SPECTRA OF THE TRANSLATIONS AND WIENER-HOPF OPERATORS ON $L^2_\omega(\mathbb{R}^+)$

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ABSTRACT. We study bounded operators T on the weighted space $L^2_\omega(\mathbb{R}^+)$ commuting either with the “right shift operators” $(R_t)_{t \geq 0}$ or “left shift operators” $(L_{-t})_{t \geq 0}$, and we establish the existence of a symbol μ of T . We characterize completely the spectrum $\sigma(R_t)$ of the operator R_t proving that

$$\sigma(R_t) = \{z \in \mathbb{C} : |z| \leq e^{\alpha_0 t}\},$$

where α_0 is the growth bound of $(R_t)_{t \geq 0}$. We obtain a similar result for the spectrum of L_{-t} , $t > 0$. Moreover, for a bounded operator T commuting with R_t , $t \geq 0$, we establish the inclusion $\overline{\mu(\mathcal{O})} \subset \sigma(T)$, where

$$\mathcal{O} = \{z \in \mathbb{C} : \operatorname{Im} z < \alpha_0\}.$$

1. INTRODUCTION

Let ω be a weight on $\mathbb{R}^+$. It means that ω is a positive, not vanishing, continuous function on $\mathbb{R}^+$ such that

$$0 < \inf_{x \geq 0} \frac{\omega(x+t)}{\omega(x)} \leq \sup_{x \geq 0} \frac{\omega(x+t)}{\omega(x)} < +\infty, \forall t \in \mathbb{R}^+.$$

For example, e^x and e^{-x} are weights on $\mathbb{R}^+$, and we will see later that

$$\sup_{x \geq 0} \frac{\omega(x+t)}{\omega(x)} \leq Ce^{mt},$$

where C and m are constants.

Let $L^2_\omega(\mathbb{R}^+)$ be the set of measurable functions on $\mathbb{R}^+$ such that

$$\int_0^\infty |f(x)|^2 \omega(x)^2 dx < +\infty.$$

The space $H = L^2_\omega(\mathbb{R}^+)$ equipped with the scalar product

$$\langle f, g \rangle = \int_{\mathbb{R}^+} f(x) \overline{g(x)} \omega(x)^2 dx, \quad f \in L^2_\omega(\mathbb{R}^+), \quad g \in L^2_\omega(\mathbb{R}^+),$$

and the related norm $\|\cdot\|$ is a Hilbert space. Let $C_c^\infty(\mathbb{R})$ (resp. $C_c^\infty(\mathbb{R}^+)$) be the space of C^∞ functions on $\mathbb{R}$ (resp. $\mathbb{R}^+$) with compact support in $\mathbb{R}$ (resp. $\mathbb{R}^+$). Notice that $C_c^\infty(\mathbb{R}^+)$ is dense in $L^2_\omega(\mathbb{R}^+)$. Let r_+ be the restriction from $L^1_{loc}(\mathbb{R}^-) \oplus L^2_\omega(\mathbb{R}^+)$ into $L^2_\omega(\mathbb{R}^+)$ and let r_+ be the restriction from $L^2(\mathbb{R})$ into

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$L^2(\mathbb{R}^+)$. Let ℓ_0 be the extension of a function defined on $\mathbb{R}^+$ into a function on $\mathbb{R}$ setting $\ell_0 f(x) = 0, \forall x \in \mathbb{R}^-$. These notations are compatible with those used for example in [5].

For $t \in \mathbb{R}$ and a function defined on $\mathbb{R}$ we denote $(S_t f)(x) = f(x - t)$. Then for $t > 0$ we introduce the “right shift” operator on $L^2_\omega(\mathbb{R}^+)$ by $R_t := r_+ S_t \ell_0$. For simplicity R_1 will be denoted by R . For $t > 0$ we define the “left shift” operator on $L^2_\omega(\mathbb{R}^+)$ by $L_{-t} f := r_+ S_{-t} \ell_0 f$. It is clear that R_t and L_{-t} map $L^2_\omega(\mathbb{R}^+)$ into $L^2_\omega(\mathbb{R}^+)$, while the shift S_t acts on functions defined on $\mathbb{R}$. The notations R_t and L_{-t} are similar to those for the operators on the spaces of sequences (see [16]). Let I be the identity operator on $L^2_\omega(\mathbb{R}^+)$.

Definition 1.1. A bounded operator T on $L^2_\omega(\mathbb{R}^+)$ is called a Wiener-Hopf operator if

$$L_{-t} T R_t f = T f, \forall t \in \mathbb{R}^+, f \in L^2_\omega(\mathbb{R}^+).$$

More general Wiener-Hopf operators have been intensively studied in the literature (see [18] and the references given there). There also exist many results about Toeplitz operators on weighted Hardy spaces (see [3], [2]). Such operators have some similarities with Wiener-Hopf operators.

Every Wiener-Hopf operator T has a representation by a convolution. The reader may find a proof in [12] where the arguments of [9], [10] are used. More precisely, there exists a distribution $\mu \in \mathcal{D}'(\mathbb{R})$ such that

$$T f(x) = (\mu * \ell_0 f)(x), \forall f \in C_c^\infty(\mathbb{R}^+), x \in \mathbb{R}^+.$$

If $\phi \in C_c^\infty(\mathbb{R})$, then the operator

$$L^2_\omega(\mathbb{R}^+) \ni f \longrightarrow r_+(\phi * \ell_0 f)$$

is a Wiener-Hopf operator and we will denote it by T_ϕ .

