EDGE RINGS SATISFYING SERRE’S CONDITION (R_1)

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ABSTRACT. A combinatorial criterion for the edge ring of a finite connected graph satisfying Serre’s condition (R_1) is studied.

INTRODUCTION

The edge polytopes and edge rings of finite connected graphs have been studied from the viewpoints of both combinatorics and computational commutative algebra ([3], [4]). Especially, a combinatorial characterization for the edge ring to be normal is obtained by both [3] and [6] independently. It follows immediately from [2, Theorem 6.4.2] that a normal edge ring is Cohen–Macaulay. However, in general it seems unclear when the edge ring is Cohen–Macaulay. Recall that a noetherian ring is normal if and only if it satisfies Serre’s conditions (R_1) and (S_2) . Thus in particular an edge ring satisfying Serre’s condition (R_1) is normal if and only if it is Cohen–Macaulay. In the present paper the problem when a given edge ring satisfies Serre’s condition (R_1) is investigated.

1. EDGE RINGS AND EDGE POLYTOPES OF FINITE CONNECTED GRAPHS

First, we recall from [3] the definitions of edge rings and edge polytopes of finite connected graphs. Let G be a finite connected graph on the vertex set $[d] = \{1, \dots, d\}$ with $E(G) = \{e_1, \dots, e_n\}$ its edge set. We always assume that G is simple, i.e., G has no loop and no multiple edge. Let $\mathbf{e}_1, \dots, \mathbf{e}_d$ denote the i th unit coordinate vectors of $\mathbb{R}^d$. We associate each edge $e = \{i, j\} \in E(G)$ with the vector $\rho(e) = \mathbf{e}_i + \mathbf{e}_j \in \mathbb{R}^d$. The *edge polytope* is the convex polytope $\mathcal{P}_G \subset \mathbb{R}^d$ which is the convex hull of the finite set $\{\rho(e_1), \dots, \rho(e_n)\}$. Let $K[\mathbf{t}] = K[t_1, \dots, t_d]$ be the polynomial ring in d variables over a field K . We associate each edge $e = \{i, j\} \in E(G)$ with the quadratic monomial $\mathbf{t}^e = t_i t_j \in K[\mathbf{t}]$. The *edge ring* is the affine semigroup ring $K[G] = K[\mathbf{t}^{e_1}, \dots, \mathbf{t}^{e_n}]$.

Let, in general, $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope, i.e., a convex polytope all of whose vertices have integer coordinates, which lies on a hyperplane $\mathcal{H} \subset \mathbb{R}^d$ with $\mathbf{0} \notin \mathcal{H}$, where $\mathbf{0}$ is the origin of $\mathbb{R}^d$. We assume that $\mathcal{P} \subset \mathbb{R}_{\geq 0}^d$, where $\mathbb{R}_{\geq 0}$ is

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the set of nonnegative real numbers. Then for each integer point $\mathbf{a} = (a_1, \dots, a_d)$ belonging to $\mathcal{P}$, we associate the monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_d^{a_d} \in K[\mathbf{t}]$. The *toric ring* of $\mathcal{P}$ is the affine semigroup ring $K[\mathcal{P}] = K[\{\mathbf{t}^{\mathbf{a}} : \mathbf{a} \in \mathcal{P} \cap \mathbb{Z}^d\}]$. Thus in particular the edge ring $K[G]$ of a finite connected graph G is the toric ring of the edge polytope $\mathcal{P}_G$ of G .

We say that an integral convex polytope $\mathcal{P}$ is *normal* if its toric ring $K[\mathcal{P}]$ is normal. It is shown in [3] and [6] that the edge ring of a finite connected graph is normal if and only if G satisfies the *odd cycle condition*. This condition states that for every two disjoint minimal odd cycles in G , there is an edge with one endpoint in each cycle [3, p. 410]. Recall that a toric ring is normal if and only if it satisfies Serre's conditions (R_1) and (S_2) and that every normal toric ring is Cohen–Macaulay. Thus in particular a toric ring satisfying Serre's condition (R_1) is normal if and only if it is Cohen–Macaulay. In the present paper the problem when a given edge ring satisfies Serre's condition (R_1) is investigated.

2. WHEN DOES AN EDGE RING SATISFY SERRE'S CONDITION (R_1) ?

Let G be a finite connected graph on the vertex set $[d] = \{1, \dots, d\}$. If G is bipartite, then $K[G]$ is normal and satisfies Serre's condition (R_1) . Thus in what follows we assume that G is nonbipartite, i.e., G possesses at least one odd cycle.

