A CLASS OF NONLINEAR DELAY EVOLUTION EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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ABSTRACT. We establish a sufficient condition for the existence, uniqueness and global uniform asymptotic stability of a C^0 -solution for the nonlinear delay differential evolution equation

$$\begin{cases} u'(t) \in Au(t) + f(t, u_t), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0], \end{cases}$$

where $\tau > 0$, X is a real Banach space, A is the infinitesimal generator of a nonlinear semigroup of contractions, $f : \mathbb{R}_+ \times C([-\tau, 0]; \overline{D(A)}) \rightarrow X$ is continuous and $g : C_b([-\tau, +\infty); \overline{D(A)}) \rightarrow C([-\tau, 0]; \overline{D(A)})$ is nonexpansive.

1. INTRODUCTION

The goal of this paper is to prove an existence and uniform asymptotic stability result of a C^0 -solution for the nonlinear delay differential evolution equation with nonlocal initial data

$$(1.1) \quad \begin{cases} u'(t) \in Au(t) + f(t, u_t), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0]. \end{cases}$$

Here $A : D(A) \subseteq X \rightsquigarrow X$ is an m -dissipative operator in the real (infinite dimensional) Banach space X , $\tau > 0$ is arbitrarily fixed, $f : \mathbb{R}_+ \times C([-\tau, 0]; \overline{D(A)}) \rightarrow X$ is continuous and $g : C_b([-\tau, +\infty); \overline{D(A)}) \rightarrow C([-\tau, 0]; \overline{D(A)})$ is nonexpansive. If I is an interval, $C_b(I; X)$ denotes the space of all bounded and continuous functions from I , equipped with the sup-norm $\|\cdot\|_{C_b(I; X)}$, while $C_b(I; \overline{D(A)})$ denotes the closed subset in $C_b(I; X)$ consisting of all elements $u \in C_b(I; X)$ satisfying $u(t) \in \overline{D(A)}$ for each $t \in I$. Further, $C([a, b]; X)$ stands for the space of all continuous functions from $[a, b]$ to X endowed with the sup-norm $\|\cdot\|_{C([a, b]; X)}$ and $C([a, b]; \overline{D(A)})$ is the closed subset of $C([a, b]; X)$ containing all $u \in C([a, b]; X)$ with $u(t) \in \overline{D(A)}$ for each $t \in [a, b]$. As usual, if $u \in C([-\tau, +\infty); X)$ and $t \in [0, +\infty)$, $u_t \in C([-\tau, 0]; X)$ is defined by $u_t(s) := u(t+s)$ for each $s \in [-\tau, 0]$.

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Remark 1.1. As observed by Hale [9, Section 2, p. 11], in the case $A = 0$ and $X = \mathbb{R}^n$, and also in our case, the class of evolution differential-difference problems of the form

$$(1.2) \quad \begin{cases} u'(t) \in Au(t) + F(t, u(t), u(t - \tau_1), \dots, u(t - \tau_n)), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0], \end{cases}$$

where $0 < \tau_1 < \tau_2 < \dots < \tau_n \leq \tau$ are some intermediate delays and the forcing term $F : \mathbb{R}_+ \times [\overline{D(A)}]^{n+1} \rightarrow X$ is continuous or Lipschitz, falls into our general framework. Indeed, if we denote by $\delta(s - \theta)$ the Dirac delta concentrated at $s = -\theta$, we have

$$\begin{aligned} F(t, u(t), u(t - \tau_1), \dots, u(t - \tau_n)) \\ = F(t, \delta(s - 0)u_t, \delta(s - \tau_1)u_t, \dots, \delta(s - \tau_n)u_t) := f(t, u_t). \end{aligned}$$

Since $\delta(s - \theta)$ is nonexpansive from $C([- \tau, 0]; X)$ to X , we easily conclude that the function $f : \mathbb{R}_+ \times C([- \tau, 0]; \overline{D(A)}) \rightarrow X$, defined as above, is continuous or Lipschitz whenever F is continuous or Lipschitz with respect to its last $n + 1$ arguments. Accordingly, our main result applies also to problems of the form (1.2), but not only.

Differential-difference evolution equations of the specific form (1.2) were recently studied by Vrabie [21] under the hypotheses that A generates a nonlinear compact semigroup which decays exponentially, F is jointly continuous and the class of those $g : C_b([- \tau, +\infty); \overline{D(A)}) \rightarrow C([- \tau, 0]; \overline{D(A)})$, allowed in (1.2), is general enough to contain elements of the form

$$(1.3) \quad g(u)(t) = \int_{\tau}^{+\infty} \mathcal{N}(u(t + \theta)) d\mu(\theta),$$

for each $u \in C_b([- \tau, +\infty); \overline{D(A)})$ and $t \in [- \tau, 0]$. In (1.3), $\mathcal{N} : X \rightarrow X$ is a (possible nonlinear) nonexpansive operator and μ is a σ -finite and complete measure on $[\tau, +\infty)$, satisfying $\text{supp } \mu = [\delta, +\infty)$, $\delta \geq \tau$ and $\mu([\tau, +\infty)) = 1$. In turn, by obvious particular choices of $\mathcal{N}$ and μ , the class of functions defined by (1.3) includes the following three important subclasses:

- (i) $g(u)(t) = u(2\pi + t)$, $t \in [- \tau, 0]$ (2π -periodicity condition);
- (ii) $g(u)(t) = -u(2\pi + t)$, $t \in [- \tau, 0]$ (2π -antiperiodicity condition);
- (iii) $g(u)(t) = \int_{\tau}^{+\infty} k(\theta)u(t + \theta) d\theta$, $t \in [- \tau, 0]$, where $k \in L^1([\tau, +\infty); \mathbb{R})$ and $\int_{\tau}^{+\infty} |k(\theta)| d\theta = 1$ (mean condition).

Vrabie [21] has also shown that one can get rid of the compactness of the semigroup if we assume, in addition, that F is Lipschitz with respect to its last $n + 1$ arguments and there exists $a > 0$ such that $\|g(u) - g(v)\|_{C([- \tau, 0]; X)} \leq \|u - v\|_{C_b([a, +\infty); X)}$ for all $u, v \in C_b([- \tau, +\infty); X)$, a condition which corresponds to $\text{supp } \mu = [\delta, +\infty)$ with $\delta > \tau$ in (1.3).