A bounded operator T on $L^2_\omega(\mathbb{R}^+)$ commuting either with $R_t, \forall t > 0$, or with $L_{-t}, \forall t > 0$, is a Wiener-Hopf operator. On the other hand, every operator $\alpha L_{-t} + \beta R_t$ with $t > 0, \alpha, \beta \in \mathbb{C}$ is a Wiener-Hopf operator. It is clear that the set of Wiener-Hopf operators is not a sub-algebra of the algebra of the bounded operators on $L^2_\omega(\mathbb{R}^+)$.

Notice also that

$$(L_{-t} R_t) f = f, \forall f \in L^2_\omega(\mathbb{R}^+), t > 0,$$

but it is obvious that $(R_t L_{-t}) f \neq f$, for all $f \in L^2_\omega(\mathbb{R}^+)$ with a support not included in $]t, +\infty[$. The fact that R_t is not invertible leads to many difficulties, in contrast to the case when we deal with the space $L^2_\omega(\mathbb{R})$. The latter space has been considered in [14] and [15], and the author has studied the operators commuting with the translations on $L^2_\omega(\mathbb{R})$ characterizing their spectrum. The group of translations on $L^2_\omega(\mathbb{R})$ is commutative and the investigation of its spectrum is easier. In this work, first we apply some ideas used in [14] and [15] to study Wiener-Hopf operators on $L^2_\omega(\mathbb{R}^+)$. For this purpose it is necessary to treat two semigroups of not invertible operators instead of a group of invertible operators. More precisely, we must deal with the semigroups $(R_t)_{t \geq 0}$ and $(L_{-t})_{t \geq 0}$ on $L^2_\omega(\mathbb{R}^+)$. For our analysis it is more convenient to replace the weight ω by another one. To do this, given a weight ω , denote by $\bar{\omega}$ the function $\bar{\omega}(t) = \sup_{x \geq 0} \frac{\omega(x+t)}{\omega(x)}$. Define

$$\omega_0(x) = \exp\left(\int_1^2 \ln(\omega(x+t)) dt\right).$$

Following [1], ω_0 is a well defined weight on $\mathbb{R}^+$ and, moreover, we have

$$(1.1) \quad \overline{\omega_0}(t) \leq e^{mt}, \quad \forall t \geq 0,$$

where m is a constant. The weights ω and ω_0 are equivalent (see [1]); that is, there exist constants C_1 and C_2 such that

$$C_1\omega_0(x) \leq \omega(x) \leq C_2\omega_0(x), \quad \forall x \in \mathbb{R}^+.$$

Taking into account that $\|R_t\| = \overline{\omega}(t)$, from (1.1) we get the estimate $\|R_t\| \leq Ce^{mt}$, $\forall t \in \mathbb{R}^+$. A similar estimate holds for the semigroup $(L_{-t})_{t \geq 0}$.

Denote by $\rho(B)$ (resp. $\sigma(B)$) the spectral radius (resp. the spectrum) of an operator B . Introduce the ground orders of the semigroups $(R_t)_{t \geq 0}$ and $(L_{-t})_{t \geq 0}$ by

$$\alpha_0 = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|R_t\|, \quad \alpha_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|L_{-t}\|.$$

Then it is well known (see for example [4]) that we have

$$\rho(R_t) = e^{\alpha_0 t}, \quad \rho(L_{-t}) = e^{\alpha_1 t}.$$

Let $\mathcal{I}$ be the interval $[-\alpha_1, \alpha_0]$ and define

$$\Omega := \left\{ z \in \mathbb{C} : e^{-\alpha_1} \leq |z| \leq e^{\alpha_0} \right\}.$$

Notice that $\alpha_1 + \alpha_0 \geq 0$. Indeed, for every $n \in \mathbb{N}$ we have $L_{-n}R_n = I$ and

$$1 \leq \limsup_{n \rightarrow \infty} \|(L_{-1})^n\|^{1/n} \limsup_{n \rightarrow \infty} \|R^n\|^{1/n} = e^{\alpha_1} e^{\alpha_0},$$

which yields the result. For a function f and $a \in \mathbb{C}$ we denote by $(f)_a$ the function

$$(f)_a : x \longrightarrow f(x)e^{ax}.$$

Denote by $\mathcal{F}$ the usual Fourier transformation on $L^2(\mathbb{R})$ and set $\hat{f} = \mathcal{F}f$ for $f \in L^2(\mathbb{R})$. For a function $f \in L^2(\mathbb{R}^+)$, we define $\tilde{f}$ as the Fourier transform of the function f extended by 0 on $\mathbb{R}^-$.

Our first result is the following.

Theorem 1.2. *Let $a \in \mathcal{I} = [-\alpha_1, \alpha_0]$ and let T be a Wiener-Hopf operator. There exists $h_a \in L^\infty(\mathbb{R})$ such that for every $f \in L^2_\omega(\mathbb{R}^+)$ satisfying $(f)_a \in L^2(\mathbb{R}^+)$, we have*

$$(1.2) \quad (Tf)_a = \mathbf{r}_+ \mathcal{F}^{-1}(\widetilde{h_a(f)_a})$$

and

$$\|h_a\|_\infty \leq C\|T\|,$$

where C is a constant independent of a . Moreover, if $\alpha_1 + \alpha_0 > 0$, the function h defined on $U = \{z \in \mathbb{C} : \operatorname{Im} z \in [-\alpha_1, \alpha_0]\}$ by $h(z) = h_{\operatorname{Im} z}(\operatorname{Re} z)$ is holomorphic on $\overset{\circ}{U}$.