If T is a nonempty subset of $[d]$, then the induced subgraph of G on T is denoted by G_T . A nonempty subset T of $[d]$ is called *independent* if $\{i, j\} \notin E(G)$ for all $i, j \in T$ with $i \neq j$. If T is independent and if $N(G; T)$ is the set of vertices $j \in [d]$ with $\{i, j\} \in E(G)$ for some $i \in T$, then the *bipartite graph induced by T* is defined to be the bipartite graph having the vertex set $T \cup N(G; T)$ and consisting of all edges $\{i, j\} \in E(G)$ with $i \in T$ and $j \in N(G; T)$. We say that a nonempty subset $T \subset [d]$ is *fundamental* if

- T is independent;
- the bipartite graph induced by T is connected;
- either $T \cup N(G; T) = [d]$ or every connected component of the induced subgraph $G_{[d] \setminus (T \cup N(G; T))}$ has at least one odd cycle.

Moreover, we call a vertex $i \in [d]$ *regular* if every connected component of $G_{[d] \setminus i}$ has at least one odd cycle. Note that a regular vertex is not the same as a fundamental set with one element.

We are now in the position to state our criterion for an edge ring to satisfy Serre's condition (R_1) .

Theorem 2.1. *Let G be a finite connected nonbipartite graph on $[d]$. Then the edge ring $K[G]$ of G satisfies Serre's condition (R_1) if and only if the following conditions are satisfied:*

- (i) *For every regular vertex $i \in [d]$, the induced subgraph $G_{[d] \setminus i}$ is connected.*
- (ii) *For every fundamental set $T \subset [d]$, one has either $T \cup N(G; T) = [d]$ or the induced subgraph $G_{[d] \setminus (T \cup N(G; T))}$ is connected.*

Example 2.2. Let G be the finite connected graph on $\{1, \dots, 8\}$ depicted in Figure 1. The graph G clearly violates the odd cycle condition, hence the edge ring $K[G]$ is not normal. The only vertices whose removal makes G disconnected are 3 and 4, but both are not regular. If $T \subset [8]$ is a set such that $G_{[d] \setminus (T \cup N(G; T))}$ is disconnected, then either 3 or 4 are contained in $T \cup N(G; T)$. But then $G_{[d] \setminus (T \cup N(G; T))}$

FIGURE 1. The graph G of Example 1

has only one odd cycle left, so T cannot be fundamental. Hence $K[G]$ satisfies Serre's condition (R_1) . More generally, the same argument shows that the graphs G_{k+6} constructed in [5] satisfy (R_1) if and only if $k \geq 2$.

3. PROOF OF THEOREM 2.1

First recall the description of the facets of $\mathcal{P}_G$. To every regular vertex i we associate the linear form $\sigma_i : \mathbb{R}^d \rightarrow \mathbb{R}$ which projects onto the i th component. Moreover, we set $\mathcal{H}_i = \{x \in \mathbb{R}^d : \sigma_i(x) = 0\}$ and $\mathcal{F}_i = \mathcal{P}_G \cap \mathcal{H}_i$. Similarly, to every fundamental set T we associate the linear form

$$\sigma_T : \mathbb{R}^d \ni (x_1, \dots, x_d) \mapsto \sum_{j \in N(G;T)} x_j - \sum_{i \in T} x_i$$

and we set $\mathcal{H}_T = \{x \in \mathbb{R}^d : \sigma_T(x) = 0\}$ and $\mathcal{F}_T = \mathcal{P}_G \cap \mathcal{H}_T$.

Lemma 3.1 ([3]). *The facets of $\mathcal{P}_G$ are exactly the sets $\mathcal{F}_i$ and $\mathcal{F}_T$ for all regular vertices i and all fundamental sets T .*

A combinatorial condition for a semigroup ring to satisfy Serre's condition (R_1) is explicitly stated in [7, Theorem 2.7]. In fact, in [7] a characterization of (R_l) for all l is given, but for our purposes we only need the case $l = 1$.

Proposition 3.2 ([7]). *Let M be an affine monoid, K a field and $K[M]$ its semigroup ring. Then $K[M]$ satisfies Serre's condition (R_1) if and only if every facet $\mathcal{F}$ of M satisfies the following two conditions:*

- (i) *There exists $x \in M$ such that $\sigma_{\mathcal{F}}(x) = 1$, where $\sigma_{\mathcal{F}}$ is a support form of $\mathcal{F}$ taking integer values on $\text{gp}(M)$.*
- (ii) *$\text{gp}(M \cap \mathcal{F}) = \text{gp}(M) \cap \mathcal{H}$, where $\mathcal{H}$ is the supporting hyperplane of $\mathcal{F}$.*

Here $\text{gp}(M)$ denotes the additive group generated by M .

