F -PURE THRESHOLDS OF BINOMIAL HYPERSURFACES

DANIEL J. HERNÁNDEZ

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ABSTRACT. In this article, we develop an algorithm that computes the F -pure threshold of a binomial hypersurface over a field of characteristic $p > 0$. This algorithm is related to earlier work of Shibuta and Takagi (e.g., both depend on properties of certain associated rational polytopes), but differs in that it works in all characteristics.

1. INTRODUCTION

Let $f \in \mathbb{L}[x_1, \dots, x_m]$ be a polynomial over a field $\mathbb{L}$ of characteristic $p > 0$ vanishing at the origin, so that $f \in \mathfrak{m} := (x_1, \dots, x_m)$. The F -pure threshold of f , denoted $\mathbf{fpt}_{\mathfrak{m}}(f)$, is an invariant that measures the singularities of f near $\mathbf{0}$. This invariant is closely related to the classical notion of F -purity, and is defined via properties of the Frobenius morphism [TW04]. Given such a polynomial f , the value of $\mathbf{fpt}_{\mathfrak{m}}(f)$ is often difficult to compute. In fact, the only explicit formulas in the literature are for the F -pure thresholds of the polynomials $x^2 + y^3$, $x^2 + y^7$, and $x^5 + y^4 + x^3y^2$ appearing in [MTW05]. In this article, we aim to understand the value of $\mathbf{fpt}_{\mathfrak{m}}(f)$ for all possible values of p in the special case that f is a binomial (i.e., a linear combination of two distinct monomials).

The main result of this article is Theorem 4.1. Using this theorem, we construct Algorithm 4.7, an effective algorithm for computing the F -pure threshold of *any* binomial over a field of *arbitrary* prime characteristic. This algorithm avoids explicitly verifying containment in arbitrarily high Frobenius powers of certain ideals (a necessary ingredient in the definition of F -pure thresholds of polynomials), and can therefore be implemented in standard computer algebra programs.¹ The statement of Theorem 4.1 is technical, so we omit it here and instead refer the reader to Examples 4.3, 4.4, 4.5 and 4.6 for explicit demonstrations of how Theorem 4.1 can be used to compute F -pure thresholds.

The expressions for $\mathbf{fpt}_{\mathfrak{m}}(f)$ given in Theorem 4.1 are functions of certain geometric invariants of the *splitting polytope* $\mathbf{P}$ associated to f ; see Definition 3.5. The splitting polytope associated to a binomial is a rational polytope contained in $[0, 1]^2$, and a variant was previously used in [ST09] to compute the log canonical threshold (an important invariant typically defined over fields of characteristic

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¹In fact, Algorithm 4.7 has recently been implemented by Emily Witt in the computer algebra program *Macaulay2*.

zero) of certain ideals $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_m]$ generated by binomials. In fact, the methods of [ST09] rely on reduction to positive characteristic, and results appearing in loc. cit. show that given a binomial f , there exists a sparse (yet infinite) set of primes p for which the value of $\mathbf{fpt}_m(f)$ is independent of p . Our results also show this, and additionally shed light on how the value of $\mathbf{fpt}_m(f)$ varies as p varies.

2. ON BASE p EXPANSIONS

Definition 2.1. Let $\alpha \in (0, 1]$, and let p be a prime number. For $d \geq 1$, there exist unique integers $\alpha^{(d)}$ such that $0 \leq \alpha^{(d)} \leq p - 1$, $\alpha = \sum_{d \geq 1} \frac{\alpha^{(d)}}{p^d}$, and such that $\alpha^{(d)}$ is not eventually zero as a function of d . The integers $\alpha^{(d)}$ are called the *digits* of α (base p), and the expression $\alpha = \sum_{d \geq 1} \frac{\alpha^{(d)}}{p^d}$ is called the *non-terminating expansion* of α (base p).

Definition 2.2. Let $\alpha \in (0, 1]$, and let p be a prime number. We use $\langle \alpha \rangle_e$ to denote $\sum_{d=1}^e \frac{\alpha^{(d)}}{p^d}$, the e^{th} *truncation* of α (base p), and $\llbracket \alpha \rrbracket_e$ to denote $\alpha - \langle \alpha \rangle_e = \sum_{d > e} \frac{\alpha^{(d)}}{p^d}$, the e^{th} *tail* of α (base p). We adopt the convention that $\langle \alpha \rangle_0 = \langle 0 \rangle_e = \llbracket 0 \rrbracket_e = 0$.

Lemma 2.3. For $\alpha, \beta \in (0, 1]$, we have the following:

- (1) $\langle \alpha \rangle_e \in \frac{1}{p^e} \cdot \mathbb{N}$ and $\langle \alpha \rangle_e < \alpha$.
- (2) $0 < \llbracket \alpha \rrbracket_e \leq \frac{1}{p^e}$, with equality if and only if $\alpha \in \frac{1}{p^e} \cdot \mathbb{N}$.
- (3) $\alpha \leq \beta$ if and only if $\langle \alpha \rangle_e \leq \langle \beta \rangle_e$ for all $e \geq 1$.
- (4) If $\beta \in [0, 1] \cap \frac{1}{p^e} \cdot \mathbb{N}$ and $\alpha > \beta$, then $\langle \alpha \rangle_e \geq \beta$.

Proof. The first two assertions follow directly from the definitions. We now prove the last point, leaving the third for the reader. As $\beta < \alpha$, we have that

$$p^e \beta < p^e \alpha = p^e \langle \alpha \rangle_e + p^e \llbracket \alpha \rrbracket_e \leq p^e \langle \alpha \rangle_e + 1,$$

where we have used above that $\llbracket \alpha \rrbracket_e \leq \frac{1}{p^e}$. Thus, $p^e \beta < p^e \langle \alpha \rangle_e + 1$, and as both terms are integers, we conclude that $p^e \beta \leq p^e \langle \alpha \rangle_e$. $\square$

Definition 2.4. Let $(\alpha, \beta) \in [0, 1]^2$, and let p be a prime number. We say that the e^{th} digits of α and β *add without carrying* (base p) if $\alpha^{(e)} + \beta^{(e)} \leq p - 1$, and we say that α and β *add without carrying* (base p) if the e^{th} digits of α and β add without carrying for all e . We define the corresponding conditions for integers in the obvious way.

Remark 2.5. It follows from the definitions that α and β add without carrying (base p) if and only if the integers $p^e \langle \alpha \rangle_e$ and $p^e \langle \beta \rangle_e$ add without carrying for all $e \geq 1$.

Example 2.6. Let $\alpha \in (0, 1]$. If $(p - 1) \cdot \alpha \in \mathbb{N}$, then multiplying the non-terminating base p expansion $1 = \sum_{e \geq 1} \frac{p-1}{p^e}$ by α shows that $\alpha^{(e)} = (p - 1) \cdot \alpha$ for all $e \geq 1$. In particular, if $(\alpha, \beta) \in [0, 1]^2$ and $(p - 1) \cdot (\alpha, \beta) \in \mathbb{N}^2$, then α and β add without carrying (base p) if and only if $\alpha + \beta \leq 1$.

Lemma 2.7 ([Luc78]). Let $k_1, k_2 \in \mathbb{N}$, and set $N = k_1 + k_2$. The binomial coefficient $\binom{N}{k_1, k_2} := \frac{N!}{k_1! k_2!} \not\equiv 0 \pmod{p}$ if and only if k_1 and k_2 add without carrying (base p).

Lemma 2.8. Consider $(\alpha, \beta) \in [0, 1]^2$, and suppose that $\alpha + \beta \leq 1$. If $\alpha^{(L+1)} + \beta^{(L+1)} \geq p$ for some integer $L \geq 0$, then $\langle \alpha \rangle_L + \langle \beta \rangle_L + \frac{1}{p^L} = \langle \alpha + \beta \rangle_L$.