Definition 1.3. The function h defined in Theorem 1.2 is called the symbol of T .

A weaker result than Theorem 1.2 has been proved in [12] where the representation (1.2) has been obtained only for functions $f \in C_c^\infty(\mathbb{R}^+)$ which is too restrictive for the applications to the spectral problems studied in Section 3 and Section 4. On the other hand, in the proof in [12] there is a gap in the approximation argument. Indeed, in the proof of Lemma 2 in [12] there is one argument from Lemma 6 in [11] which can be applied only if the function $\mathbb{R}^+ \ni t \longrightarrow R_t$ is continuous with respect

to the operator norm topology on the set of bounded operators on $L^2_\omega(\mathbb{R}^+)$. Guided by the approach in [15], in this work we prove a stronger version of the result of [12] applying other techniques based essentially on the spectral theory of semigroups. On the other hand, in many interesting cases as $\omega(x) = e^x$, $\omega(x) = e^{-x}$, we have $\alpha_0 + \alpha_1 = 0$, and the result of Theorem 1.2 is not satisfied since the symbol of T is defined only on the line $\operatorname{Im} z = \alpha_0$. To obtain more complete results we introduce the following class of operators.

Definition 1.4. Denote by $\mathcal{M}$ (resp. $\mathcal{W}$) the set of bounded operators on $L^2_\omega(\mathbb{R}^+)$ commuting with R_t , $\forall t > 0$ (resp. L_{-t} , $\forall t > 0$).

For operators in $\mathcal{M}$ or $\mathcal{W}$ we obtain a stronger version of Theorem 1.2.

Theorem 1.5. *Let T be a bounded operator commuting with $(R_t)_{t>0}$ (resp. $(L_{-t})_{t>0}$). Let $a \in J =]-\infty, \alpha_0]$ (resp. $K = [-\alpha_1, +\infty[$). There exists $h_a \in L^\infty(\mathbb{R})$ such that for every $f \in L^2_\omega(\mathbb{R}^+)$ satisfying $(f)_a \in L^2(\mathbb{R}^+)$, we have*

$$(Tf)_a = \mathbf{r}_+ \mathcal{F}^{-1}(\widetilde{h_a(f)_a})$$

and

$$\|h_a\|_\infty \leq C\|T\|,$$

where C is a constant independent of a . Moreover, the function h defined on $\mathcal{O} = \{z \in \mathbb{C} : \operatorname{Im} z < \alpha_0\}$ (resp. $\mathcal{V} = \{z \in \mathbb{C} : \operatorname{Im} z > -\alpha_1\}$) by $h(z) = h_{\operatorname{Im} z}(\operatorname{Re} z)$ is holomorphic on $\mathcal{O}$ (resp. $\mathcal{V}$).

Our main spectral result is the following

Theorem 1.6. *We have*

$$(1.3) \quad (i) \quad \sigma(R_t) = \{z \in \mathbb{C}, |z| \leq e^{\alpha_0 t}\}, \forall t > 0.$$

$$(1.4) \quad (ii) \quad \sigma(L_{-t}) = \{z \in \mathbb{C}, |z| \leq e^{\alpha_1 t}\}, \forall t > 0.$$

(iii) *Let $T \in \mathcal{M}$ and let μ_T be the symbol of T . Then we have*

$$(1.5) \quad \overline{\mu_T(\mathcal{O})} \subset \sigma(T).$$

(iv) *Let $T \in \mathcal{W}$ and let μ_T be the symbol of T . Then we have*

$$(1.6) \quad \overline{\mu_T(\mathcal{V})} \subset \sigma(T).$$

It is important to note that for $T \in \mathcal{M}$ (resp. $\mathcal{W}$) and $\lambda \in \mathbb{C}$, if the resolvent $(T - \lambda I)^{-1}$ exists, then $(T - \lambda I)^{-1}$ is also in $\mathcal{M}$ (resp. $\mathcal{W}$). In general, if T is a Wiener-Hopf operator and $\lambda \in \mathbb{C}$, even if the resolvent $(T - \lambda I)^{-1}$ exists, $(T - \lambda I)^{-1}$ could be not a Wiener-Hopf operator. For more information about Wiener-Hopf operators the reader may consult [5] and [7].

The result in Theorem 1.6 cannot be obtained from a spectral calculus which is unknown and quite difficult to construct for the operators in $\mathcal{M}$ or $\mathcal{W}$. On the other hand, our analysis shows the importance of the existence of symbols, and this was our main motivation to establish Theorem 1.2 and Theorem 1.5.

The spectrum of the weighted right and left shifts on $l^2(\mathbb{R}^+)$, denoted respectively by $\mathbf{R}$ and $\mathbf{L}$, has been studied in [16]. In particular, it was shown that

$$(1.7) \quad \sigma(\mathbf{R}) = \sigma(\mathbf{L}) = \{z \in \mathbb{C}, |z| \leq \rho(\mathbf{R})\}.$$

In this special case the operators $\mathbf{R}$ and $\mathbf{L}$ are adjoint, while this property in general is not true for R and L_{-1} .