We apply Proposition 3.2 to the affine monoid

$$M_G = \mathbb{N}(\mathcal{P}_G \cap \mathbb{Z}^d)$$

generated by the integer points in $\mathcal{P}_G$. Note that the support hyperplanes $\mathcal{H}_i$ and $\mathcal{H}_T$ of $\mathcal{P}_G$ are also the support hyperplanes of M_G . We start proving Theorem 2.1 by the following.

Lemma 3.3. *Let G be a finite connected nonbipartite graph on the vertex set $[d]$. Then the facets of M_G satisfy the first condition of Proposition 3.2.*

Proof. First, let $i \in [d]$ be a regular vertex. Since G is connected, there exists an edge $e = \{i, j\} \in E(G)$ to another vertex j . Then $\sigma_i(\rho(e)) = 1$.

Second, let $T \subset [d]$ be a fundamental set. If $T \cup N(G; T) \subsetneq [d]$, then there exists an edge $e = \{i, j\} \in E(G)$ such that $i \in N(G; T)$ and $j \in [d] \setminus (T \cup N(G; T))$. This edge satisfies $\sigma_T(\rho(e)) = 1$. If instead $T \cup N(G; T) = [d]$, then every edge of G has either both endpoints in $N(G; T)$, or one in $N(G; T)$ and one in T . Hence $\sigma_T(e) \in \{0, 2\}$ for every edge e of G . It then follows that $\frac{1}{2}\sigma_T$ satisfies the condition of Proposition 3.2. $\square$

To check the second condition of Proposition 3.2 we need to compute the lattice generated by M_G . The following Lemma 3.4 appears in [3, p. 426] without an explicit proof. However, for the sake of completeness, we give its detailed proof.

Lemma 3.4. *Let G be a finite connected nonbipartite graph on the vertex set $[d]$. Then the lattice $gp(M_G)$ is the set of all integer vectors in $\mathbb{Z}^d$ with an even coordinate sum.*

Proof. Since every generator of M_G has an even coordinate sum, it follows that the lattice $gp(M_G)$ is contained in the set of all integer vectors in $\mathbb{Z}^d$ with an even coordinate sum.

To prove the converse, assume the edges $e_1, \dots, e_\ell$ form an odd cycle of G and let i be the common vertex of e_1 and e_ℓ . Then

$$2\mathbf{e}_i = \sum_{j=1}^{\ell} (-1)^{j+1} \rho(e_j) \in gp(M_G).$$

Now consider a spanning tree G' of G . The set $\{\rho(e) \mid e \in E(G')\}$ together with $2\mathbf{e}_i$ forms a $\mathbb{Z}$ -basis for the space of all integer vectors in $\mathbb{Z}^d$ with an even coordinate sum. $\square$

Now, we can prove two propositions which complete our proof of Theorem 2.1.

Proposition 3.5. *Let G be a finite connected nonbipartite graph on the vertex set $[d]$ and let $i \in [d]$ be a regular vertex of G . Then $\mathcal{F}_i$ satisfies the second condition of Proposition 3.2 if and only if $G_{[d] \setminus i}$ is connected.*

Proof. We denote the connected components of $G_{[d] \setminus i}$ with G'_j . Then it is easy to see that $M_{G_{[d] \setminus i}} = \bigoplus_j M_{G'_j}$ and hence $gp(M_G \cap \mathcal{F}_i) = gp(M_{G_{[d] \setminus i}}) = \bigoplus_j gp(M_{G'_j})$. Since every G'_j is connected and contains an odd cycle, we can use Lemma 3.4 to describe $gp(M_{G'_j})$. If $G_{[d] \setminus i}$ is connected, then $gp(G_{[d] \setminus i})$ and $gp(M_G) \cap \mathcal{H}_i$ are both the set of integer vectors in $\mathbb{Z}^d$ with even coordinate sum and i th coordinate equal to zero; thus these sets coincide.