Proof. By definition,

$$(2.1) \quad \langle \alpha + \beta \rangle_L + \llbracket \alpha + \beta \rrbracket_L = \alpha + \beta = \langle \alpha \rangle_L + \langle \beta \rangle_L + \llbracket \alpha \rrbracket_L + \llbracket \beta \rrbracket_L.$$

As the digits appearing in a base p expansion are less than or equal to $p - 1$,

$$\begin{aligned} \frac{1}{p^L} < \llbracket \alpha \rrbracket_L + \llbracket \beta \rrbracket_L &= \frac{1}{p^L} + \frac{\alpha^{(L+1)} + \beta^{(L+1)} - p}{p^{L+1}} + \llbracket \alpha \rrbracket_{L+1} + \llbracket \beta \rrbracket_{L+1} \\ &\leq \frac{1}{p^L} + \frac{2p - 2 - p}{p^{L+1}} + \frac{1}{p^{L+1}} + \frac{1}{p^{L+1}} = \frac{2}{p^L}, \end{aligned}$$

so that $\lceil p^L \llbracket \alpha \rrbracket_L + p^L \llbracket \beta \rrbracket_L \rceil = 2$. Thus, multiplying (2.1) by p^L and rounding up, keeping in mind that $0 < \llbracket \alpha + \beta \rrbracket_L \leq \frac{1}{p^L}$, shows that $p^L \langle \alpha + \beta \rangle_L + 1 = p^L \langle \alpha \rangle_L + p^L \langle \beta \rangle_L + 2$. $\square$

3. F-PURE THRESHOLDS AND SPLITTING POLYTOPES

3.1. F-pure thresholds of polynomials. Fix a field $\mathbb{L}$ of prime characteristic $p > 0$, and let $f \in \mathbb{L}[x_1, \dots, x_m]$ be a polynomial vanishing at the origin, so that $f \in \mathfrak{m} := (x_1, \dots, x_m)$. For every $e \geq 1$, let $\mathfrak{m}^{[p^e]}$ denote $(x_1^{p^e}, \dots, x_m^{p^e})$, the e^{th} Frobenius power of $\mathfrak{m}$.

Definition 3.1 ([BMS08]). The limit $\mathbf{fpt}_{\mathfrak{m}}(f) := \lim_{e \rightarrow \infty} \frac{1}{p^e} \cdot \max \{l : f^l \notin \mathfrak{m}^{[p^e]}\}$ exists and is contained in $(0, 1] \cap \mathbb{Q}$. We call $\mathbf{fpt}_{\mathfrak{m}}(f)$ the *F-pure threshold* of f at $\mathfrak{m}$.

Lemma 3.2 allows us to recover the terms $\max \{l : f^l \notin \mathfrak{m}^{[p^e]}\}$ from their limiting value.

Lemma 3.2. For every $e \geq 1$, $\langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e = \frac{1}{p^e} \cdot \max \{l : f^l \notin \mathfrak{m}^{[p^e]}\}$.

Proof. This is a restatement of [MTW05] in the language of base p expansions. $\square$

Lemma 3.3. Let $\mathfrak{m} \subseteq \mathbb{L}[x_1, \dots, x_m]$ and $\mathfrak{n} \subseteq \mathbb{L}[y_1, \dots, y_n]$ denote the ideals generated by the variables in each ring, and $\mathfrak{m} + \mathfrak{n}$ denote the ideal generated by the variables in $\mathbb{L}[\mathbf{x}, \mathbf{y}]$. If $f \in \mathfrak{m}$ and $g \in \mathfrak{n}$, then $\mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) = \min\{\mathbf{fpt}_{\mathfrak{m}}(f), \mathbf{fpt}_{\mathfrak{n}}(g)\}$.

Proof. Let $\lambda_1 = \mathbf{fpt}_{\mathfrak{m}}(f)$ and $\lambda_2 = \mathbf{fpt}_{\mathfrak{n}}(g)$, and suppose that $\lambda_1 \leq \lambda_2$. We now show that $\mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) = \lambda_1$. As $\lambda_1 \leq \lambda_2$, it follows from Lemma 2.3 that $\langle \lambda_1 \rangle_e \leq \langle \lambda_2 \rangle_e$ for all $e \geq 1$, and so by Lemma 3.2 we have that $f^{p^e \langle \lambda_1 \rangle_e} \notin \mathfrak{m}^{[p^e]}$ and $g^{p^e \langle \lambda_1 \rangle_e} \notin \mathfrak{n}^{[p^e]}$. As f and g are in different sets of variables, it follows that $(fg)^{p^e \langle \lambda_1 \rangle_e} \notin (\mathfrak{m} + \mathfrak{n})^{[p^e]}$. By Lemma 3.2, we know that $f^{p^e \langle \lambda_1 \rangle_e + 1} \in \mathfrak{m}^{[p^e]}$, and so $(fg)^{p^e \langle \lambda_1 \rangle_e + 1} \in \mathfrak{m}^{[p^e]} \subseteq (\mathfrak{m} + \mathfrak{n})^{[p^e]}$. We conclude that

$$p^e \langle \lambda_1 \rangle_e = \max \{l : (fg)^l \notin (\mathfrak{m} + \mathfrak{n})^{[p^e]}\} = p^e \langle \mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) \rangle_e,$$

where the last inequality holds by Lemma 3.2. The claim follows by letting $e \rightarrow \infty$. $\square$

3.2. Splitting polytopes in dimension two. We begin by reviewing some standard conventions from convex geometry. Recall that a set $\mathcal{P} \subseteq \mathbb{R}^2$ is called a *rational polyhedron* if there exist finitely many linear forms $L_1, \dots, L_d \in \mathbb{Q}[t_1, t_2]$ such that

$$\mathcal{P} = \{s \in \mathbb{R}^2 : L_i(s) \leq 1 \text{ for all } 1 \leq i \leq d\}.$$

A bounded rational polyhedron is called a *rational polytope*. If $L \in \mathbb{Q}[t_1, t_2]$ is a linear form and $\beta \in \mathbb{Q}$, the set $H_L^\beta := \{s : L(s) \leq \beta\}$ is called a (lower) *supporting halfspace* of $\mathcal{P}$ if $\mathcal{P} \subseteq H_L^\beta$ and $L^{-1}(\beta) \cap \mathcal{P}$ is non-empty. In this case, we call the set $L^{-1}(\beta) \cap \mathcal{P}$ a *face* of $\mathcal{P}$. An element v is called a *vertex* of $\mathcal{P}$ if $\{v\}$ is a face of $\mathcal{P}$.

Notation 3.4 (Vectors in $\mathbb{R}^m$). We use $\prec$ and $\preceq$ to denote componentwise inequality in $\mathbb{R}^m$, and $\bullet$ to denote the standard Euclidean product on $\mathbb{R}^2$. That is, $(a, b) \bullet (s, t) = as + bt$. If $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, we will use $|\alpha|$ to denote the coordinate sum $\alpha_1 + \alpha_2$. We use $\mathbf{1}_m$ to denote the element $(1, \dots, 1) \in \mathbb{R}^m$. Finally, we will often use vector notation for monomials: if $u_1, \dots, u_n$ are elements of an ambient ring and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we will use u^α to denote the element $u_1^{\alpha_1} \cdots u_n^{\alpha_n}$.

Definition 3.5. Fix elements $a \neq b$ of $\mathbb{N}^m$. We call $\mathbf{E} = (a \ b) \in M_{m \times 2}(\mathbb{Z})$ the *splitting matrix* determined by a and b , and $\mathbf{P} = \{s \in \mathbb{R}^2 : s \succcurlyeq \mathbf{0} \text{ and } \mathbf{E}s \preceq \mathbf{1}_m\}$ the *splitting polytope* determined by a, b . As $a, b \in \mathbb{N}^m$, we see that $\mathbf{P} \subseteq [0, 1]^2$ is a rational polytope.

An example of a simple splitting polytope appears in Figure 1.

FIGURE 1. The polytope $\mathbf{P}$ determined by $a = (1, 4, 7)$ and $b = (9, 8, 4)$.

Setup for the remainder of Section 3. For the remainder of this section, we fix distinct elements a and b of $\mathbb{N}^m$. Throughout, $\mathbf{E}$ and $\mathbf{P}$ will refer to the corresponding associated objects appearing in Definition 3.5.

Definition 3.6. We call the set $\{s \in \mathbf{P} : \mathbf{E}s \prec \mathbf{1}_m\}$ the *lower interior* of $\mathbf{P}$, and we denote it by $\mathbf{P}_{\text{lower}}^\circ$. An element $\eta \in \mathbf{P}$ is called a *maximal point* of $\mathbf{P}$ if $|\eta| = \max\{|\mathbf{s}| : \mathbf{s} \in \mathbf{P}\}$.

Set $\alpha = \max\{|\mathbf{s}| : \mathbf{s} \in \mathbf{P}\}$. By definition, $H_{\max} := \{s : (\frac{1}{\alpha}, \frac{1}{\alpha}) \bullet s \leq 1\}$ is a supporting halfspace of $\mathbf{P}$, but it is not one of the defining halfspaces of $\mathbf{P}$ unless there exists a row (a_i, b_i) of $\mathbf{E}$ with $a_i = b_i = \frac{1}{\alpha}$. It follows that whenever $a_i \neq b_i$ for all rows (a_i, b_i) of $\mathbf{E}$, the face determined by $H_{\max}$ must be a vertex of $\mathbf{P}$. We record this observation below.