Our main goal is to extend Theorem 3.5 in [21] from problems of the form (1.2) to problems of the form (1.1) under the very same general hypotheses on A and on g imposed by [21]. We notice that the main results in [21] generalize in several directions those of Li [12]. More precisely, [12] analyzes the semilinear periodic

case, i.e., when g is of the form (i) above, in a Hilbert space H , by assuming that $A : D(A) \subseteq H \rightarrow H$ is a linear operator which generates an analytic, compact semigroup which decays exponentially and F is Lipschitz with respect to the last $n + 1$ arguments.

Related results concerning nondelayed equations of the form (1.1) with nonlocal initial conditions were previously obtained by Byszewski [4], Garcia and Reich [8] and Paicu and Vrabie [15]. For some mathematical models described by evolution equations subjected to nonlocal initial conditions, see Deng [7] and McKibben [13, Section 10.2, pp. 394–398].

For periodic problems without delay related to our main results, see Cascaval and Vrabie [5], Hirano [11], Paicu [14] and Vrabie [16]. As far as antiperiodic problems are concerned, see Aizicovici, Pavel and Vrabie [1]. For a periodic, semilinear problem with infinite delay, see the recent paper of Henríquez and Lizama [10].

A multi-valued version of (1.2), i.e., for F a multi-function, was considered by Vrabie [19], while similar results in the multi-valued case of (1.1) were obtained by Vrabie [20] and [22] under the additional hypothesis that A generates a compact semigroup.

This paper is organized into 5 sections. In Section 2 we briefly review some concepts and results referring to evolutions governed by m -dissipative operators. In Section 3 we state the main result, i.e. Theorem 3.1, and one important consequence, i.e. Theorem 3.2, while in Section 4 we prove the above mentioned theorems. In Section 5, as applications of our abstract theory, we obtain two existence results for nonlinear parabolic problems with delay, subject either to general nonlocal initial conditions or to periodic conditions.

2. PRELIMINARIES

The reader is expected to be familiar with the basic theory concerning m -dissipative operators and nonlinear evolution equations in Banach spaces. For details, we refer the reader to Barbu [2] and Vrabie [17]. As far as functional differential equations with delay are concerned, we refer to [9]. For the sake of clarity, we recall some concepts and results we will frequently use in the sequel.

Let X be a real Banach space with norm $\|\cdot\|$. Let $x, y \in X$ and $h \in \mathbb{R} \setminus \{0\}$. We denote

$$[x, y]_h := \frac{1}{h}(\|x + hy\| - \|x\|)$$

and we recall that there exists

$$\lim_{h \downarrow 0} [x, y]_h = \inf\{[x, y]_h; h > 0\} := [x, y]_+.$$

Remark 2.1. One may easily see that for each $x, y \in X$ and $\alpha > 0$, we have

- (i) $[\alpha x, y]_+ = [x, y]_+$,
- (ii) $|[x, y]_+| \leq \|y\|$.

For other remarkable properties of the mapping $(x, y) \mapsto [x, y]_+$, see [2, Proposition 3.7, p. 103].

An operator $A : D(A) \subseteq X \rightsquigarrow X$ is called *dissipative* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, we have

$$[x_1 - x_2, y_2 - y_1]_+ \geq 0.$$

It is called *m-dissipative* if it is dissipative and, in addition, $R(I - \lambda A) = X$, for each $\lambda > 0$.

Let $f \in L^1(a, b; X)$ and let us consider the evolution equation

$$(2.1) \quad u'(t) \in Au(t) + f(t).$$

A function $u : [a, b] \rightarrow \overline{D(A)}$ is called a C^0 -solution, or *integral solution* of (2.1) on $[a, b]$, if $u \in C([a, b]; X)$ and it satisfies

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) + y]_+ d\tau$$

for each $x \in D(A)$, $y \in Ax$ and $a \leq s \leq t \leq b$.

Theorem 2.1. *Let $\omega > 0$ and let $A : D(A) \subseteq X \rightsquigarrow X$ be an m -dissipative operator such that $A + \omega I$ is dissipative. Then, for each $\xi \in \overline{D(A)}$ and $f \in L^1(a, b; X)$, there exists a unique C^0 -solution of (2.1) on $[a, b]$ which satisfies $u(a) = \xi$. If $f, g \in L^1(a, b; X)$ and u, v are two C^0 -solutions of (2.1) corresponding to f and g respectively, then*

$$(2.2) \quad \|u(t) - v(t)\| \leq e^{-\omega(t-s)} \|u(s) - v(s)\| + \int_s^t e^{-\omega(t-\theta)} \|f(\theta) - g(\theta)\| d\theta$$

for each $a \leq s \leq t \leq b$.

In particular, if $x \in D(A)$ and $y \in Ax$, we have

$$(2.3) \quad \|u(t) - x\| \leq e^{-\omega(t-s)} \|u(s) - x\| + \int_s^t e^{-\omega(t-\theta)} \|f(\theta) + y\| d\theta$$

for each $a \leq s \leq t \leq b$.

See [2, Theorem 4.1, p. 130].

Let $\xi \in X$, $\tau \in [a, b]$ and $f \in L^1(a, b; X)$. Thereafter, $u(\cdot, \tau, \xi, f)$ denotes the unique C^0 -solution $v : [\tau, b] \rightarrow \overline{D(A)}$ of the problem (2.1) which satisfies the initial condition $v(\tau) = \xi$. We denote by $\{S(t) : \overline{D(A)} \rightarrow \overline{D(A)}, t \geq 0\}$ the semigroup of nonexpansive mappings generated by A on $\overline{D(A)}$ in the sense of Crandall and Liggett [6]. Then

$$S(t)\xi = u(t, 0, \xi, 0)$$

for each $\xi \in X$ and $t \geq 0$.

3. THE MAIN RESULTS

Our main assumptions on A , f and g are listed below.

