain

$$(22) \quad \sum_{\substack{(x_1, \dots, x_n) \in \mathcal{J}[n, n] \\ x_1 = bA_2}} \phi(x_n \cdots x_1) = \sum_{\substack{p, k-1 > 0 \\ p+k < n}} E(p, k),$$

and the conclusion follows, since (22) and (20) imply (18). $\square$

Corollary 4.2. *Let $\mathcal{A}$ be a unital algebra, $\Phi, \Psi : \mathcal{A} \rightarrow \mathbb{C}$ be two linear maps with $\Phi(1) = \Psi(1) = 1$ and let X, Y be two c -free (with respect to the maps Φ, Ψ) elements from $\mathcal{A}$.*

- (i) $R_{X+Y} = R_X + R_Y$ and ${}^c R_{X=Y} = {}^c R_X + {}^c R_Y$ as formal power series.
- (ii) If $\Psi(X), \Psi(Y)$ are non-zero, then $T_{XY} = T_X \cdot T_Y$ and ${}^c T_{XY} = {}^c T_X \cdot {}^c T_Y$ as formal power series.

Proof. The equalities for R - and T -transforms are basic properties in the Free Probability Theory (see [21], [13]). We need to prove (i) and (ii) for the cR - and cT -transforms.

As in Corollary 2.7, we will consider two complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ of dimension at least two and $\mathcal{E} = \mathcal{T}(\mathcal{H}) \oplus [\mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K})]$, where

$$\begin{aligned}\mathcal{T}(\mathcal{H}) &= \mathbb{C}\omega \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \dots, \\ \mathcal{T}(\mathcal{K}) &= \mathbb{C}\omega \oplus \mathcal{K} \oplus (\mathcal{K} \otimes \mathcal{K}) \oplus \dots.\end{aligned}$$

We fix e_1, e_2 , respectively η_1, η_2 , as two pairs of orthogonal unit vectors from $\mathcal{H}$, respectively $\mathcal{K}$. From Corollary 2.7, the algebras $\mathfrak{A}_1 = \mathcal{D}(\eta_1) \vee \pi(\mathcal{A}(e_1))$ and $\mathfrak{A}_2 = \mathcal{D}(\eta_2) \vee \pi(\mathcal{A}(e_2))$ are c-free with respect to $\phi(\cdot) = \langle \cdot \omega \otimes \omega, \omega \otimes \omega \rangle$ and $\psi(\cdot) = \langle \cdot \omega, \omega \rangle$.

Also note that according to the relations defining the free, respectively c-free, independence (see subsection 2.1), the moments up to order N of $X + Y$ and XY with respect to Φ and Ψ are uniquely determined by the moments of order up to N of X and Y .

For (i), consider $f_1(z)$, $F_1(z)$, respectively $f_2(z)$, $F_2(z)$ to be the polynomials obtained by the truncation of order N of R_X , cR_X , respectively R_Y , cR_Y (i.e. if ${}^cR_X(z) = \sum_{k=1}^{\infty} l_k \cdot z^k$, then $F_1(z) = \sum_{k=1}^N l_k \cdot z^k$ and the analogues).

With the notation from Section 2, take ($i = 1, 2$)

$$\alpha_i = \pi(a_{e_i}^* + f_i(a_{e_i})) + A_{\eta_i}^* + A_{\eta_i, F_i^{\otimes}}.$$

We have that $\alpha_i \in \mathfrak{A}_i$, so α_1, α_2 are c-free with respect to ϕ and ψ ; hence, from Theorem 4.1,

$$(23) \quad {}^cR_{\alpha_1 + \alpha_2}(z) = {}^cR_{\alpha_1}(z) + {}^cR_{\alpha_2}(z).$$

From Theorem 2.1(2) and Theorem 3.1, we have that $R_{\alpha_i}(z) = f_i(z)$ and ${}^cR_{\alpha_i}(z) = F_i(z)$; therefore R_{α_1} , ${}^cR_{\alpha_1}$, and R_X , cR_X coincide up to order N . Then equations (1) and (5) imply that the moments up to order N of X with respect to Φ , respectively Ψ , coincide to the moments up to order N of α_1 with respect to ϕ and ψ . The same holds true for Y and α_2 ; therefore the moments up to order N of $X + Y$ and $\alpha_1 + \alpha_2$ do coincide. Henceforth, from equation (5), the first N coefficients of ${}^cR_{\alpha_1 + \alpha_2}$ and ${}^cR_{X+Y}$ do coincide. Since N is arbitrary, equation (23) gives the conclusion.