Lemma 3.7. *If $\mathbf{E}$ has no constant rows, then $\mathbf{P}$ has a unique maximal point.*

Definition 3.8. Suppose that $\mathbf{P}$ has a unique maximal point $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \mathbf{P}$, and set

$${}^*\mathbf{P} := \{\mathbf{s} \in \mathbf{P} : s_2 \geq \eta_2\} \text{ and } \mathbf{P}_\star := \{\mathbf{s} \in \mathbf{P} : s_1 \geq \eta_1\}.$$

Note that this gives a decomposition $\mathbf{P} = {}^*\mathbf{P} \cup \mathbf{P}_\star \cup \{\mathbf{s} \in \mathbf{P} : \mathbf{s} \preceq \boldsymbol{\eta}\}$.

FIGURE 2. The decomposition of $\mathbf{P}$ from Figure 1.

In Figure 2, we see that the faces of $\mathbf{P}$ with slope less than -1 correspond to faces of $\mathbf{P}_\star$, while those with slope between 0 and -1 correspond to faces of ${}^*\mathbf{P}$. This observation holds in general, and is the key idea behind the following lemma.

Lemma 3.9. *Suppose that $\mathbf{P}$ contains a unique maximal point $\boldsymbol{\eta}$. If $\mathbf{s} \in {}^*\mathbf{P}$, then $\mathbf{s} \in \mathbf{P}_{\text{lower}}^\circ$ if and only if $(a_i, b_i) \bullet \mathbf{s} < 1$ for all i with $b_i > a_i$.*

Proof. We may assume that $\mathbf{s} \neq \boldsymbol{\eta}$, as $\boldsymbol{\eta}$ does not satisfy either of the conditions. It suffices to show that if $\boldsymbol{\eta} \neq \mathbf{s} \in {}^*\mathbf{P}$, then $\mathbf{s}$ automatically satisfies the conditions $(a_i, b_i) \bullet \mathbf{s} < 1$ for all i with $a_i > b_i$. By means of contradiction, suppose otherwise, so that there exists a row (a_i, b_i) with $a_i > b_i$ such that $(a_i, b_i) \bullet \mathbf{s} \geq 1$. For simplicity of notation, we denote this row by (a, b) with $a > b$. As $\boldsymbol{\eta} \in \mathbf{P}$,

$$(3.1) \quad (a, b) \bullet \boldsymbol{\eta} \leq 1 \leq (a, b) \bullet \mathbf{s} = (a - b, 0) \bullet \mathbf{s} + (b, b) \bullet \mathbf{s} < (a - b, 0) \bullet \mathbf{s} + (b, b) \bullet \boldsymbol{\eta},$$

where we have used in (3.1), keeping in mind that $\mathbf{s} \neq \boldsymbol{\eta}$, that

$$(b, b) \bullet \mathbf{s} = b \cdot |\mathbf{s}| < b \cdot |\boldsymbol{\eta}| = (b, b) \bullet \boldsymbol{\eta}.$$

It follows from (3.1) that $(a - b, 0) \bullet \boldsymbol{\eta} < (a - b, 0) \bullet \mathbf{s}$, and as $a > b$, we conclude that $\eta_1 < s_1$. However, substituting this into the inequality $|\mathbf{s}| < |\boldsymbol{\eta}|$ implies that $s_2 < \eta_2$, contradicting the fact that $\mathbf{s} \in {}^*\mathbf{P}$. $\square$

Lemma 3.10. *Suppose that $\mathbf{P}$ contains a unique maximal point $\boldsymbol{\eta}$, and let $\delta \geq 0$. There exists $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \mathbf{P}_{\text{lower}}^\circ$ with $\gamma_2 \geq \langle \eta_2 \rangle_d + \frac{1}{p^d}$ and $|\boldsymbol{\gamma}| = \langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \frac{1}{p^d} + \delta$ if and only if $\langle \boldsymbol{\eta} \rangle_d + \left(\delta, \frac{1}{p^d}\right) \in \mathbf{P}_{\text{lower}}^\circ$.*

Proof. Set $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) := \langle \boldsymbol{\eta} \rangle_d + \left(\delta, \frac{1}{p^d}\right)$. If $\boldsymbol{\lambda} \in \mathbf{P}_{\text{lower}}^\circ$, one may take $\boldsymbol{\gamma} = \boldsymbol{\lambda}$.

Next, suppose there exists a γ that satisfies the given properties, but that $\lambda \notin P_{\text{lower}}^\circ$. As $\lambda \in {}^*P$, Lemma 3.9 implies that there exists a row (a_i, b_i) of E such that $b_i > a_i$ and $(a_i, b_i) \bullet \lambda \geq 1$. To simplify notation, we denote this row by (a, b) with $b > a$. Then

$$(3.2) \quad (a, a) \bullet \gamma + (0, b-a) \bullet \gamma = (a, b) \bullet \gamma < 1 \leq (a, b) \bullet \lambda = (a, a) \bullet \lambda + (0, b-a) \bullet \lambda.$$

As $|\gamma| = |\lambda|$, we may cancel the summands $(a, a) \bullet \gamma$ and $(a, a) \bullet \lambda$ in (3.2) to obtain $(0, b-a) \bullet \gamma < (0, b-a) \bullet \lambda$. As $b > a$, we conclude that $\gamma_2 < \lambda_2 = \langle \eta_2 \rangle_d + \frac{1}{p^a}$, a contradiction. $\square$

3.3. Connections with F -pure thresholds of binomials.

Definition 3.11. If $g \in \mathbb{L}[x_1, \dots, x_m]$ for some field $\mathbb{L}$, we use $\text{Supp}(g)$ to denote the unique collection of monomials $\mathcal{N}$ such that g is a $\mathbb{L}^*$ -linear combination of the elements of $\mathcal{N}$.

Lemma 3.12. Suppose that P contains a unique maximal point η with $|\eta| > 1$. Then, after possibly exchanging its columns, E contains rows of the form $(1, 0)$ and $(0, n)$ for some $n \geq 1$. In particular, $\mathbf{fpt}_m(f) = 1$ for all polynomials with $\text{Supp}(f) = \mathcal{M}$.

Proof. As $P \subseteq [0, 1]^2$, the assumption that $\eta_1 + \eta_2 > 1$ implies that η_1 and η_2 are non-zero, so that *P and $P_\star$ are non-empty. By Definition 3.8, η is a vertex of both *P and $P_\star$, so

$$(3.3) \quad (a_i, b_i) \bullet \eta = (a_j, b_j) \bullet \eta = 1$$

for some (a_i, b_i) with $a_i < b_i$ and (a_j, b_j) with $b_j > a_j$. As $a_i < b_i$, it follows that

$$a_i < a_i \cdot |\eta| = (a_i, a_i) \bullet \eta < (a_i, b_i) \bullet \eta = 1.$$

As $a_i \in \mathbb{N}$, it follows that $a_i = 0$. Similarly, $b_j = 0$. Substituting these values into (3.3) shows that $\eta = \left(\frac{1}{a_j}, \frac{1}{b_i}\right)$, and the equality $\frac{1}{a_j} + \frac{1}{b_i} = |\eta| > 1$ shows that either a_j or b_i must equal 1. Thus, after possibly swapping the columns, we see that $(a_j, 0) = (1, 0)$ and $(0, b_i) = (0, n)$ for some $n \geq 1$. Rename the variables so that the variable corresponding to $(1, 0)$ is y and the variable corresponding to $(0, n)$ is z . It follows that $\mathcal{M} = \{y\mu_1, z^n\mu_2\}$ for monomials μ_1 and μ_2 containing no powers of y or z . If $f = u_1y\mu_1 + u_2z^n\mu_2$, the linear factor y implies that $\max\{l : f^l \notin (y^{p^e}, z^{p^e})\} = p^e - 1$. It follows from Lemma 3.2 that $\mathbf{fpt}_{(y,z)}(f) = 1$. $\square$

4. F -PURE THRESHOLDS OF BINOMIALS

In this section, we present Algorithm 4.7, an effective algorithm for computing the F -pure threshold of an arbitrary binomial. The main result used in this algorithm is Theorem 4.1.

Setup for Section 4. Fix distinct elements $\mathbf{a}$ and $\mathbf{b}$ of $\mathbb{N}^m$, and let $P \subseteq [0, 1]^2$ be the associated splitting polytope; recall that if $E = (\mathbf{a} \ \mathbf{b})$ is the associated splitting matrix, then $P = \{\mathbf{s} \in \mathbb{R}^2 : \mathbf{s} \succcurlyeq \mathbf{0} \text{ and } E\mathbf{s} \preccurlyeq \mathbf{1}_m\}$ and $P_{\text{lower}}^\circ = \{\mathbf{s} \in P : E\mathbf{s} \prec \mathbf{1}_m\}$. We use $\mathcal{M} = \{\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}\}$ to denote the corresponding set of (distinct) monomials in the variables $x_1, \dots, x_m$.