(H_A) the operator $A : D(A) \subseteq X \rightsquigarrow X$ is m -dissipative, $0 \in D(A)$, $0 \in A0$ and there exists $\omega > 0$ such that $A + \omega I$ is dissipative;

(H_f) the function $f : \mathbb{R}_+ \times C([- \tau, 0]; \overline{D(A)}) \rightarrow X$ is continuous and satisfies the following:

(f_1) there exists $\ell > 0$ such that

$$\|f(t, v) - f(t, \tilde{v})\| \leq \ell \|v - \tilde{v}\|_{C([- \tau, 0]; X)}$$

for each $t \in \mathbb{R}_+$ and $v, \tilde{v} \in C([- \tau, 0]; \overline{D(A)})$;

(f_2) there exists $m > 0$ such that

$$\|f(t, 0)\| \leq m$$

for each $t \in [0, +\infty)$;

(H_c) the constants ℓ , $\tau > 0$ and $\omega > 0$ satisfy the nonresonance condition

$$\ell < \omega;$$

(H_g) the function $g : C_b([-\tau, +\infty); \overline{D(A)}) \rightarrow C([-\tau, 0]; \overline{D(A)})$ satisfies that:

(g_1) for each $u \in C_b([-\tau, +\infty); \overline{D(A)})$, we have

$$\|g(u)\|_{C([-\tau, 0]; X)} \leq \|u\|_{C_b([0, +\infty); X)};$$

(g_2) there exists $a > 0$ such that for each $u, v \in C_b([-\tau, +\infty); \overline{D(A)})$, we have

$$\|g(u) - g(v)\|_{C([-\tau, 0]; X)} \leq \|u - v\|_{C_b([a, +\infty); X)}.$$

Remark 3.1. We notice that whenever the function g is defined as in (i)–(iii) in the Introduction and $\tau < 2\pi$, then g satisfies (g_1) , (g_2) . If the support of the measure μ is in $(\tau, +\infty)$, then the function g given by (1.3) satisfies (g_1) and (g_2) . Moreover, if g satisfies (g_1) and (g_2) , then it depends only on the restriction of u to $[0, +\infty)$, i.e., $g(u) = g(\tilde{u})$ whenever $u(t) = \tilde{u}(t)$ for each $t \in [0, +\infty)$. Consequently, in this case, g can be defined only on $C_b([0, +\infty); \overline{D(A)})$ and not on the whole $C_b([-\tau, +\infty); \overline{D(A)})$. However, due to the fact that we are looking for C^0 -solutions of the problem (1.1) which belong to $C_b([-\tau, +\infty); \overline{D(A)})$, we prefer to consider that g is defined on the latter set.

Our main result, which is an extension of Theorem 3.5 in [21], is:

Theorem 3.1. *If (H_A) , (H_f) , (H_g) and (H_c) are satisfied, then (1.1) has a unique C^0 -solution, $u \in C_b([-\tau, +\infty); \overline{D(A)})$, satisfying*

$$(3.1) \quad \|u\|_{C_b([-\tau, +\infty); X)} \leq \frac{m}{\omega - \ell}.$$

If instead of (H_c) the stronger condition $\ell e^{\omega\tau} < \omega$ is satisfied, then u is globally asymptotically stable.

We recall that u is *globally asymptotically stable* if for each $\varphi \in C([-\tau, 0]; \overline{D(A)})$ the unique C^0 -solution $v \in C([-\tau, +\infty); \overline{D(A)})$ of the problem

$$\begin{cases} v'(t) \in Av(t) + f(t, v_t), & t \in [0, +\infty), \\ v(t) = \varphi(t), & t \in [-\tau, 0], \end{cases}$$

satisfies

$$\lim_{t \rightarrow +\infty} \|u(t) - v(t)\| = 0.$$

If we choose $\tau \in (0, 2\pi)$ and $g(u)(t) = u(t + 2\pi)$, $t \in [-\tau, 0]$, we obtain the next result for the existence of the solution of the periodic problem

$$(3.2) \quad \begin{cases} u'(t) \in Au(t) + f(t, u_t), & t \in \mathbb{R}, \\ u(t) = u(t + 2\pi), & t \in \mathbb{R}. \end{cases}$$

More precisely, we have:

Theorem 3.2. *If (H_A) , (H_f) and (H_c) are satisfied and $f : \mathbb{R} \times C([-\tau, 0]; \overline{D(A)}) \rightarrow X$ is 2π -periodic with respect to its first argument, then (3.2) has a unique 2π -periodic C^0 -solution, $u : \mathbb{R} \rightarrow \overline{D(A)}$, satisfying*

$$(3.3) \quad \|u\|_{C([0, 2\pi]; X)} \leq \frac{m}{\omega - \ell}.$$

If, in addition, $\ell e^{\omega\tau} < \omega$, then u is globally asymptotically stable.

4. PROOFS OF THEOREMS 3.1 AND 3.2

The proof, inspired from the proof of Theorem 3.5 in [21], is based on:

Lemma 4.1. *If (H_A) , (H_f) , (H_g) and (H_c) are satisfied, then, for each $v \in C_b([-\tau, +\infty); \overline{D(A)})$, the problem*

$$(4.4) \quad \begin{cases} u'(t) \in Au(t) + f(t, v_t), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0], \end{cases}$$

has a unique C^0 -solution $u \in C_b([-\tau, +\infty); \overline{D(A)})$.