We consider the case that $G_{[d] \setminus i}$ has at least two different connected components G'_1, G'_2 . Then we can choose a vector $x \in \mathbb{Z}^d$ such that (i) its coordinate sum is even, (ii) $\sigma_i(x) = 0$, and (iii) the restricted coordinate sum over the vertices in G'_1 is odd. This x is contained in $gp(M_G) \cap \mathcal{H}_i$ but not in $gp(G_{[d] \setminus i})$; thus $\mathcal{F}_i$ violates the condition. $\square$

Proposition 3.6. *Let G be a finite connected nonbipartite graph on the vertex set $[d]$ and let $T \subset [d]$ be a fundamental set of G . Then $\mathcal{F}_T$ satisfies the second condition of Proposition 3.2 if and only if one has either $T \cup N(G; T) = [d]$ or the induced subgraph $G_{[d] \setminus (T \cup N(G; T))}$ is connected.*

Proof. Again, we denote the connected components of $G_{[d]\setminus(T\cup N(G;T))}$ with G'_j . We claim that

(1)

$$gp(M_G \cap \mathcal{F}_T) = \bigoplus_j gp(M_{G'_j}) \oplus \{x \in \mathbb{Z}^d \mid \text{supp}(x) \subset T \cup N(G;T), \sigma_T(x) = 0\}.$$

Here, $\text{supp}(\cdot)$ denotes the support of a vector. The sum is direct because the supports of the summands are disjoint. $M_G \cap \mathcal{F}_T$ (and thus $gp(M_G \cap \mathcal{F}_T)$) is generated by the set $\{\rho(e) \mid e \in E(G), \sigma_T(\rho(e)) = 0\}$. For an edge $e \in E(G)$, it holds that $\sigma_T(\rho(e)) = 0$ if and only if either both endpoints lie in $T \cup N(G;T)$ or both are not contained in this set. Thus, a set of generators on the left side of (1) is contained in the right side of the equation, and hence one inclusion follows. Furthermore $\bigoplus_j gp(M_{G'_j}) \subset gp(M_G \cap \mathcal{F}_T)$. Thus it remains to show that

$$\{x \in \mathbb{Z}^d \mid \text{supp}(x) \subset T \cup N(G;T), \sigma_T(x) = 0\} \subset gp(M_G \cap \mathcal{F}_T).$$

For this we consider a spanning tree of the induced bipartite graph on $T \cup N(G;T)$. Its edges form a $\mathbb{Z}$ -basis for the left set, hence it is contained in $gp(M_G \cap \mathcal{F}_T)$. Next, we note that

$$\begin{aligned} gp(M_G) \cap \mathcal{H}_T &= \left\{x \in \mathbb{Z}^d \mid \text{supp}(x) \cap (T \cup N(G;T)) = \emptyset, \sum x_i \text{ even}\right\} \\ &\quad \oplus \{x \in \mathbb{Z}^d \mid \text{supp}(x) \subset (T \cup N(G;T)), \sigma_T(x) = 0\}. \end{aligned}$$

Now the reasoning is completely analogous to the proof of Proposition 3.5. $\square$

REFERENCES

- [1] Winfried Bruns and Joseph Gubeladze, *Polytopes, rings, and K-theory*, Springer Monographs in Mathematics, Springer, Dordrecht, 2009. MR2508056 (2010d:19001)
- [2] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956 (95h:13020)
- [3] Hidefumi Ohsugi and Takayuki Hibi, *Normal polytopes arising from finite graphs*, J. Algebra **207** (1998), no. 2, 409–426, DOI 10.1006/jabr.1998.7476. MR1644250 (2000a:13010)
- [4] Hidefumi Ohsugi and Takayuki Hibi, *Toric ideals generated by quadratic binomials*, J. Algebra **218** (1999), no. 2, 509–527, DOI 10.1006/jabr.1999.7918. MR1705794 (2000f:13055)
- [5] Takayuki Hibi, Akihiro Higashitani, Kyouko Kimura, and Augustine B. O’Keefe, *Depth of edge rings arising from finite graphs*, Proc. Amer. Math. Soc. **139** (2011), no. 11, 3807–3813, DOI 10.1090/S0002-9939-2011-11083-9. MR2823027 (2012j:13033)
- [6] Aron Simis, Wolmer V. Vasconcelos, and Rafael H. Villarreal, *The integral closure of subrings associated to graphs*, J. Algebra **199** (1998), no. 1, 281–289, DOI 10.1006/jabr.1997.7171. MR1489364 (99c:13004)
- [7] Marie A. Vitulli, *Serre’s condition R_1 for affine semigroup rings*, Comm. Algebra **37** (2009), no. 3, 743–756, DOI 10.1080/00927870802231262. MR2503176 (2010e:13002)

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