The equalities (1.3), (1.4) are the analogue of (1.7) in $L_\omega^2(\mathbb{R}^+)$; however, our proof is quite different from that in [16] and we use essentially Theorem 1.5. Moreover, these results agree with the spectrum of composition operator studied in [17] and the circular symmetry about 0. In the standard case $\omega = 1$, the spectral results (1.3), (1.4) are well known (see, for example Chapter V, [4]). Their proof in this special case is based on the fact that the spectrum of the generator A of $(R_t)_{t \geq 0}$ is in $\{z \in \mathbb{C}, \operatorname{Re} z \leq 0\}$ and the spectral mapping theorem for semigroups yields $\sigma(R_t) = \{z \in \mathbb{C}, |z| \leq 1\}$. Notice also that in this case we have

$$s(A) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\} = \alpha_0 = 0,$$

so the spectral bound $s(A)$ of A is equal to the ground order and there is no *spectral gap*. In the general setting we deal with it is quite difficult to describe the spectrum of A . Consequently, we cannot obtain (1.3) from the spectrum of A and our techniques are not based on $\sigma(A)$. Moreover, if for the semigroup $(R_t)_{t \geq 0}$ on $L_\omega^2(\mathbb{R}^+)$ we can apply the spectral mapping theorem, since R_t preserves positive functions (see [19], [20]), in general this is not true for other Hilbert spaces of functions and we could have a spectral gap $s(A) < \alpha_0$. This shows the importance of our approach which works also for more general Hilbert spaces H of functions (see the conditions on H listed below). To the best of our knowledge it seems that Theorem 1.6 is the first result in the literature giving a complete characterization of $\sigma(R_t)$ and $\sigma(L_{-t})$ on the spaces $L_\omega^2(\mathbb{R}^+)$. On the other hand, for the weighted two-sided shift $\mathbf{S}$ in $L_\omega^2(\mathbb{R})$ a similar result has been established in [15] saying that

$$\sigma(\mathbf{S}) = \{z \in \mathbb{C} : \frac{1}{\rho(\mathbf{S}^{-1})} \leq |z| \leq \rho(\mathbf{S})\}.$$

Following the arguments in [14], the results of this paper may be extended to a larger setup. Indeed, instead of $L_\omega^2(\mathbb{R}^+)$ we may consider a Hilbert space H of functions on $\mathbb{R}^+$ satisfying the following conditions:

(H1) $C_c(\mathbb{R}^+) \subset H \subset L_{loc}^1(\mathbb{R}^+)$, with continuous inclusions, and $C_c(\mathbb{R}^+)$ is dense in H .

(H2) For every $t \in \mathbb{R}$, we have $R_t(H) \subset H$ (resp. $L_{-t}(H) \subset H$) and $\sup_{t \in K} \|R_t\| < +\infty$ (resp. $\sup_{t \in K} \|L_{-t}\| < +\infty$), for every compact set $K \subset \mathbb{R}^+$.

(H3) For every $\alpha \in \mathbb{R}$, let E_α be the operator defined by

$$E_\alpha : H \ni f \longrightarrow \left(\mathbb{R} \ni x \longrightarrow f(x)e^{i\alpha x} \right).$$

We have $E_\alpha(H) \subset H$ and, moreover, $\sup_{\alpha \in \mathbb{R}} \|E_\alpha\| < +\infty$.

(H4) There exist $C_1 > 0$ and $a_1 \geq 0$ such that $\|R_t\| \leq C_1 e^{a_1|t|}$, $\forall t \in \mathbb{R}^+$.

(H5) There exist $C_2 > 0$ and $a_2 \geq 0$ such that $\|L_{-t}\| \leq C_2 e^{a_2|t|}$, $\forall t \in \mathbb{R}^+$.

Taking into account (H3), without loss of generality we may consider that in H we have $\|fe^{i\alpha \cdot}\| = \|f\|$. For the simplicity of the exposition we deal with the case $H = L_\omega^2(\mathbb{R}^+)$ and the reader may consult [14] for the changes necessary to cover the more general setup.

2. PROOF OF THEOREM 1.2

Denote by A the generator of the semigroup $(R_t)_{t \geq 0}$. By using the arguments based on the spectral results for semigroups (see [6], [8]), we will prove the following.

Lemma 2.1. *Let λ be such that $e^\lambda \in \sigma(R)$ and $\operatorname{Re} \lambda = \alpha_0$. Then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k \in \mathbb{N}}$ of functions of H such that*

$$(2.1) \quad \lim_{k \rightarrow \infty} \| (e^{tA} - e^{(\lambda + 2\pi i n_k)t}) f_{m_k} \| = 0, \quad \forall t \in \mathbb{R}^+, \quad \|f_{m_k}\| = 1, \quad \forall k \in \mathbb{N}.$$

Proof. We have to deal with two cases: (i) $\lambda \in \sigma(A)$, (ii) $\lambda \notin \sigma(A)$. In case (i) λ is in the approximative point spectrum of A . This follows from the fact that for any $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > \alpha_0$ we have $\mu \notin \sigma(A)$, since $s(A) \leq \alpha_0$. Let μ_m be a sequence such that $\mu_m \rightarrow \lambda$, $\operatorname{Re} \mu_m > \alpha_0$. Then $\|(\mu_m I - A)^{-1}\| \geq (\operatorname{dist}(\mu_m, \operatorname{spec}(A)))^{-1}$, hence $\|(\mu_m I - A)^{-1}\| \rightarrow \infty$. Applying the uniform boundedness principle and passing to a subsequence μ_{m_k} , we may find $f \in H$ such that

$$\lim_{k \rightarrow \infty} \|(\mu_{m_k} I - A)^{-1} f\| = \infty.$$

Introduce $f_{m_k} \in D(A)$ defined by

$$f_{m_k} = \frac{(\mu_{m_k} I - A)^{-1} f}{\|(\mu_{m_k} I - A)^{-1} f\|}.$$

The identity

$$(\lambda - A)f_{m_k} = (\lambda - \mu_{m_k})f_{m_k} + (\mu_{m_k} - A)f_{m_k}$$

implies that $(\lambda - A)f_{m_k} \rightarrow 0$ as $k \rightarrow \infty$. Then the equality

$$(e^{tA} - e^{t\lambda})f_{m_k} = \left(\int_0^t e^{\lambda(t-s)} e^{sA} ds \right) (A - \lambda)f_{m_k}$$

yields (2.1), where we take $n_k = 0$.