Proof. Let $v \in C_b([-\tau, +\infty); \overline{D(A)})$ be arbitrary but fixed. Let us further consider $w \in C_b([-\tau, +\infty); \overline{D(A)})$ and the auxiliary problem

$$(4.5) \quad \begin{cases} u'(t) \in Au(t) + f(t, v_t), & t \in \mathbb{R}_+, \\ u(t) = g(w)(t), & t \in [-\tau, 0]. \end{cases}$$

Clearly, (4.5) has a unique C^0 -solution $u \in C_b([-\tau, +\infty); \overline{D(A)})$. Since the existence is immediate, we have merely to check that u is bounded. Indeed, if $t \geq 0$, using (2.3) we deduce that

$$\begin{aligned} \|u(t)\| &\leq e^{-\omega t} \|g(w)(0)\| + \int_0^t e^{-\omega(t-s)} \|f(s, v_s)\| ds \\ &\leq \|g(w)\|_{C([-\tau, 0]; X)} + \int_0^t e^{-\omega(t-s)} (\|f(s, v_s) - f(s, 0)\| + \|f(s, 0)\|) ds. \end{aligned}$$

From (H_f) and (g_1) we get

$$\|u(t)\| \leq \|w\|_{C_b([0, +\infty); X)} + \frac{1}{\omega} (\ell \|v\|_{C_b([-\tau, +\infty); X)} + m),$$

for each $t \geq 0$. For $t \in [-\tau, 0]$, by (g_1) , we have

$$\|u(t)\| \leq \|g(w)\|_{C([-\tau, 0]; X)} \leq \|w\|_{C_b([0, +\infty); X)}.$$

So, we can define the operator $T : C_b([-\tau, +\infty); \overline{D(A)}) \rightarrow C_b([-\tau, +\infty); \overline{D(A)})$ by $T(w) = u$, where $u \in C_b([-\tau, +\infty); \overline{D(A)})$ is the unique C^0 -solution of the problem (4.5). Obviously, (4.4) has a unique C^0 -solution if and only if T has a unique fixed point. So, to complete the proof, it suffices to show that T^2 is a strict contraction. To this aim, let $w, z \in C_b([-\tau, +\infty); \overline{D(A)})$. We have

$$(4.6) \quad \|Tw - Tz\|_{C_b([a, +\infty); X)} \leq e^{-\omega a} \|w - z\|_{C_b([a, +\infty); X)}.$$

Indeed, if $t \geq a$ using (2.3) and (g_2) , we obtain

$$\begin{aligned} \|(Tw)(t) - (Tz)(t)\| &\leq e^{-\omega t} \|g(w)(0) - g(z)(0)\| \\ &\leq e^{-\omega t} \|g(w) - g(z)\|_{C([-\tau, 0]; X)} \leq e^{-\omega a} \|w - z\|_{C_b([a, +\infty); X)}. \end{aligned}$$

Now we prove that

$$(4.7) \quad \|T^2 w - T^2 z\|_{C_b([-\tau, +\infty); X)} \leq e^{-\omega a} \|w - z\|_{C_b([-\tau, +\infty); X)}.$$

First, let $t \geq 0$. By (g_2) and (4.6), we have

$$\begin{aligned} \|(T^2 w)(t) - (T^2 z)(t)\| &\leq e^{-\omega t} \|(g(Tw))(0) - (g(Tz))(0)\| \\ &\leq e^{-\omega t} \|g(Tw) - g(Tz)\|_{C([-\tau, 0]; X)} \leq \|Tw - Tz\|_{C_b([a, +\infty); X)} \\ &\leq e^{-\omega a} \|w - z\|_{C_b([a, +\infty); X)} \leq e^{-\omega a} \|w - z\|_{C_b([-\tau, +\infty); X)}. \end{aligned}$$

Second, if $t \in [-\tau, 0]$, using the same inequalities, i.e., (g_2) and (4.6), we get

$$\begin{aligned} \|(T^2w)(t) - (T^2z)(t)\| &= \|(g(Tw))(t) - (g(Tz))(t)\| \leq \|g(Tw) - g(Tz)\|_{C([-\tau, 0]; X)} \\ &\leq \|Tw - Tz\|_{C_b([a, +\infty); X)} \leq e^{-\omega a} \|w - z\|_{C_b([-\tau, +\infty); X)}. \end{aligned}$$

So, for each $t \in [-\tau, +\infty)$ we have

$$\|(T^2w)(t) - (T^2z)(t)\| \leq e^{-\omega a} \|w - z\|_{C_b([-\tau, +\infty); X)},$$

which implies (4.7). But (4.7) shows that T^2 is a contraction of the constant $e^{-\omega a}$. So, T has a unique fixed point u which is a C^0 -solution of problem (4.4). This completes the proof. $\square$

Lemma 4.2. *Let $x, y \in C_b([-\tau, +\infty); \mathbb{R}_+)$ and let us assume that there exist $a > 0$, $\omega > 0$, $\ell > 0$ and $m \geq 0$ such that*

$$(4.8) \quad x(t) \leq \|x\|_{C_b([a, +\infty); \mathbb{R})}$$

for each $t \in [-\tau, 0]$ and

$$(4.9) \quad x(t) \leq e^{-\omega t} \|x\|_{C_b([a, +\infty); \mathbb{R})} + \frac{\ell}{\omega} (1 - e^{-\omega t}) \left(\|y\|_{C_b([-\tau, +\infty); \mathbb{R})} + \frac{m}{\ell} \right)$$

for each $t \in [0, +\infty)$. Then

$$(4.10) \quad \|x\|_{C_b([-\tau, +\infty); \mathbb{R})} \leq \frac{\ell}{\omega} \left(\|y\|_{C_b([-\tau, +\infty); \mathbb{R})} + \frac{m}{\ell} \right).$$

Proof. We prove first that

$$(4.11) \quad \|x\|_{C_b([a, +\infty); \mathbb{R})} \leq \frac{\ell}{\omega} \left(\|y\|_{C_b([-\tau, +\infty); \mathbb{R})} + \frac{m}{\ell} \right).$$

We distinguish between two possible complementary cases.

Case 1. There exists $\tilde{t} \in [a, +\infty)$ such that

$$(4.12) \quad \|x\|_{C_b([a, +\infty); \mathbb{R})} = x(\tilde{t}).$$

Setting $t = \tilde{t}$ in (4.9), we get

$$\|x\|_{C_b([a, +\infty); \mathbb{R})} \leq e^{-\omega \tilde{t}} \|x\|_{C_b([a, +\infty); \mathbb{R})} + \frac{\ell}{\omega} (1 - e^{-\omega \tilde{t}}) \left(\|y\|_{C_b([-\tau, +\infty); \mathbb{R})} + \frac{m}{\ell} \right)$$

and, since $\tilde{t} \geq a > 0$, we deduce (4.11).