The proof for (ii) is similar, taking ($i = 1, 2$)

$$\beta_i = \pi((\mathbf{Id} + a_{e_i}^*)f_i(a_{e_i})) + \pi((\mathbf{Id} + a_{e_i}^*)f_i(a_{e_i}))A_{\eta_i}^* + A_{\eta_i, F_i^{\otimes}},$$

where f_1, F_1 , respectively f_2, F_2 , are now the polynomials given by the truncation of order N of T_X and cT_X , respectively T_Y and cT_Y . $\square$

REFERENCES

- [1] Serban Teodor Belinschi, *C-free convolution for measures with unbounded support*, Von Neumann algebras in Sibiu, Theta Ser. Adv. Math., vol. 10, Theta, Bucharest, 2008, pp. 1–7. MR2512322 (2010k:46068)
- [2] Anis Ben Ghorbal and Michael Schürmann, *Quantum stochastic calculus on Boolean Fock space*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **7** (2004), no. 4, 631–650, DOI 10.1142/S0219025704001815. MR2105916 (2005j:81089)
- [3] Florin Boca, *Free products of completely positive maps and spectral sets*, J. Funct. Anal. **97** (1991), no. 2, 251–263, DOI 10.1016/0022-1236(91)90001-L. MR1111181 (92f:46064)
- [4] Marek Bożejko, *Uniformly bounded representations of free groups*, J. Reine Angew. Math. **377** (1987), 170–186, DOI 10.1515/crll.1987.377.170. MR887407 (89a:22009)

- [5] Marek Bożejko, *Positive definite functions on the free group and the noncommutative Riesz product* (English, with Italian summary), Boll. Un. Mat. Ital. A (6) **5** (1986), no. 1, 13–21. MR833375 (88a:43007)
- [6] Marek Bożejko and Roland Speicher, *ψ -independent and symmetrized white noises*, Quantum probability & related topics, QP-PQ, VI, World Sci. Publ., River Edge, NJ, 1991, pp. 219–236. MR1149828
- [7] Marek Bożejko, Michael Leinert, and Roland Speicher, *Convolution and limit theorems for conditionally free random variables*, Pacific J. Math. **175** (1996), no. 2, 357–388. MR1432836 (98j:46069)
- [8] Kenneth J. Dykema, *Multilinear function series and transforms in free probability theory*, Adv. Math. **208** (2007), no. 1, 351–407, DOI 10.1016/j.aim.2006.02.011. MR2304321 (2008k:46193)
- [9] Etienne F. Blanchard and Kenneth J. Dykema, *Embeddings of reduced free products of operator algebras*, Pacific J. Math. **199** (2001), no. 1, 1–19, DOI 10.2140/pjm.2001.199.1. MR1847144 (2002f:46115)
- [10] Uffe Haagerup, *On Voiculescu’s R - and S -transforms for free non-commuting random variables*, Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI, 1997, pp. 127–148. MR1426838 (98c:46137)
- [11] Anna Dorota Krystek, *Infinite divisibility for the conditionally free convolution*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **10** (2007), no. 4, 499–522, DOI 10.1142/S0219025707002919. MR2376439 (2009d:46118)
- [12] Wojciech Młotkowski, *Operator-valued version of conditionally free product*, Studia Math. **153** (2002), no. 1, 13–30, DOI 10.4064/sm153-1-2. MR1948925 (2003j:46102)
- [13] Alexandru Nica and Roland Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series, vol. 335, Cambridge University Press, Cambridge, 2006. MR2266879 (2008k:46198)
- [14] Mihai Popa, *Multilinear function series in conditionally free probability with amalgamation*, Commun. Stoch. Anal. **2** (2008), no. 2, 307–322. MR2446696 (2010i:46102)
- [15] Mihai Popa, *Realization of conditionally monotone independence and monotone products of completely positive maps*, J. Operator Theory **68** (2012), no. 1, 257–274. MR2966045
- [16] Mihai Popa, *Non-crossing linked partitions and multiplication of free random variables* (English summary), Operator theory live, Theta Ser. Adv. Math., vol. 12, pp. 135–143, Theta, Bucharest, 2010. MR2731869 (2012a:46128)
- [17] Mihai Popa and Victor Vinnikov, *Non-commutative functions and the non-commutative free Lévy–Hinčin formula*, Adv. Math. **236** (2013), 131–157, DOI 10.1016/j.aim.2012.12.013. MR3019719
- [18] Mihai Popa and Jiun-Chau Wang, *On multiplicative conditionally free convolution*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6309–6335, DOI 10.1090/S0002-9947-2011-05242-6. MR2833556 (2012f:46134)
- [19] Dan Voiculescu, *Addition of certain noncommuting random variables*, J. Funct. Anal. **66** (1986), no. 3, 323–346, DOI 10.1016/0022-1236(86)90062-5. MR839105 (87j:46122)
- [20] Dan Voiculescu, *Multiplication of certain noncommuting random variables*, J. Operator Theory **18** (1987), no. 2, 223–235. MR915507 (89b:46076)
- [21] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables*, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups, CRM Monograph Series, vol. 1, American Mathematical Society, Providence, RI, 1992. MR1217253 (94c:46133)
- [22] Jiun-Chau Wang, *Limit theorems for additive conditionally free convolution*, Canad. J. Math. **63** (2011), no. 1, 222–240, DOI 10.4153/CJM-2010-075-4. MR2779139 (2012b:46145)

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