To deal with case (ii), we repeat the argument in [14] and for the sake of completeness we present the details. We have $e^\lambda \in \sigma(e^A) \setminus e^{\sigma(A)}$. Applying the results for the spectrum of a semigroup in Hilbert space (see [6], [8]), we conclude that there exists a sequence of integers (n_k) such that $|n_k| \rightarrow \infty$ and

$$\|(A - (\lambda + 2\pi i n_k)I)^{-1}\| \geq k, \quad \forall k \in \mathbb{N}.$$

We choose a sequence $(g_{m_k}) \in H$, $\|g_{m_k}\| = 1$ so that

$$\|(A - (\lambda + 2\pi i n_k)I)^{-1} g_{m_k}\| \geq k/2, \quad \forall k \in \mathbb{N}$$

and define

$$f_{m_k} = \frac{(A - (\lambda + 2\pi i n_k)I)^{-1} g_{m_k}}{\|(A - (\lambda + 2\pi i n_k)I)^{-1} g_{m_k}\|}.$$

Next we have

$$(e^{tA} - e^{(\lambda + 2\pi i n_k)t})f_{m_k} = \left(\int_0^t e^{(\lambda + 2\pi i n_k)(t-s)} e^{sA} ds \right) (A - (2\pi i n_k + \lambda)I)f_{m_k}$$

and we deduce (2.1). □

Lemma 2.2. *Let λ be such that $e^\lambda \in \sigma(R)$ and $\operatorname{Re} \lambda = \alpha_0$. Then, there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k \in \mathbb{N}}$ of functions of H such that for all $t \in \mathbb{R}$,*

$$(2.2) \quad \lim_{k \rightarrow \infty} \left\| \left(r_+ S_t \ell_0 - e^{(\lambda + 2\pi i n_k)t} \right) f_{m_k} \right\| = 0, \quad \|f_{m_k}\| = 1, \quad \forall k \in \mathbb{N}.$$

Proof. Clearly, for $t \geq 0$ we get (2.2) by (2.1). Moreover, we have

$$\begin{aligned} \|(L_{-t} - e^{-(\lambda + 2\pi i n_k)t})f_{m_k}\| &= \|(L_{-t} - e^{-(\lambda + 2\pi i n_k)t} L_{-t} R_t)f_{m_k}\| \\ &\leq \|L_{-t}\| \|e^{-(\lambda + 2\pi i n_k)t}\| \left\| \left(e^{(\lambda + 2\pi i n_k)t} - R_t \right) f_{m_k} \right\|, \quad \forall t \in \mathbb{R}^+. \end{aligned}$$

Thus

$$\lim_{k \rightarrow \infty} \|(L_{-t} - e^{-(\lambda + 2\pi i n_k)t})f_{m_k}\| = 0,$$

and this completes the proof of (2.2). $\square$

Recall that for $\phi \in C_c^\infty(\mathbb{R})$, T_ϕ is the operator on $L_\omega^2(\mathbb{R}^+)$ given by

$$T_\phi(f) = r_+(\phi * \ell_0 f), \quad \forall f \in L_\omega^2(\mathbb{R}^+).$$

Lemma 2.3. *For all $\phi \in C_c^\infty(\mathbb{R})$ and λ such that $e^\lambda \in \sigma(R)$ with $\operatorname{Re} \lambda = \alpha_0$, we have*

$$(2.3) \quad |\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|, \quad \forall a \in \mathbb{R}.$$

Proof. Let $\lambda \in \mathbb{C}$ be such that $e^\lambda \in \sigma(R)$ with $\operatorname{Re} \lambda = \alpha_0$ and let $(f_{m_k})_{k \in \mathbb{N}}$ be a sequence satisfying (2.2). Fix $\phi \in C_c^\infty(\mathbb{R})$ and consider

$$\begin{aligned} |\hat{\phi}(i\lambda + a)| &= \left| \int_{\mathbb{R}} \langle \phi(t) e^{(\lambda - ia)t} f_{m_k}, f_{m_k} \rangle dt \right| \\ &\leq \left| \int_{\mathbb{R}} \langle \phi(t) \left(e^{(\lambda + i2\pi n_k)t} - r_+ S_t \ell_0 \right) e^{-i(a + 2\pi n_k)t} f_{m_k}, f_{m_k} \rangle dt \right| \\ &\quad + \left| \int_{\mathbb{R}} \langle \phi(t) r_+ S_t \ell_0 e^{-i(a + 2\pi n_k)t} f_{m_k}, f_{m_k} \rangle dt \right|. \end{aligned}$$

The first term on the right side of the last inequality goes to 0 as $k \rightarrow \infty$ since by Lemma 2.2, for every fixed t we have

$$\lim_{k \rightarrow +\infty} \|e^{-i(a + 2\pi n_k)t} (e^{(\lambda + i2\pi n_k)t} - r_+ S_t \ell_0) f_{m_k}\| = 0.$$