Case 2. If there is no $\tilde{t} \in [a, +\infty)$ such that (4.12) holds true, then there does exist at least one sequence $(t_k)_k \subseteq [a, +\infty)$ with $\lim_k t_k = +\infty$ and

$$\|x\|_{C_b([a, +\infty); \mathbb{R})} = \lim_k x(t_k).$$

Setting $t = t_k$ in (4.9) and letting $k \rightarrow +\infty$, we again obtain (4.11).

Now, let us prove (4.10).

Here, we distinguish between three possible cases.

Case 1. There exists $\bar{t} \in [-\tau, 0]$ such that

$$\|x\|_{C_b([-\tau, +\infty); \mathbb{R})} = x(\bar{t}).$$

By (4.8), we get

$$x(\bar{t}) \leq \|x\|_{C_b([a, +\infty); \mathbb{R})}$$

and thus

$$\|x\|_{C_b([-\tau, +\infty); \mathbb{R})} = \|x\|_{C_b([a, +\infty); \mathbb{R})},$$

and by (4.11), we deduce (4.10).

Case 2. There exists $\bar{t} > 0$ such that

$$\|x\|_{C_b([- \tau, +\infty); \mathbb{R})} = x(\bar{t}).$$

Using (4.9), we conclude that

$$\|x\|_{C_b([- \tau, +\infty); \mathbb{R})} \leq e^{-\omega \bar{t}} \|x\|_{C_b([- \tau, +\infty); \mathbb{R})} + \frac{\ell}{\omega} (1 - e^{-\omega \bar{t}}) \left(\|y\|_{C_b([- \tau, +\infty); \mathbb{R})} + \frac{m}{\ell} \right).$$

Since $\bar{t} > 0$, we easily get (4.10).

Case 3. If, for each $t \in [-\tau, +\infty)$,

$$x(t) < \|x\|_{C_b([- \tau, +\infty); \mathbb{R})},$$

then there exists $(t_k)_k$ with $\lim_k t_k = +\infty$ such that

$$\lim_k x(t_k) = \|x\|_{C_b([- \tau, +\infty); \mathbb{R})}.$$

Setting $t = t_k$ in (4.9) and letting $k \rightarrow +\infty$, we obtain (4.10).

So, in all possible cases, x and y satisfy (4.10), and this completes the proof. $\square$

The next lemma is perhaps known, but since we did not find any reference to it in the form we need later, we include it with a complete proof.

Lemma 4.3. *Let $y : [-\tau, +\infty) \rightarrow \mathbb{R}_+$ and $\alpha_0, \beta : [0, +\infty) \rightarrow \mathbb{R}_+$ be continuous functions with α_0 nondecreasing. If*

$$(4.13) \quad y(t) \leq \alpha_0(t) + \int_0^t \beta(s) \|y_s\|_{C([- \tau, 0]; \mathbb{R})} ds$$

for each $t \in [0, +\infty)$, then

$$(4.14) \quad y(t) \leq \alpha(t) + \int_0^t \alpha(s) \beta(s) e^{\int_s^t \beta(\sigma) d\sigma} ds$$

for each $t \in [0, +\infty)$, where

$$\alpha(t) := \|y_0\|_{C([- \tau, 0]; \mathbb{R})} + \alpha_0(t)$$

for each $t \in [0, +\infty)$.

Proof. Let us denote by $z : [-\tau, +\infty) \rightarrow \mathbb{R}_+$ the function

$$z(t) := \sup\{\|y_s\|_{C([- \tau, 0]; \mathbb{R})}; s \in [0, t]\} = \sup\{y(\theta); \theta \in [-\tau, t]\}.$$

Clearly $y(t) \leq z(t)$ for each $t \in [0, +\infty)$. Since y is continuous, for each $t \in [0, +\infty)$ there exists $s_t \in [-\tau, t]$ such that $z(t) = y(s_t)$.

We distinguish between two cases.

Case 1. If $s_t \in [-\tau, 0]$ and $y(s) < z(s)$ for each $s \in (0, t]$, then

$$(4.15) \quad z(t) = y(s_t) \leq \|y_0\|_{C([- \tau, 0]; \mathbb{R})} \leq \alpha(t) + \int_0^t \beta(s) z(s) ds.$$

Case 2. If $s_t \in (0, t]$, then

$$\begin{aligned} z(t) &= y(s_t) \leq \alpha_0(s_t) + \int_0^{s_t} \beta(\sigma) \|y_\sigma\|_{C([- \tau, 0]; \mathbb{R})} d\sigma \\ &\leq \alpha_0(t) + \|y_0\|_{C([- \tau, 0]; \mathbb{R})} + \int_0^t \beta(\sigma) z(\sigma) d\sigma = \alpha(t) + \int_0^t \beta(\sigma) z(\sigma) d\sigma. \end{aligned}$$

So, (4.15) holds true for each $t \in [0, +\infty)$. From Bellman's Inequality (see Vrabie [18, Problem 1.16, p. 48]) we get

$$z(t) \leq \alpha(t) + \int_0^t \alpha(s) \beta(s) e^{\int_s^t \beta(\sigma) d\sigma} ds$$

for each $t \in [0, +\infty)$. Since $y(t) \leq z(t)$ for each $t \in [0, +\infty)$, the inequality above implies (4.14). The proof is complete. $\square$

4.1. Proof of Theorem 3.1.

Proof. The proof is based on a fixed point argument. Namely, let us define the operator $Q : C_b([-\tau, +\infty); \overline{D(A)}) \rightarrow C_b([-\tau, +\infty); \overline{D(A)})$ by $Q(v) = u$, where $u \in C_b([-\tau, +\infty); \overline{D(A)})$ is the unique C^0 -solution of the problem

$$\begin{cases} u'(t) \in Au(t) + f(t, v_t), & t \in \mathbb{R}_+, \\ u(t) = g(u)(t), & t \in [-\tau, 0]. \end{cases}$$