Theorem 4.1. Suppose that $\mathbf{P}$ contains a unique maximal point $\boldsymbol{\eta} = (\eta_1, \eta_2)$ with $\eta_1 + \eta_2 \leq 1$. Fix a prime p , and let $L = \inf \left\{ e \geq 0 : \eta_1^{(e+1)} + \eta_2^{(e+1)} \geq p \right\}$. Consider a field $\mathbb{L}$ with $\text{char } \mathbb{L} = p$, and let f be any binomial over $\mathbb{L}$ with $\text{Supp}(f) = \mathcal{M}$.

(1) Note that $L = \infty$ if and only if η_1 and η_2 add without carrying (base p). In this case, $\mathbf{fpt}_m(f) = \eta_1 + \eta_2$.

If $L < \infty$, set $d := \max \left\{ e \leq L : \eta_1^{(e)} + \eta_2^{(e)} \leq p - 2 \right\}$. We will see that $1 \leq d \leq L$.

(2) If neither $\left(\langle \eta_1 \rangle_d + \frac{1}{p^d}, \langle \eta_2 \rangle_d \right)$ nor $\left(\langle \eta_1 \rangle_d, \langle \eta_2 \rangle_d + \frac{1}{p^d} \right)$ is contained in $\mathbf{P}_{\text{lower}}^\circ$, then $\mathbf{fpt}_m(f) = \langle \eta_1 + \eta_2 \rangle_L$.

(3) Otherwise, either $\left(\langle \eta_1 \rangle_d + \frac{1}{p^d}, \langle \eta_2 \rangle_d \right)$ or $\left(\langle \eta_1 \rangle_d, \langle \eta_2 \rangle_d + \frac{1}{p^d} \right)$ is in $\mathbf{P}_{\text{lower}}^\circ$, so that

$$\varepsilon = \max \left\{ \delta : \left(\langle \eta_1 \rangle_d + \frac{1}{p^d}, \langle \eta_2 \rangle_d + \delta \right) \text{ or } \left(\langle \eta_1 \rangle_d + \delta, \langle \eta_2 \rangle_d + \frac{1}{p^d} \right) \text{ is in } \mathbf{P} \right\} > 0.$$

Then $0 < \varepsilon \leq \llbracket \eta_1 + \eta_2 \rrbracket_L$, with equality $\iff$ either η_1 or η_2 is in $\frac{1}{p^d} \cdot \mathbb{N}$.

Furthermore, $\mathbf{fpt}_m(f) = \langle \eta_1 + \eta_2 \rangle_L + \varepsilon$.

Remark 4.2 (On effective computation). Before describing Algorithm 4.7, we indicate how each of the quantities appearing in Theorem 4.1 may be effectively computed (e.g., via a computer program). We continue to use the notation established in Theorem 4.1; we also adopt standard decimal notation to denote base p expansions of numbers in $(0, 1]$.

Determination of membership in $\mathbf{P}$ and $\mathbf{P}_{\text{lower}}^\circ$. As the defining inequalities of $\mathbf{P}$ and $\mathbf{P}_{\text{lower}}^\circ$ are explicitly determined by the vectors $\mathbf{a}$ and $\mathbf{b}$, determining whether or not an element of $\mathbb{R}^2$ is in $\mathbf{P}$ or $\mathbf{P}_{\text{lower}}^\circ$ is computationally trivial.

Determination of $\boldsymbol{\eta}$. If $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_m)$, then the defining hyperplanes of $\mathbf{P}$ are of the form $a_i s_1 + b_i s_2 = 1$, $s_1 = 0$, and $s_2 = 0$, the latter two coming from the fact that the $\mathbf{P} \subset [0, 1]^2$. Given two such distinct and intersecting lines, one can easily compute their point of intersection. Such an intersection point can be tested for membership in $\mathbf{P}$, and the set of all intersection points contained in $\mathbf{P}$ determines the vertices of $\mathbf{P}$. Among this finite set of vertices, the unique maximal point $\boldsymbol{\eta}$ is the vertex with the largest coordinate sum, and can therefore easily be determined.

Determination of L . As $\mathbf{P}$ is a rational polyhedron, $\boldsymbol{\eta} \in \mathbb{Q}^2$. As a consequence, there exist integers $c, \gamma \geq 0$ such that $p^c(p^\gamma - 1) \cdot \boldsymbol{\eta} \in \mathbb{N}^2$, and it follows that

$$\eta_i = . \eta_i^{(1)} : \dots : \eta_i^{(c)} : \overline{\eta_i^{(c+1)}} : \dots : \eta_i^{(c+\gamma)} \text{ (base } p)$$

for $i = 1, 2$. From this expression, it is apparent that L (if finite) is bounded above by $c + \gamma$. We see that $L < \infty$ if and only if $L \leq \min \left\{ c + \gamma : p^c(p^\gamma - 1) \cdot \boldsymbol{\eta} \in \mathbb{N}^2 \right\}$. Thus, L may be determined in a finite number of steps (which depends only on $\boldsymbol{\eta}$).

Determination of d . Once L has been determined, $d \leq L$ can easily be determined by definition. That $d \geq 1$ is demonstrated in Lemma 4.16.

Determination of ε . For $e \geq 1$, let $\langle \boldsymbol{\eta} \rangle_e$ denote $(\langle \eta_1 \rangle_e, \langle \eta_2 \rangle_e) \in \frac{1}{p^e} \cdot \mathbb{N}^e$. Given d , we may easily determine whether $\langle \boldsymbol{\eta} \rangle_d + \left(\frac{1}{p^d}, 0 \right)$ or $\langle \boldsymbol{\eta} \rangle_d + \left(0, \frac{1}{p^d} \right)$ is in $\mathbf{P}_{\text{lower}}^\circ$.

Next, we assume that at least one of these points is contained in $\mathbf{P}_{\text{lower}}^\circ$, so that the computation of ε is necessary.

Suppose that $\langle \boldsymbol{\eta} \rangle_d + \left(\frac{1}{p^d}, 0\right)$ is in $\mathbf{P}_{\text{lower}}^\circ$. We now indicate how to effectively compute

$$(4.1) \quad \max \left\{ \delta : \langle \boldsymbol{\eta} \rangle_d + \left(\frac{1}{p^d}, \delta\right) \in \mathbf{P} \right\}.$$

For every $1 \leq i \leq m$, substitute $(s_1, s_2) = \langle \boldsymbol{\eta} \rangle_d + \left(\frac{1}{p^d}, \delta_i\right)$ into the equation $a_i s_1 + b_i s_2 = 1$ of $\mathbf{P}$, and solve for $\delta_i \in \mathbb{Q}$. It follows that $\langle \boldsymbol{\eta} \rangle_d + \left(\frac{1}{p^d}, \min\{\delta_i\}\right)$ is contained in $\mathbf{P}$, but $\langle \boldsymbol{\eta} \rangle_d + \left(\frac{1}{p^d}, \min\{\delta_i\} + \rho\right)$ violates one of the defining equations of $\mathbf{P}$ for every real number $\rho > 0$. Thus, the quantity appearing in (4.1) is equal to $\min\{\delta_1, \dots, \delta_m\}$. It follows that this maximum value, and therefore ε , may be effectively computed.

Setup for Examples 4.3–4.6. We now present a series of examples which show Theorem 4.1 in action. We continue to adopt standard decimal notation to denote base p expansions of numbers in $(0, 1]$. In each of the following examples, we set $\mathbf{a} = (7, 2)$ and $\mathbf{b} = (5, 6)$, so that f is an $\mathbb{L}^*$ -linear combination of $x^7 y^2$ and $x^5 y^6$. It follows that

$$\mathbf{P} = \left\{ (s_1, s_2) \in \mathbb{R}_{\geq 0}^2 : \begin{array}{l} 2s_1 + 6s_2 \leq 1 \\ 7s_1 + 5s_2 \leq 1 \end{array} \right\}.$$

Note that $\mathbf{P}$ contains a unique maximal point $\boldsymbol{\eta} = \left(\frac{1}{32}, \frac{5}{32}\right)$ (see Figures 3 and 4).