On the other hand,

$$\begin{aligned} I_k &= \left| \int_{\mathbb{R}} \langle \phi(t) r_+ S_t \ell_0 e^{-i(a + 2\pi n_k)t} f_{m_k}, f_{m_k} \rangle dt \right| \\ &= \left| \left\langle r_+ \left[\int_{\mathbb{R}} \phi(t) e^{-i(a + 2\pi n_k)t} \ell_0 f_{m_k}(\cdot - t) dt \right], f_{m_k}(\cdot) \right\rangle \right| \\ &= \left| \left\langle r_+ \int_{\mathbb{R}} \phi(\cdot - y) e^{i(a + 2\pi n_k)y} \ell_0 f_{m_k}(y) dy, e^{i(a + 2\pi n_k)\cdot} f_{m_k}(\cdot) \right\rangle \right| \\ &= \left| \left\langle \left(T_\phi(e^{i(a + 2\pi n_k)\cdot} f_{m_k}) \right), e^{i(a + 2\pi n_k)\cdot} f_{m_k}(\cdot) \right\rangle \right| \end{aligned}$$

and $I_k \leq \|T_\phi\|$. Consequently, we deduce that $|\hat{\phi}(i\lambda + a)| \leq \|T_\phi\|$. $\square$

Notice that property (2.3) implies that

$$|\hat{\phi}(\lambda)| \leq \|T_\phi\|, \quad \forall \lambda \in \mathbb{C}, \text{ provided } \operatorname{Im} \lambda = \alpha_0.$$

Lemma 2.4. *Let $\phi \in C_c^\infty(\mathbb{R})$ and let λ be such that $e^{-\bar{\lambda}} \in \sigma((L_{-1})^*)$ with $\operatorname{Re} \lambda = -\alpha_1$. Then we have*

$$(2.4) \quad |\hat{\phi}(i\lambda + a)| \leq \|(T_\phi)\|, \quad \forall a \in \mathbb{R}.$$

Proof. Consider the semigroup $(L_{-t})_{t \geq 0}^*$ and let B be its generator. We identify H and its dual space H' . So the semigroup $(L_{-t})^*, t \geq 0$, is acting on H .

Let $\lambda \in \mathbb{C}$ be such that $e^{-\bar{\lambda}} \in \sigma((L_{-1})^*)$ and $|e^{-\bar{\lambda}}| = \rho(L_{-1}) = \rho((L_{-1})^*) = e^{\alpha_1}$. Then, by the same argument as in Lemma 2.1, we prove that there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k \in \mathbb{N}}$ of functions of H such that for all $t \in \mathbb{R}^+$,

$$\lim_{k \rightarrow \infty} \|(e^{tB} - e^{(-\bar{\lambda} + i2\pi n_k)t})f_{m_k}\| = 0$$

and $\|f_{m_k}\| = 1$. Thus we deduce

$$\lim_{k \rightarrow \infty} \|(L_{-t})^* f_{m_k} - e^{(-\bar{\lambda} - i2\pi n_k)t} f_{m_k}\| = 0, \quad t \geq 0.$$

Since for $t \geq 0$ we have $L_{-t}R_t = I$, we get $(R_t)^*(L_{-t})^* = I$. Then, for $t \geq 0$ we get

$$\begin{aligned} & \|(R_t)^* f_{m_k} - e^{(\bar{\lambda} - i2\pi n_k)t} f_{m_k}\| \\ &= \|(R_t)^* f_{m_k} - e^{(\bar{\lambda} - i2\pi n_k)t} (R_t)^* (L_{-t})^* f_{m_k}\| \\ &\leq \|(R_t)^* \|e^{(\bar{\lambda} - i2\pi n_k)t} \| (e^{(-\bar{\lambda} - i2\pi n_k)t} f_{m_k} - (L_{-t})^* f_{m_k})\|. \end{aligned}$$

This implies that

$$(2.5) \quad \lim_{k \rightarrow \infty} \|((r_+ S_t \ell_0)^* - e^{(\bar{\lambda} - i2\pi n_k)t})f_{m_k}\| = 0, \quad \forall t \in \mathbb{R}.$$

We write

$$\begin{aligned} \hat{\phi}(i\lambda + a) &= \int_{\mathbb{R}} \langle \phi(t) e^{-i(a+2\pi n_k)t} f_{m_k}, e^{\bar{\lambda}t - 2i\pi n_k t} f_{m_k} \rangle dt \\ &= \int_{\mathbb{R}} \langle \phi(t) e^{-i(a+2\pi n_k)t} f_{m_k}, (e^{(\bar{\lambda} - i2\pi n_k)t} - (r_+ S_t \ell_0)^*) f_{m_k} \rangle dt \\ &\quad + \int_{\mathbb{R}} \langle \phi(t) e^{-i(a+2\pi n_k)t} (r_+ S_t \ell_0) f_{m_k}, f_{m_k} \rangle dt = J'_k + I'_k. \end{aligned}$$

From (2.5) we deduce that $J'_k \rightarrow 0$ as $k \rightarrow \infty$. For I'_k we apply the same argument as in the proof of Lemma 2.3 and we get $|\hat{\phi}(i\lambda)| \leq \|T_\phi\|$. $\square$