To complete the proof, it would suffice to show that Q is a strict contraction. To this aim, let $v, w \in C_b([-\tau, +\infty); \overline{D(A)})$ be arbitrary and let us observe that for each $t \geq 0$, from (2.3) and (f_1) we get

$$\begin{aligned} \|Q(v)(t) - Q(w)(t)\| &\leq e^{-\omega t} \|Q(v)(0) - Q(w)(0)\| + \frac{\ell}{\omega} (1 - e^{-\omega t}) \sup_{s \in [-\tau, t]} \|v_s - w_s\| \\ &\leq e^{-\omega t} \|g(Q(v))(0) - g(Q(w))(0)\| + \frac{\ell}{\omega} (1 - e^{-\omega t}) \|v - w\|_{C_b([-\tau, +\infty); X)} \\ &\leq e^{-\omega t} \|g(Q(v)) - g(Q(w))\|_{C([-\tau, 0]; X)} + \frac{\ell}{\omega} (1 - e^{-\omega t}) \|v - w\|_{C_b([-\tau, +\infty); X)}. \end{aligned}$$

Thanks to (g_2) , we conclude that for each $t \geq 0$, we have

$$\begin{aligned} &\|Q(v)(t) - Q(w)(t)\| \\ (4.16) \quad &\leq e^{-\omega t} \|Q(v) - Q(w)\|_{C_b([a, +\infty); X)} + \frac{\ell}{\omega} (1 - e^{-\omega t}) \|v - w\|_{C_b([-\tau, +\infty); X)}. \end{aligned}$$

Since, by (g_2) , we have

$$\|Q(v)(t) - Q(w)(t)\| \leq \|Q(v) - Q(w)\|_{C_b([a, +\infty); X)}$$

for each $t \in [-\tau, 0]$, denoting by $x(t) = \|Q(v)(t) - Q(w)(t)\|$, $y(t) = \|v(t) - w(t)\|$ for $t \in [-\tau, +\infty)$, $\omega > 0$, $\ell > 0$ as above and $m = 0$, we have merely to observe that we are in the hypotheses of Lemma 4.2. Then, we conclude that

$$(4.17) \quad \|Q(v) - Q(w)\|_{C_b([-\tau, +\infty); X)} \leq \frac{\ell}{\omega} \|v - w\|_{C_b([-\tau, +\infty); X)},$$

which, in view of (H_c) , i.e., $\ell < \omega$, shows that Q is a strict contraction.

In order to prove the inequality (3.1), let us observe first that by (g_1) we have $g(0) = 0$, while from (g_2) we get

$$(4.18) \quad \|g(u)\|_{C([-\tau, 0]; X)} \leq \|u\|_{C_b([a, +\infty); X)}$$

for each $u \in C_b([-\tau, +\infty); \overline{D(A)})$.

Since $0 \in A0$, from (2.3) in Theorem 2.1, we have

$$\|u(t)\| \leq e^{-\omega t} \|g(u)\|_{C([-\tau, 0]; X)} + \int_0^t e^{-\omega(t-s)} (\|f(s, u_s) - f(s, 0)\| + \|f(s, 0)\|) ds,$$

for each $t \geq 0$. Next, using (H_f) and (4.18), we deduce that

$$(4.19) \quad \|u(t)\| \leq e^{-\omega t} \|u\|_{C_b([a, +\infty); X)} + \frac{\ell}{\omega} (1 - e^{-\omega t}) \left(\|u\|_{C_b([- \tau, +\infty); X)} + \frac{m}{\ell} \right).$$

Since, by the initial nonlocal condition and (g_2) we have

$$\|u(t)\| \leq \|u\|_{C_b([a, +\infty); X)}$$

for each $t \in [-\tau, 0]$, it follows that all the hypotheses of Lemma 4.2 are satisfied by $x(t) = y(t) = \|u(t)\|$ and $\omega > 0$, $\ell > 0$ and $m > 0$ as above. So

$$\|u\|_{C_b([- \tau, +\infty); X)} \leq \frac{\ell}{\omega} \left(\|u\|_{C_b([- \tau, +\infty); X)} + \frac{m}{\ell} \right),$$

from which we get (3.1).

Finally, let $u \in C_b([- \tau, +\infty); \overline{D(A)})$ be the unique C^0 -solution of (1.1), let $\varphi \in C([- \tau, 0]; \overline{D(A)})$ be arbitrary but fixed, and let $v \in C_b([- \tau, +\infty); \overline{D(A)})$ be the unique C^0 -solution of the problem

$$\begin{cases} v'(t) \in Av(t) + f(t, v_t), & t \in [0, +\infty), \\ v(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases}$$

In view of (2.2) and of (f_1) in (H_f) , we have

$$\begin{aligned} \|u(t) - v(t)\| &\leq e^{-\omega t} \|u(0) - v(0)\| + \int_0^t e^{-\omega(t-s)} \|f(s, u_s) - f(s, v_s)\| ds \\ &\leq e^{-\omega t} \|u(0) - v(0)\| + \int_0^t \ell e^{-\omega(t-s)} \|u_s - v_s\|_{C([- \tau, 0]; X)} ds \end{aligned}$$

for each $t \geq 0$. Multiplying both sides by $e^{\omega t}$ and observing that

$$e^{\omega s} \|u_s - v_s\|_{C([- \tau, 0]; X)} \leq e^{\omega \tau} \sup_{\theta \in [-\tau, 0]} \left\{ e^{\omega(s+\theta)} \|u(s+\theta) - v(s+\theta)\| \right\},$$

we deduce that

$$e^{\omega t} \|u(t) - v(t)\| \leq \|u(0) - v(0)\| + \int_0^t \ell e^{\omega \tau} \sup_{\theta \in [-\tau, 0]} \left\{ e^{\omega(s+\theta)} \|u(s+\theta) - v(s+\theta)\| \right\} ds$$

for each $t \geq 0$. From Lemma 4.3 with

$$\begin{cases} y(t) = e^{\omega t} \|u(t) - v(t)\|, & t \in [-\tau, +\infty), \\ \alpha_0(t) = \|u(0) - v(0)\| := c, & t \in [0, +\infty), \\ \beta(t) = \ell e^{\omega \tau} := b, & t \in [0, +\infty), \end{cases}$$

we get

$$e^{\omega t} \|u(t) - v(t)\| \leq c + \|u - v\|_{C([- \tau, 0]; X)} + b (c + \|u - v\|_{C([- \tau, 0]; X)}) \int_0^t e^{b(t-s)} ds$$

for each $t \geq 0$. Accordingly, we have

$$\|u(t) - v(t)\| \leq (c + \|u - v\|_{C([- \tau, 0]; X)}) e^{-(\omega - b)t}$$

for each $t \geq 0$. Since $b = \ell e^{\omega \tau} < \omega$, this shows that

$$\lim_{t \rightarrow +\infty} \|u(t) - v(t)\| = 0,$$

which means that u is globally asymptotically stable. This completes the proof. $\square$