Example 4.3 ($\mathbf{fpt}_m(f) = |\boldsymbol{\eta}|$). By Theorem 4.1, $\mathbf{fpt}_m(f) = |\boldsymbol{\eta}| = \frac{3}{16}$ whenever $\frac{1}{32}$ and $\frac{5}{32}$ add without carrying (base p). By Example 2.6, $\frac{1}{32}$ and $\frac{5}{32}$ add without carrying whenever $p \equiv 1 \pmod{32}$. Note that there are infinitely many such primes by Dirichlet's result on primes in arithmetic progressions. However, there exist p such that $\mathbf{fpt}_m(f) = \frac{3}{16}$ with $p \not\equiv 1 \pmod{32}$. For example, if $p = 47$, then $\frac{1}{32} = \overline{.1 : 22}$ (base 47) and $\frac{5}{32} = \overline{.7 : 16}$ (base 47), and so $\mathbf{fpt}_m(f) = \frac{3}{16}$.

Example 4.4 ($\mathbf{fpt}_m(f) = \langle |\boldsymbol{\eta}| \rangle_L$). We now compute $\mathbf{fpt}_m(f)$ when $p = 43$. As

$$\frac{1}{32} = \overline{.1 : 14 : 33 : 25 : 22 : 36 : 12 : 4} \text{ and } \frac{5}{32} = \overline{.6 : 30 : 38 : 41 : 28 : 9 : 17 : 20}$$

(base 43), we see that the first instance of carrying occurs with the second digits, and that $d = L = 1$. These expansions also show that $\langle \boldsymbol{\eta} \rangle_1 = \left(\frac{1}{43}, \frac{6}{43}\right)$, and, as illustrated in Figure 3, neither $\langle \boldsymbol{\eta} \rangle_1 + \left(\frac{1}{43}, 0\right)$ nor $\langle \boldsymbol{\eta} \rangle_1 + \left(0, \frac{1}{43}\right)$ is contained in $\mathbf{P}_{\text{lower}}^\circ$. Theorem 4.1 allows us to conclude that $\mathbf{fpt}_m(f) = \langle |\boldsymbol{\eta}| \rangle_1 = \left\langle \frac{3}{16} \right\rangle_1 = \frac{8}{43}$.

Example 4.5 ($\mathbf{fpt}_m(f) = \langle |\boldsymbol{\eta}| \rangle_L + \varepsilon$). We compute $\mathbf{fpt}_m(f)$ when $p = 37$. As

$$\frac{1}{32} = \overline{.1 : 5 : 28 : 33 : 19 : 24 : 10 : 15} \text{ and } \frac{5}{32} = \overline{.5 : 28 : 33 : 19 : 24 : 10 : 15 : 1}$$

(base 37), we see that the first instance of carrying occurs with the third digits, and that $d = L = 2$. These expansions also show that $\langle \boldsymbol{\eta} \rangle_2 = \left(\frac{1}{37} + \frac{5}{37^2}, \frac{5}{37} + \frac{28}{37^2}\right)$. From Figure 4, we see that both $\langle \boldsymbol{\eta} \rangle_2 + \left(\frac{1}{37^2}, 0\right)$ and $\langle \boldsymbol{\eta} \rangle_2 + \left(0, \frac{1}{37^2}\right)$ are contained in

FIGURE 3. Computation of $\mathbf{fpt}_m(f)$ when $p = 43$.

FIGURE 4. Computation of $\mathbf{fpt}_m(f)$ when $p = 37$.

P_{lower}° . From Figure 4, we also see that $\varepsilon = \max \left\{ \delta : \langle \eta \rangle_2 + \left(\frac{1}{37^2}, \delta \right) \in P \right\}$. Moreover, Figure 4 shows that the point $\langle \eta \rangle_2 + \left(\frac{1}{37^2}, \varepsilon \right)$ lies on the hyperplane $7s_1 + 5s_2 = 1$, and an easy calculation shows that $\varepsilon = \frac{3}{6845} = .0 : 0 : 22 : 7 : 14 : 29$ (base 37). Thus, Theorem 4.1 shows that

$$\mathbf{fpt}_m(f) = \langle \eta \rangle_2 + \varepsilon = .6 : 34 : \overline{22 : 7 : 14 : 29} \text{ (base 37)} = \frac{1283}{6845}.$$

Example 4.6 (Odd behavior in small characteristic). One of the main advantages of Theorem 4.1 is that it allows us to compute $\mathbf{fpt}_m(f)$ for *any* value of p . In particular, it allows us to observe some atypical behavior when p is small. We gather three interesting results for $p = 2, 3$ and 5 below.

p	L	d	$\langle \eta \rangle_L$ (base p)	ε (base p)	$\mathbf{fpt}_m(f)$
2	5	5	$.0 : 0 : 1 : 0 : 1$	$.0 : 0 : 0 : 0 : 0 : \overline{1}$	$.0 : 0 : 1 : 0 : 1 : \overline{1} = \frac{3}{16}$
3	5	3	$.0 : 1 : 2 : 1 : 2$	— — —	$.0 : 1 : 2 : 1 : 2 = \frac{5}{27}$
5	2	2	$.0 : 4$	$.0 : 0 : 2 : \overline{4}$	$.0 : 4 : 2 : \overline{4} = \frac{23}{125}$

When $p = 3$, we see that d need not equal L . When $p = 5$, we see that $\mathbf{fpt}_m(f) = \langle \eta_1 + \eta_2 \rangle_L + \varepsilon$, yet the denominator of $\mathbf{fpt}_m(f)$ is still a power of p . The case $p = 2$

is especially interesting: as the entries of η both have 2^5 in their denominator, they cannot add without carrying. However, the values of d, L and ε align perfectly to produce the identity $\mathbf{fpt}_{\mathbf{m}}(f) = \langle |\eta| \rangle_L + \varepsilon = |\eta| = \frac{3}{16}$. We refer the reader to Figure 5 to better appreciate the symmetry present when $p = 2$.

FIGURE 5. Computation of $\mathbf{fpt}_{\mathbf{m}}(f) = \langle |\eta| \rangle_L + \varepsilon = |\eta|$ when $p = 2$.

4.1. An algorithm for computing F -pure thresholds of binomial hypersurfaces. We now indicate how Theorem 4.1 may be used to construct an algorithm for computing the F -pure threshold at $\mathbf{m}$ of *any* binomial over a field of arbitrary (prime) characteristic.

Algorithm 4.7. Let g be a binomial in $\mathbb{L}[x_1, \dots, x_m]$.

Step 1. Let μ_1 and μ_2 denote the supporting monomials of g . By gathering all of the variables appearing with the same exponent in both μ_1 and μ_2 , we may factor $g = \mu \cdot h$ for some monomial μ and binomial h with the property that no variable appearing in μ appears in h , and so that no variable appears with the same exponent in both supporting monomials of h . For example, if $g = x^2y^3z^5 + x^4y^3z^{11}$, then $\mu = y^3$ and $h = x^2z^5 + x^4z^{11}$. Observe that, by Lemma 3.7, the polytope P associated to the exponent vectors of h will contain a unique maximal point $\eta \in P$.

Step 2. Renumber the variables so that $\mu = x_1^{a_1} \cdots x_d^{a_d}$. By Lemma 3.3,

$$\mathbf{fpt}_{\mathbf{m}}(g) = \mathbf{fpt}_{\mathbf{m}}(\mu \cdot h) = \min \{ \mathbf{fpt}_{(x_1, \dots, x_d)}(\mu), \mathbf{fpt}_{(x_{d+1}, \dots, x_m)}(h) \}.$$

It is an easy exercise to verify that $\mathbf{fpt}_{(x_1, \dots, x_d)}(\mu) = \min \left\{ \frac{1}{a_1}, \dots, \frac{1}{a_d} \right\}$.

Step 3. If $|\eta| > 1$, it follows from Lemma 3.12 that $\mathbf{fpt}_{\mathbf{m}}(h) = 1$.

Step 4. If $|\eta| \leq 1$, we may compute $\mathbf{fpt}_{\mathbf{m}}(h)$ using Theorem 4.1.

Remark 4.8. Algorithm 4.7 was recently implemented by Emily Witt in the computer algebra program `Macaulay2`.

4.2. Further observations.

Remark 4.9 (Relation with log canonical thresholds). Let $f_{\mathbb{Z}}$ be a binomial over $\mathbb{Z}$ and for each prime p , let f_p denote the polynomial over $\mathbb{F}_p$ obtained by reducing each coefficient of f modulo p . The *log canonical threshold* of $f_{\mathbb{Z}}$ at $\mathbf{0}$, denoted $\mathbf{lct}_{\mathbf{0}}(f_{\mathbb{Z}})$, is an invariant of $f_{\mathbb{Z}}$ defined via resolution of singularities, and the relationship

between the values $\mathbf{lct}_0(f_{\mathbb{Z}})$ and $\mathbf{fpt}_m(f_p)$ has long been studied. In particular, it has been shown in [MTW05, Theorem 3.4] that $\lim_{p \rightarrow \infty} \mathbf{fpt}_m(f_p) = \mathbf{lct}_0(f_{\mathbb{Z}})$. Furthermore, it has been conjectured that $\mathbf{lct}_0(f_{\mathbb{Z}}) = \mathbf{fpt}_m(f_p)$ for infinitely many p (see, e.g., [MTW05, Conjecture 3.6]).