Lemma 2.5. *For every function $\phi \in C_c^\infty(\mathbb{R})$ and for $z \in U = \{z \in \mathbb{C}, \operatorname{Im} z \in [-\alpha_1, \alpha_0]\}$ we have*

$$|\hat{\phi}(z)| \leq \|T_\phi\|.$$

Proof. We will use the Phragmén-Lindelöf theorem and start by proving the estimations on the bounding lines. There exists $\alpha = e^{-iz} \in \sigma(R)$ such that $|\alpha| = e^{\operatorname{Im} z} = e^{\alpha_0}$. Following (2.3), we obtain

$$|\hat{\phi}(z)| \leq \|T_\phi\|,$$

for every z such that $\operatorname{Im} z = \alpha_0$. Next notice that $\rho(L_{-1}) = \rho((L_{-1})^*)$. So there exists $\beta = e^{-iz} = e^{-\overline{(-iz)}} \in \sigma((L_{-1})^*)$ such that $|\beta| = e^{\alpha_1}$ and

$$-\operatorname{Im} z = \ln |\beta| = \alpha_1.$$

Then taking into account (2.4), we get

$$|\hat{\phi}(z)| \leq \|T_\phi\|,$$

for every z such that $\operatorname{Im} z = -\alpha_1$. In the case $\alpha_1 + \alpha_0 = 0$ the result is obvious. So assume that $\alpha_0 + \alpha_1 > 0$. Since $\phi \in C_c^\infty(\mathbb{R})$ we have

$$|\hat{\phi}(z)| \leq C\|\phi\|_\infty e^{k|\operatorname{Im} z|} \leq K\|\phi\|_\infty, \quad \forall z \in U,$$

where $C > 0$, $k > 0$ and $K > 0$ are constants. An application of the Phragmén-Lindelöf theorem for the holomorphic function $\hat{\phi}$, yields

$$|\hat{\phi}(\alpha)| \leq \|T_\phi\|$$

for $\alpha \in \{z \in \mathbb{C} : \operatorname{Im} z \in [-\alpha_1, \alpha_0]\}$. $\square$

Notice that for multipliers in $L^p(\mathbb{R})$ studied in [9] some relations concerning the norm $\|T_\phi\|$ and ϕ hold. Combining the results in Lemma 2.3-2.5, we get

Lemma 2.6. *For every $\phi \in C_c^\infty(\mathbb{R})$ and every $a \in [-\alpha_1, \alpha_0]$ we have*

$$|(\widehat{(\phi)_a})(x)| \leq \|T_\phi\|, \quad \forall x \in \mathbb{R}.$$

Proof of Theorem 1.2. Let T be a Wiener-Hopf operator. Let $a \in [-\alpha_1, \alpha_0]$. The proof follows the approach in [12]. Notice that in [12], we establish (1.2) for $f \in C_c^\infty(\mathbb{R}^+)$; what is new here is that we prove (1.2) for $f \in L_\omega^2(\mathbb{R}^+)$ such that $(f)_a \in L^2(\mathbb{R}^+)$. Following [12], there exists a sequence $(\phi_n)_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R})$ such that T is the limit of $(T_{\phi_n})_{n \in \mathbb{N}}$ with respect to the strong operator topology and we have $\|T_{\phi_n}\| \leq C\|T\|$, where C is a constant independent of n . According to Lemma 2.6, we have

$$(2.6) \quad |(\widehat{(\phi_n)_a})(x)| \leq \|T_{\phi_n}\| \leq C\|T\|, \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N},$$

and we replace $(\widehat{(\phi_n)_a})_{n \in \mathbb{N}}$ by a suitable subsequence, also denoted by $(\widehat{(\phi_n)_a})_{n \in \mathbb{N}}$, converging with respect to the weak topology $\sigma(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$ to a function $h_a \in L^\infty(\mathbb{R})$ such that $\|h_a\|_\infty \leq C\|T\|$. We have

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}} \left(\widehat{(\phi_n)_a}(x) - h_a(x) \right) g(x) dx = 0, \quad \forall g \in L^1(\mathbb{R}).$$

Fix $f \in L_\omega^2(\mathbb{R}^+)$ so that $(f)_a \in L^2(\mathbb{R}^+)$. Then we get

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}} \left(\widehat{(\phi_n)_a}(x) \widehat{(f)_a}(x) - h_a(x) \widehat{(f)_a}(x) \right) g(x) dx = 0,$$

for all $g \in L^2(\mathbb{R})$. We conclude that $\left(\widehat{(\phi_n)_a} \widehat{(f)_a} \right)_{n \in \mathbb{N}}$ converges weakly in $L^2(\mathbb{R})$ to $h_a \widehat{(f)_a}$.

On the other hand, we have

$$(T_{\phi_n} f)_a = \mathbf{r}_+((\phi_n)_a * \ell_0(f)_a) = \mathbf{r}_+ \mathcal{F}^{-1}(\widehat{(\phi_n)_a} \widehat{(f)_a}),$$

and thus $(T_{\phi_n} f)_a$ converges weakly in $L^2(\mathbb{R}^+)$ to $\mathbf{r}_+ \mathcal{F}^{-1}(h_a \widehat{(f)_a})$.