4.2. Proof of Theorem 3.2.

Proof. We have only to observe that (3.2) can be equivalently rewritten as (1.1) where $g : C_b([-\tau, +\infty); \overline{D(A)}) \rightarrow C([-\tau, 0]; \overline{D(A)})$ is defined by

$$g(u)(t) = u(2\pi + t)$$

for each $u \in C_b([-\tau, +\infty); \overline{D(A)})$ and each $t \in [-\tau, 0]$. Let $u \in C_b([-\tau, +\infty); \overline{D(A)})$ be arbitrary but fixed. Then, there exists $t_u \in [-\tau, 0]$ such that

$$\|g(u)\|_{C([-\tau, 0]; X)} = \|g(u)(t_u)\| = \|u(2\pi + t_u)\|$$

and, since $2\pi + t_u \geq 2\pi - \tau > 0$, we deduce that

$$\|g(u)\|_{C([-\tau, 0]; X)} \leq \|u\|_{C_b([0, +\infty); X)},$$

which shows that g satisfies (g_1) in (H_g) .

To check (g_2) , let $u, v \in C_b([-\tau, +\infty); \overline{D(A)})$ be arbitrary but fixed and observe that there exists $t_{u,v} \in [-\tau, 0]$ such that

$$\|g(u) - g(v)\|_{C([-\tau, 0]; X)} = \|g(u)(t_{u,v}) - g(v)(t_{u,v})\| = \|u(2\pi + t_{u,v}) - v(2\pi + t_{u,v})\|.$$

But $2\pi + t_{u,v} \geq 2\pi - \tau = a > 0$, and therefore

$$\|g(u) - g(v)\|_{C([-\tau, 0]; X)} \leq \|u - v\|_{C_b([a, +\infty); X)}.$$

So, g satisfies (g_2) in (H_g) too. Thus we are in the hypotheses of Theorem 3.1. Since (3.3) follows from (3.1) in conjunction with the periodicity condition, this completes the proof. $\square$

5. EXAMPLES

Let Ω be a nonempty, bounded domain in $\mathbb{R}^n$, $n \geq 2$, with C^1 boundary Γ , let $\tau > 0$, $\omega > 0$ and let Δ be the Laplace operator in the sense of distributions over Ω . Let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal-monotone, let $f : \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$ be continuous, let $N : L^1(\Omega) \rightarrow L^1(\Omega)$ be a (possible nonlinear) nonexpansive operator and μ a σ -finite and complete measure on $[\tau, +\infty)$ with $\text{supp } \mu = [\delta, +\infty)$, $\delta > \tau$. We consider the following problem with nonlocal initial conditions:

$$(5.20) \quad \begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta \varphi(u(t, x)) - \omega u(t, x) + f(t, u_t)(x), & (t, x) \in \mathbb{R}_+ \times \Omega, \\ \varphi(u(t, x)) = 0, & (t, x) \in \mathbb{R}_+ \times \Gamma, \\ u(t, x) = \int_{\tau}^{+\infty} N(u(t + \theta, \cdot))(x) d\mu(\theta), & (t, x) \in [-\tau, 0] \times \Omega. \end{cases}$$

If $u : \Omega \rightarrow D(\varphi)$, we denote

$$\mathcal{S}_{\varphi}(u) = \{v \in L^1(\Omega); v(x) \in \varphi(u(x)), \text{ a.e. for } x \in \Omega\}.$$

We recall the following result of Brezis and Strauss [3].

Theorem 5.1. *Let Ω be a nonempty, bounded domain in $\mathbb{R}^n$, $n \geq 2$, with C^1 boundary Γ and let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal-monotone with $0 \in \varphi(0)$. Then the operator $\Delta \varphi : D(\Delta \varphi) \subseteq L^1(\Omega) \rightsquigarrow L^1(\Omega)$, defined by*

$$\begin{aligned} D(\Delta \varphi) &= \{u \in L^1(\Omega); \exists v \in \mathcal{S}_{\varphi}(u) \cap W_0^{1,1}(\Omega), \Delta v \in L^1(\Omega)\}, \\ \Delta \varphi(u) &= \{\Delta v; v \in \mathcal{S}_{\varphi}(u) \cap W_0^{1,1}(\Omega)\} \cap L^1(\Omega) \end{aligned}$$

for each $u \in D(\Delta \varphi)$, is m -dissipative on $L^1(\Omega)$.

Theorem 5.2. *Let Ω be a nonempty, bounded domain in $\mathbb{R}^n$ with C^1 boundary Γ , let $\tau > 0$, $\omega > 0$ and let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal-monotone with $0 \in \varphi(0)$. Let $f : \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$ be continuous, let $\mathcal{N} : L^1(\Omega) \rightarrow L^1(\Omega)$ be a (possible nonlinear) operator with $\mathcal{N}(0) = 0$ and let μ be a σ -finite and complete measure on $[\tau, +\infty)$ with $\text{supp } \mu = [\delta, +\infty)$, $\delta > \tau$. If the following hypotheses are satisfied:*

(F₁) *there exists $\ell \in (0, \omega)$ such that*

$$\|f(t, v) - f(t, \tilde{v})\|_{L^1(\Omega)} \leq \ell \|v - \tilde{v}\|_{C([-\tau, 0]; L^1(\Omega))}$$

for each $t \in \mathbb{R}_+$ and each $v, \tilde{v} \in C([-\tau, 0]; L^1(\Omega))$,

(F₂) *there exists $m > 0$ such that for each $t \in \mathbb{R}_+$ we have*

$$\|f(t, 0)\|_{L^1(\Omega)} \leq m,$$

(N) *for each $u, \tilde{u} \in L^1(\Omega)$ we have*

$$\|\mathcal{N}(u) - \mathcal{N}(\tilde{u})\|_{L^1(\Omega)} \leq \|u - \tilde{u}\|_{L^1(\Omega)},$$

(μ) $\mu([\tau, +\infty)) = 1$,

then the problem (5.20) has a unique C^0 -solution, $u \in C_b([-\tau, +\infty); L^1(\Omega))$, satisfying

$$\|u\|_{C_b([-\tau, +\infty); L^1(\Omega))} \leq \frac{m}{\omega - \ell}.$$

If, in addition, instead of $\ell \in (0, \omega)$, the stronger condition $\ell e^{\omega\tau} < \omega$ is satisfied, then u is globally asymptotically stable.