Suppose that the splitting polytope associated to the exponent vectors in $\text{Supp}(f_{\mathbb{Z}})$ satisfies the hypotheses of Theorem 4.1, and let (η_1, η_2) denote its unique maximal point. For all large primes p , f_p will be a binomial with $\text{Supp}(f_p) = \text{Supp}(f_{\mathbb{Z}})$, and Theorem 4.1 implies that

$$\langle \eta_1 + \eta_2 \rangle_L \leq \mathbf{fpt}_m(f_p) \leq \eta_1 + \eta_2.$$

As $L \geq 1$, it follows that the distance between $\eta_1 + \eta_2$ and $\mathbf{fpt}_m(f_p)$ is at most $\frac{1}{p}$, and taking the limit as $p \rightarrow \infty$ shows that $\lim_{p \rightarrow \infty} \mathbf{fpt}_m(f_p) = \eta_1 + \eta_2$. We conclude that $\mathbf{lct}_0(f_{\mathbb{Z}}) = \eta_1 + \eta_2$, an equality that also follows from [ST09, Theorem 2.4]. Thus, Theorem 4.1 shows that

$$\mathbf{lct}_0(f_{\mathbb{Z}}), \langle \mathbf{lct}_0(f_{\mathbb{Z}}) \rangle_L \quad \text{and} \quad \langle \mathbf{lct}_0(f_{\mathbb{Z}}) \rangle_L + \varepsilon$$

are all the possible values for $\mathbf{fpt}_m(f_p)$, and characterizes when each occurs. In particular, we see that for all large primes (e.g., for all primes such that the denominators of η_1 and η_2 are not powers of p), $\mathbf{fpt}_m(f_p)$ and $\mathbf{lct}_0(f_{\mathbb{Z}})$ are equal if and only if η_1 and η_2 add without carrying (base p). However, we saw in Example 4.6 that for small primes p , $\mathbf{fpt}_m(f_p)$ may equal $\mathbf{lct}_0(f_{\mathbb{Z}})$, even if η_1 and η_2 add with carrying.

Remark 4.10 (On elimination of variables). We now show that, in large characteristic, extraneous variables may be eliminated from a binomial without affecting its F -pure threshold. Let $f = u_1 x_1^{a_1} \cdots x_m^{a_m} + u_2 x_1^{b_1} \cdots x_m^{b_m}$ be a binomial over a field of characteristic p , and let $\mathbf{P}$ be the polytope associated to the exponent vectors of f . For every $1 \leq \ell \leq m$, let $H_{\ell} = \{(s_1, s_2) : a_{\ell} s_1 + b_{\ell} s_2 = 1\}$ denote the defining hyperplane of $\mathbf{P}$ determined by variable x_{ℓ} , and for $N = 1, 2$, let $H_{\text{axis}}^N = \{(s_1, s_2) : s_N = 0\}$ denote the corresponding coordinate hyperplane. When determining which variables to eliminate, there are three cases one must consider:

Case 1. $\mathbf{P}$ contains a unique maximal point $\boldsymbol{\eta} \in H_k \cap H_{\ell}$ for some $k \neq \ell$.

Case 2. $\mathbf{P}$ contains a unique maximal point $\boldsymbol{\eta} \in H_k \cap H_{\text{axis}}^N$ for some k, N .

Case 3. $\mathbf{P}$ does not contain a unique maximal point.

The first two cases are similar, while the third case can be reduced to the first two by factoring f as in Algorithm 4.7. Thus, we will only focus on the first case, leaving the remaining details to the reader.

Let g denote the binomial $g = u_1 x_k^{a_k} x_{\ell}^{a_{\ell}} + u_2 x_k^{b_k} x_{\ell}^{b_{\ell}}$, and let $\mathbf{Q}$ denote the polytope associated to the exponent vectors of g ; by definition, $\boldsymbol{\eta}$ is also the unique maximal point of $\mathbf{Q}$. This situation is depicted in Figure 6.

If $|\boldsymbol{\eta}| > 1$, it follows from Lemma 3.12 that $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(g) = 1$. If $|\boldsymbol{\eta}| \leq 1$, Theorem 4.1 shows that $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(g) = \eta_1 + \eta_2$ whenever these summands add without carrying (base p). If $|\boldsymbol{\eta}| \leq 1$ but η_1 and η_2 do not add without carrying, then the values of $1 \leq d \leq L < \infty$ appearing in Theorem 4.1 are the same for $\mathbf{P}$ as they are for $\mathbf{Q}$. As $d \geq 1$, $\langle \boldsymbol{\eta} \rangle_d = (\langle \eta_1 \rangle_d, \langle \eta_2 \rangle_d)$ gets arbitrarily close to $\boldsymbol{\eta}$ as p gets large. Furthermore, once $\langle \boldsymbol{\eta} \rangle_d$ is sufficiently close to $\boldsymbol{\eta}$, it is apparent that the issue of whether $\mathbf{fpt}_m(f)$ and $\mathbf{fpt}_m(g)$ take the value of $\eta_1 + \eta_2, \langle \eta_1 + \eta_2 \rangle_L$ or $\langle \eta_1 + \eta_2 \rangle_L + \varepsilon$ depends only on the proximity of $\langle \boldsymbol{\eta} \rangle_d$ to the hyperplanes H_k and H_{ℓ} . In particular, we see that $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(g)$ whenever p is large enough.

FIGURE 6. For large p , $\langle \eta \rangle_d$ will be close to η , and the value of $\mathbf{fpt}_m(f)$ will depend only on the geometry of P near η .

We record the preceding observations in Corollary 4.11 below.

Corollary 4.11. *Fix elements $\mathbf{a} = (a_1, \dots, a_m) \neq \mathbf{b} = (b_1, \dots, b_m)$ of $\mathbb{N}^m$, and let $f = u_1 \mathbf{x}^{\mathbf{a}} + u_2 \mathbf{x}^{\mathbf{b}}$ be a binomial over a field of characteristic p . There exist integers N and $1 \leq k \neq \ell \leq m$, all depending only on $\mathbf{a}$ and $\mathbf{b}$, with the following property: $g := u_1 x_k^{a_k} x_\ell^{a_\ell} + u_2 x_k^{b_k} x_\ell^{b_\ell}$ is a binomial, and $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(g)$ whenever $p > N$.*

Remark 4.12. By letting $p \rightarrow \infty$, we obtain an analogous result for log canonical thresholds of binomials over $\mathbb{Z}$.

4.3. Proof of Theorem 4.1. The rest of this section is dedicated to the proof of Theorem 4.1. We will rely on Lemma 3.10, Lemma 4.14 (proven below), and on estimates for F -pure thresholds given in [Her11, Main Theorem]. We break up the proof into three parts.

Notation 4.13. We will continue to use the notation established in Theorem 4.1. In particular, f will denote a polynomial over $\mathbb{L}$ with $\text{Supp}(f) = \{\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}\}$, and we fix $u_1, u_2 \in \mathbb{L}^*$ such that $f = u_1 \mathbf{x}^{\mathbf{a}} + u_2 \mathbf{x}^{\mathbf{b}}$. We use $\eta = (\eta_1, \eta_2)$ to denote the unique maximal point of P and $\langle \eta \rangle_e$ to denote $(\langle \eta_1 \rangle_e, \langle \eta_2 \rangle_e) \in \frac{1}{p^e} \cdot \mathbb{N}^2$.