Proof. We will apply Theorem 3.1 with $X = L^1(\Omega)$; $A : D(A) \subseteq L^1(\Omega) \rightsquigarrow L^1(\Omega)$, defined by $D(A) = D(\Delta\varphi)$; $Au = \Delta\varphi(u) - \omega u$ for each $u \in D(A)$, where $\Delta\varphi$ is as in Theorem 5.1; $f : \mathbb{R}_+ \times C([-\tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$ as above and $g : C_b([-\tau, +\infty); L^1(\Omega)) \rightarrow C([-\tau, 0]; L^1(\Omega))$ given by (1.3), i.e.,

$$(g(u)(t))(x) = \int_{\tau}^{+\infty} \mathcal{N}(u(t+\theta)(\cdot))(x) d\mu(\theta)$$

for each $u \in C([-\tau, 0]; L^1(\Omega))$, $t \in [-\tau, 0]$ and a.e. for $x \in \Omega$.

In view of Theorem 5.1, the operator A satisfies (H_A) in Theorem 3.1. From (F_1) and (F_2) we deduce that f satisfies (H_f) . As $\mathcal{N}(0) = 0$, from (N) we conclude that $\|\mathcal{N}(u)\|_{L^1(\Omega)} \leq \|u\|_{L^1(\Omega)}$ which, together with (μ) , shows that g satisfies (g_1) . If we consider $a = \delta - \tau > 0$, where $\text{supp } \mu = [\delta, +\infty)$, the nonexpansivity condition on $\mathcal{N}$ proves that g also satisfies (g_2) . So, the problem (5.20) can be rewritten as an abstract problem of the form (1.1) satisfying the hypotheses of Theorem 3.1, whence the conclusion of Theorem 5.2. $\square$

Remark 5.1. We indicate two important examples of forcing terms f defined by starting from a real function $\tilde{f} : \mathbb{R} \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and satisfying the hypotheses (F_1) and (F_2) .

(i) $f(t, u)(x) := \tilde{f}(t, x, u(-\tau, x))$ for each $t \in \mathbb{R}_+$, each $u \in C([-\tau, L^1(\Omega)])$ and a.e. for $x \in \Omega$. In this case, $f(t, u_t)(x) = \tilde{f}(t, x, u(t - \tau, x))$ is the usual delay difference forcing term.

(ii) $f(t, u)(x) := \tilde{f}\left(t, x, \int_{-\tau}^0 k(s)u(s, x) ds\right)$, which represents the neutral type forcing term.

It is easy to check that whenever $\tilde{f}$ is continuous and there exist $\tilde{\ell} > 0$ and $\tilde{m} > 0$ such that $|\tilde{f}(t, x, u) - \tilde{f}(t, x, v)| \leq \tilde{\ell}|u - v|$ and $|\tilde{f}(t, x, 0)| \leq \tilde{m}$ for each $t \in \mathbb{R}_+$, each $u, v \in \mathbb{R}$ and each $x \in \Omega$, then $\tilde{f}$, defined as in (i), satisfies (F_1) and (F_2) with $\ell = \tilde{\ell}$ and $m = \tilde{m}$. Furthermore, if $\tilde{f}$ is as above and $k \in L^1(-\tau, 0; \mathbb{R})$, then $\tilde{f}$, defined as in (ii), satisfies (F_1) and (F_2) with $\ell = \tilde{\ell}\|k\|_{L^1(-\tau, 0; \mathbb{R})}^{-1}$ and $m = \tilde{m}\|k\|_{L^1(-\tau, 0; \mathbb{R})}^{-1}$.

As an application of Theorem 3.2, we consider the following nonlinear delay parabolic equation subjected to periodic conditions:

$$(5.21) \quad \begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta \varphi(u(t, x)) - \omega u(t, x) + f(t, u_t)(x), & (t, x) \in \mathbb{R} \times \Omega, \\ \varphi(u(t, x)) = 0, & (t, x) \in \mathbb{R} \times \Gamma, \\ u(t, x) = u(t + 2\pi, x), & (t, x) \in \mathbb{R} \times \Omega. \end{cases}$$

Theorem 5.3. *Let Ω be a nonempty, bounded domain in $\mathbb{R}^n$ with C^1 boundary Γ , let $\omega > 0$ and let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be a maximal-monotone operator with $0 \in \varphi(0)$. Let $f : \mathbb{R}_+ \times C([- \tau, 0]; L^1(\Omega)) \rightarrow L^1(\Omega)$ be continuous and 2π -periodic with respect to its first argument. If $2\pi > \tau$ and f satisfies (F_1) and (F_2) in Theorem 5.2, then the problem (5.21) has a unique C^0 -solution $u \in C(\mathbb{R}; L^1(\Omega))$. If, in addition, instead of $\ell \in (0, \omega)$, the stronger condition $\ell e^{\omega\tau} < \omega$ is satisfied, then u is globally asymptotically stable.*

Remark 5.2. It should be emphasized that, as mentioned in [17, Remark 2.7.1, p. 72], under the hypotheses of both Theorems 5.2 and 5.3, the semigroup generated by A on $L^1(\Omega)$ is not necessarily compact. Therefore, Theorems 5.2 and 5.3 cannot be obtained as single-valued corollaries of the main results in [20] and [22].

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