Lemma 4.14. *If $\alpha \in \frac{1}{p^e} \cdot \mathbb{N}$, then $f^{p^e \alpha} \equiv \sum (p^e \alpha)_{\gamma} u^{p^e \gamma} x^{p^e \mathbf{E} \gamma} \pmod{\mathfrak{m}^{[p^e]}}$, where we sum over all $\gamma \in \frac{1}{p^e} \cdot \mathbb{N}^2 \cap P_{\text{lower}}^\circ$ such that $|\gamma| = \alpha$. In particular, $f^{p^e \alpha} \notin \mathfrak{m}^{[p^e]}$ if and only if there exists $\gamma \in \frac{1}{p^e} \cdot \mathbb{N} \cap P_{\text{lower}}^\circ$ with $|\gamma| = \alpha$ and $(p^e \alpha)_{\gamma} \neq 0 \pmod{p}$.*

Proof. We know that $f^{p^e \alpha} = \sum_{|\mathbf{k}|=p^e \alpha} (p^e \alpha)_{\mathbf{k}} u^{\mathbf{k}} x^{\mathbf{E} \mathbf{k}}$. If $\mathbf{x}^{\mathbf{E} \mathbf{k}} \notin \mathfrak{m}^{[p^e]}$, then $\mathbf{E} \mathbf{k} \prec p^e \cdot \mathbf{1}_m$, so that $\gamma = \frac{1}{p^e} \cdot \mathbf{k} \in P_{\text{lower}}^\circ$ and $|\gamma| = \alpha$. As f is a binomial, it is easy to see that there is no gathering of terms when raising f to powers. Thus $f^{p^e \alpha} \notin \mathfrak{m}^{[p^e]}$ if and only if one of the summands given by the binomial theorem is not in $\mathfrak{m}^{[p^e]}$. $\square$

Remark 4.15. In proving Theorem 4.1 we will use Lemma 4.14 to compute $\max \{l : f^l \notin \mathfrak{m}^{[p^e]}\}$, and hence $\mathbf{fpt}_m(f)$, in terms of the geometry of P , which explains why the statements in Theorem 4.1 do not depend on the coefficients of f .

Theorem 4.1: Part I. *If $L = \infty$, then $\mathbf{fpt}_m(f) = |\eta|$.*

Proof. This follows from [Her11, Main Theorem]. $\square$

Suppose that $L < \infty$, so that $\langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{1}{p^L} = \langle \eta_1 + \eta_2 \rangle_L$ by Lemma 2.8. This fact is crucial to many of the arguments that follow, and we will apply it without further mention. In fact, the same statement holds after replacing L by d , as defined in Theorem 4.1.

Lemma 4.16. *Suppose that $L < \infty$, and let $d := \max \{e \leq L : \eta_1^{(e)} + \eta_2^{(e)} \leq p - 2\}$. Then $1 \leq d \leq L$. Furthermore, $\langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \frac{1}{p^d} = \langle \eta_1 + \eta_2 \rangle_L$. In particular, $\langle \eta_1 + \eta_2 \rangle_L \in \frac{1}{p^d} \cdot \mathbb{N}$.*

Proof. If $L = 0$, then $\eta_1^{(1)} + \eta_2^{(1)} \geq p$, which is impossible as $|\eta| \leq 1$. Thus, $L \geq 1$. If $d = 0$, then $\eta_1^{(e)} + \eta_2^{(e)} = p - 1$ for $1 \leq e \leq L$, and it follows that $\langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L = \sum_{e=1}^L \frac{p-1}{p} = \frac{p^L-1}{p^L}$. By definition, $\eta_1^{(L+1)} + \eta_2^{(L+1)} \geq p$, and so

$$\eta_1 + \eta_2 > \langle \eta_1 \rangle_{L+1} + \langle \eta_2 \rangle_{L+1} = \langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{\eta_1^{(L+1)} + \eta_2^{(L+1)}}{p^{L+1}} \geq \frac{p^L-1}{p^L} + \frac{1}{p^L} = 1,$$

which again contradicts the fact that $|\eta| \leq 1$. We conclude that $d \geq 1$.

For the remainder, we assume $d < L$. Then $\langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L = \langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \sum_{e=d+1}^L \frac{p-1}{p^e}$, so $\langle \eta_1 + \eta_2 \rangle_L = \langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{1}{p^L} = \langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \sum_{e=d+1}^L \frac{p-1}{p^e} + \frac{1}{p^L} = \langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \frac{1}{p^d}$. $\square$

We now continue with our proof of Theorem 4.1.

Theorem 4.1: Part II. *Suppose $L < \infty$. Then $1 \leq d \leq L$. Furthermore, if neither $\langle \eta \rangle_d + (\frac{1}{p^d}, 0)$ nor $\langle \eta \rangle_d + (0, \frac{1}{p^d})$ is in $\mathbf{P}_{\text{lower}}^\circ$, then $\mathbf{fpt}_m(f) = \langle \eta_1 + \eta_2 \rangle_L$.*

Proof. We saw $1 \leq d \leq L$ in Lemma 4.16, and it follows from [Her11, Main Theorem] that

$$(4.2) \quad \langle \eta_1 + \eta_2 \rangle_L = \langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{1}{p^L} \leq \mathbf{fpt}_m(f).$$

We claim the inequality in (4.2) is strict. By means of contradiction, suppose it's not. Applying Lemma 2.3 shows that

$$(4.3) \quad \langle \eta_1 + \eta_2 \rangle_L \leq \langle \mathbf{fpt}_m(f) \rangle_L = \max \left\{ l : f^l \notin \mathfrak{m}^{[p^L]} \right\},$$

where we have used Lemma 3.2 to obtain the equality in (4.3). Apparently, (4.3) shows that $f^{p^L \langle \eta_1 + \eta_2 \rangle_L} \notin \mathfrak{m}^{[p^L]}$. However, Lemma 4.16 allows us to rewrite this as

$$f^{p^L \langle \eta_1 + \eta_2 \rangle_L} = \left(f^{p^d \langle \eta_1 + \eta_2 \rangle_L} \right)^{p^{L-d}} \notin \mathfrak{m}^{[p^L]},$$

and the flatness of Frobenius then implies that

$$(4.4) \quad f^{p^d \langle \eta_1 + \eta_2 \rangle_L} \notin \mathfrak{m}^{[p^d]}.$$

Applying Lemma 4.14 to (4.4) shows that there exists an element

$$(4.5) \quad \gamma = (\gamma_1, \gamma_2) \in \frac{1}{p^d} \cdot \mathbb{N} \cap \mathbf{P}_{\text{lower}}^\circ \text{ with } |\gamma| = \langle \eta_1 + \eta_2 \rangle_L = \langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \frac{1}{p^d},$$

where we have used Lemma 4.16 to obtain the last equality in (4.5). We see that it is impossible for both $\gamma_1 \leq \langle \eta_1 \rangle_d$ and $\gamma_2 \leq \langle \eta_2 \rangle_d$. Without loss of generality, we will assume that $\gamma_2 > \langle \eta_2 \rangle_d$, and as $\gamma_2 \in \frac{1}{p^d} \cdot \mathbb{N}$, we may even assume that $\gamma_2 \geq \langle \eta_2 \rangle_d + \frac{1}{p^d}$.

Finally, combining this inequality with the properties of γ recorded in (4.5), we see that γ satisfies the conditions of Lemma 3.10 (with $\delta = 0$), which implies that $\langle \eta \rangle_d + \left(0, \frac{1}{p^d}\right) \in \mathbf{P}_{\text{lower}}^\circ$, a contradiction. We conclude that equality holds in (4.2), i.e. $\mathbf{fpt}_m(f) = \langle \eta_1 + \eta_2 \rangle_L$. $\square$

Lemma 4.17. *Suppose that either $\langle \eta \rangle_d + \left(\frac{1}{p^d}, 0\right)$ or $\langle \eta \rangle_d + \left(0, \frac{1}{p^d}\right)$ is in $\mathbf{P}_{\text{lower}}^\circ$ and set*

$$\varepsilon = \max \left\{ \delta : \langle \eta \rangle_d + \left(\frac{1}{p^d}, \delta\right) \text{ or } \langle \eta \rangle_d + \left(\delta, \frac{1}{p^d}\right) \text{ is in } \mathbf{P} \right\}.$$

Then, $0 < \varepsilon \leq \llbracket \eta_1 + \eta_2 \rrbracket_L$, with equality if and only if either η_1 or η_2 is in $\frac{1}{p^d} \cdot \mathbb{N}$.

Proof. It is obvious that $\varepsilon > 0$. Without loss of generality, we assume that $\langle \eta \rangle_d + \left(\varepsilon, \frac{1}{p^d}\right) \in \mathbf{P}$. By Lemma 4.16, this element has coordinate sum $\langle \eta_1 + \eta_2 \rangle_L + \varepsilon$, which must be bounded above by $|\eta| = \langle \eta_1 + \eta_2 \rangle_L + \llbracket \eta_1 + \eta_2 \rrbracket_d$, as η is maximal. We conclude that $\varepsilon \leq \llbracket \eta_1 + \eta_2 \rrbracket_L$. In addition, if we set $\alpha := \langle \eta \rangle_d + \left(\llbracket \eta_1 + \eta_2 \rrbracket_L, \frac{1}{p^d}\right)$, it follows that $\varepsilon = \llbracket \eta_1 + \eta_2 \rrbracket_L \iff \alpha \in \mathbf{P}$. We now wish to derive a more useful expression for α . By Lemma 4.16,

$$\begin{aligned} \llbracket \eta_1 + \eta_2 \rrbracket_L &= \eta_1 + \eta_2 - \langle \eta_1 + \eta_2 \rangle_L = \eta_1 + \eta_2 - \left(\langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \frac{1}{p^d} \right) \\ &= \llbracket \eta_1 \rrbracket_d + \llbracket \eta_2 \rrbracket_d - \frac{1}{p^d}. \end{aligned}$$

Substituting this into the definition of α shows that

$$(4.6) \quad \alpha = \langle \eta \rangle_d + \left(\llbracket \eta_1 \rrbracket_d + \llbracket \eta_2 \rrbracket_d - \frac{1}{p^d}, \frac{1}{p^d} \right) = \left(\eta_1 + \llbracket \eta_2 \rrbracket_d - \frac{1}{p^d}, \langle \eta_2 \rangle_d + \frac{1}{p^d} \right).$$

It follows from (4.6) that $|\alpha| = |\eta|$. As η is the unique maximal point of $\mathbf{P}$, we conclude that $\alpha \in \mathbf{P}$ if and only if $\alpha = \eta$, which by (4.6) happens if and only if $\llbracket \eta_2 \rrbracket_d = \frac{1}{p^d}$ and $\eta_2 = \langle \eta_2 \rangle_d + \frac{1}{p^d}$, which is clearly redundant and equivalent to the single condition that $\llbracket \eta_2 \rrbracket_d = \frac{1}{p^d}$. In summary, we have just demonstrated that

$$\varepsilon = \llbracket \eta_1 + \eta_2 \rrbracket_L \iff \alpha \in \mathbf{P} \iff \alpha = \eta \iff \llbracket \eta_2 \rrbracket_d = \frac{1}{p^d} \iff \eta_2 \in \frac{1}{p^d} \cdot \mathbb{N},$$

where the last equivalence follows from Lemma 2.3. $\square$

Theorem 4.1: Part III. *Suppose either $\langle \eta \rangle_d + \left(\frac{1}{p^d}, 0\right)$ or $\langle \eta \rangle_d + \left(0, \frac{1}{p^d}\right)$ is in $\mathbf{P}_{\text{lower}}^\circ$.*

- (1) *Then $0 < \varepsilon \leq \llbracket |\eta| \rrbracket_L$, with equality $\iff$ either η_1 or η_2 is in $\frac{1}{p^d} \cdot \mathbb{N}$.*
- (2) *Furthermore, we have that $\frac{1}{p^e} \cdot \max \{ l : f^l \in \mathbf{m}^{[p^e]} \} = \langle \eta_1 + \eta_2 \rangle_L + \langle \varepsilon \rangle_e$ for every $e \geq L$. In particular, $\mathbf{fpt}_m(f) = \langle \eta_1 + \eta_2 \rangle_L + \varepsilon$.*

Proof. The first point is the content of Lemma 4.17. We assume once and for all that

$$(4.7) \quad \langle \eta \rangle_d + \left(0, \frac{1}{p^d}\right) \in \mathbf{P}_{\text{lower}}^\circ \quad \text{and} \quad \lambda := \langle \eta \rangle_d + \left(\varepsilon, \frac{1}{p^d}\right) \in \mathbf{P}.$$

By the definition of d , $\eta_1^{(e)} + \eta_2^{(e)} \leq 1$ for $1 \leq e < d$ while $\eta_1^{(d)} + \eta_2^{(d)} + 1 \leq p - 1$, and the bound $\varepsilon \leq \llbracket \eta_1 + \eta_2 \rrbracket_L$ implies that $\varepsilon^{(e)} = 0$ for all $1 \leq e \leq L$. Fix $e \geq L$, and set

$$\lambda^e := \langle \eta \rangle_d + \left(\langle \varepsilon \rangle_e, \frac{1}{p^d} \right).$$

The preceding arguments show that the entries of $p^e \cdot \lambda^e$ add without carrying (base p), so that $\binom{p^e \lambda^e}{p^e \lambda^e} \neq 0 \pmod{p}$. As $\langle \varepsilon \rangle_e < \varepsilon$, it follows from (4.7) that $\lambda^e \in \mathbf{P}_{\text{lower}}^\circ$. By Lemma 4.16, $|\lambda^e| = \langle \eta_1 + \eta_2 \rangle_L + \langle \varepsilon \rangle_e$, and applying Lemma 4.14 with $\gamma = \lambda^e$ shows that

$$(4.8) \quad f^{p^e \langle \eta_1 + \eta_2 \rangle_L + p^e \langle \varepsilon \rangle_e} \notin \mathfrak{m}^{[p^e]} \text{ for } e \geq L.$$

We will now show that

$$(4.9) \quad f^{p^e \langle \eta_1 + \eta_2 \rangle_L + p^e \langle \varepsilon \rangle_e + 1} \in \mathfrak{m}^{[p^e]} \text{ for } e \geq L.$$

Assume the statement in (4.9) is false. By Lemma 4.16, we may write

$$\left(f^{p^d \langle \eta_1 + \eta_2 \rangle_L} \right)^{p^{e-d}} \cdot f^{p^e \langle \varepsilon \rangle_e + 1} \notin \mathfrak{m}^{[p^e]}.$$

Choose monomials $\mu_1 \in \text{Supp} \left(f^{p^d \langle \eta_1 + \eta_2 \rangle_L} \right)$ and $\mu_2 \in \text{Supp} \left(f^{p^e \langle \varepsilon \rangle_e + 1} \right)$ such that $\mu_1^{p^{e-d}} \mu_2 \notin \mathfrak{m}^{[p^e]}$. Note that $\mu_1 \notin \mathfrak{m}^{[p^d]}$. By Lemma 4.14, it follows that

$$(4.10) \quad \mu_2 = \mathbf{x}^{p^e \mathbf{E} \beta} \text{ with } \beta \in \frac{1}{p^e} \cdot \mathbb{N}^2 \cap \mathbf{P}_{\text{lower}}^\circ \text{ and } |\beta| = \langle \varepsilon \rangle_e + \frac{1}{p^e} \text{ and}$$

$$(4.11) \quad \mu_1 = \mathbf{x}^{p^d \mathbf{E} \alpha} \text{ with } \alpha \in \frac{1}{p^d} \cdot \mathbb{N}^2 \cap \mathbf{P}_{\text{lower}}^\circ \text{ and } |\alpha| = \langle \eta_1 + \eta_2 \rangle_L = \langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \frac{1}{p^d}.$$

The condition that $\mu_1^{p^{e-d}} \cdot \mu_2 = \mathbf{x}^{p^e \mathbf{E} \alpha + p^e \mathbf{E} \beta} \notin \mathfrak{m}^{[p^e]}$ then implies that

$$(4.12) \quad \gamma := \alpha + \beta \in \mathbf{P}_{\text{lower}}^\circ.$$

It follows from (4.11) that either $\alpha_1 > \langle \eta_1 \rangle_d$ or $\alpha_2 > \langle \eta_2 \rangle_d$. Without loss of generality, we will assume that $\alpha_2 > \langle \eta_2 \rangle_d$, and as $\alpha_2 \in \frac{1}{p^d} \cdot \mathbb{N}$, we may even assume that $\alpha_2 \geq \langle \eta_2 \rangle_d + \frac{1}{p^d}$. It follows that $\gamma_2 \geq \langle \eta_2 \rangle_d + \frac{1}{p^d}$. Furthermore, both (4.10) and (4.11) show that $|\gamma| = \langle \eta_1 \rangle_d + \langle \eta_2 \rangle_d + \frac{1}{p^d} + \left(\langle \varepsilon \rangle_e + \frac{1}{p^e} \right)$. Combining these observations with (4.12), we apply Lemma 3.10 (with $\delta = \langle \varepsilon \rangle_e + \frac{1}{p^e}$) to obtain that

$$\langle \eta \rangle_d + \left(\langle \varepsilon \rangle_e + \frac{1}{p^e}, \frac{1}{p^d} \right) \in \mathbf{P}_{\text{lower}}^\circ.$$

As $\varepsilon \leq \langle \varepsilon \rangle_e + \frac{1}{p^e}$, this is impossible by the definition of ε . Thus, (4.12) is also impossible, and we conclude that (4.9) holds. $\square$

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MATHEMATICAL SCIENCES RESEARCH INSTITUTE, 17 GAUSS WAY, BERKELEY, CALIFORNIA 94720-5070

Current address: Department of Mathematics, The University of Utah, Salt Lake City, UT 84112

E-mail address: dhernan@math.utah.edu