For $g \in C_c^\infty(\mathbb{R})$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^+} \left| (T_{\phi_n} f)_a(x) - (Tf)_a(x) \right| |g(x)| dx \\ & \leq C_{a,g} \|T_{\phi_n} f - Tf\|, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $C_{a,g}$ is a constant depending only on g and a . Since $(T_{\phi_n}f)_{n \in \mathbb{N}}$ converges to Tf in $L^2_\omega(\mathbb{R}^+)$, we get

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^+} (T_{\phi_n} f)_a(x) g(x) dx = \int_{\mathbb{R}^+} (Tf)_a(x) g(x) dx, \quad \forall g \in C_c^\infty(\mathbb{R}).$$

Thus we deduce that $(Tf)_a = \mathbf{r}_+ \mathcal{F}^{-1}(h_a(\widetilde{f})_a)$. The symbol h is holomorphic on $\overset{\circ}{U}$ following the same arguments as in [12]. $\square$

3. PRELIMINARY SPECTRAL RESULT

As a first step to our spectral analysis in this section we prove the following:

Proposition 3.1. *Let $T \in \mathcal{M}$ and suppose that the symbol μ of T is continuous on U . Then $\mu(U) \subset \sigma(T)$.*

Proof of Proposition 3.1. Let T be a bounded operator on H commuting with R_t , $t \geq 0$, or L_{-t} , $t \geq 0$. For $a \in [-\alpha_1, \alpha_0]$, we have

$$(Tf)_a = \mathbf{r}_+ \mathcal{F}^{-1}(\mu_a(\widetilde{f})_a), \quad \forall f \in L^2_\omega(\mathbb{R}^+),$$

where $\mu_a \in L^\infty(\mathbb{R})$, provided $(f)_a \in L^2(\mathbb{R}^+)$. Suppose that $\lambda \notin \sigma(T)$. Then, it follows easily that the resolvent $(T - \lambda I)^{-1}$ also commutes with $(R_t)_{t \in \mathbb{R}^+}$ or $(L_{-t})_{t \in \mathbb{R}^+}$. Consequently, $(T - \lambda I)^{-1}$ is a Wiener-Hopf operator and for $a \in [-\alpha_1, \alpha_0]$ there exists a function $h_a \in L^\infty(\mathbb{R})$ such that

$$((T - \lambda I)^{-1}g)_a = \mathbf{r}_+ \mathcal{F}^{-1}(h_a(\widetilde{g})_a),$$

for $g \in L^2_\omega(\mathbb{R}^+)$ such that $(g)_a \in L^2(\mathbb{R}^+)$. If f is such that $(f)_a \in L^2(\mathbb{R}^+)$, set $g = (T - \lambda I)f$. Then following Theorem 1.2, we deduce that $(Tf)_a \in L^2(\mathbb{R}^+)$ and $(g)_a = ((T - \lambda I)f)_a \in L^2(\mathbb{R}^+)$. Thus applying Theorem 1.2 once more, we get

$$\begin{aligned} ((T - \lambda I)^{-1}(T - \lambda I)f)_a &= \mathbf{r}_+ \mathcal{F}^{-1}(h_a(\widetilde{(T - \lambda I)f})_a) \\ &= \mathbf{r}_+ \mathcal{F}^{-1}\left(h_a \mathcal{F} \mathbf{r}_+ [\mathcal{F}^{-1}((\mu_a - \lambda)(\widetilde{f})_a)]\right). \end{aligned}$$

We have

$$\|(f)_a\|_{L^2} \leq \|h_a \mathcal{F} \mathbf{r}_+ \mathcal{F}^{-1}((\mu_a - \lambda)(\widetilde{f})_a)\|_{L^2} \leq \|h_a\|_\infty \|\mathcal{F} \mathbf{r}_+ \mathcal{F}^{-1}((\mu_a - \lambda)(\widetilde{f})_a)\|_{L^2}$$

and we deduce

$$(3.1) \quad \|(\widetilde{f})_a\|_{L^2} \leq C \|(\mu_a - \lambda)(\widetilde{f})_a\|_{L^2}$$

for all $f \in L^2_\omega(\mathbb{R}^+)$ such that $(f)_a \in L^2(\mathbb{R}^+)$. Let $\lambda = \mu_a(\eta_0) = \mu(\eta_0 + ia) \in \mu(U)$ for $a \in [-\ln \rho(L_{-1}), \ln \rho(R)]$ and some $\eta_0 \in \mathbb{R}$. Since the symbol μ of T is continuous, the function $\mu_a(\eta) = \mu(\eta + ia)$ is continuous on $\mathbb{R}$. We will construct a function $f(x) = F(x)e^{-ax}$ with $\text{supp}(F) \subset \mathbb{R}^+$ for which (3.1) is not fulfilled. Consider

$$g(t) = e^{-\frac{b^2(t-t_0)^2}{2}} e^{i(t-t_0)\eta_0}, \quad b > 0, t_0 > 1,$$

with Fourier transform

$$\hat{g}(\xi) = \frac{1}{b} e^{-\frac{(\xi - \eta_0)^2}{2b^2}} e^{-it_0\xi}.$$

Fix a small $0 < \epsilon < \frac{1}{2}C^{-2}$, where C is the constant in (3.1) and let $\delta > 0$ be fixed so that $|\mu_a(\xi) - \lambda| \leq \sqrt{\epsilon}$ for $\xi \in V = \{\xi \in \mathbb{R} : |\xi - \eta_0| \leq \delta\}$. Moreover, assume that

$$|\mu_a(\xi) - \lambda|^2 \leq C_1, \quad \text{a.e. } \xi \in \mathbb{R}.$$

We have for $0 < b \leq 1$ small enough,

$$\begin{aligned} \int_{\mathbb{R} \setminus V} |\hat{g}(\xi)|^2 d\xi &\leq \frac{1}{b^2} \int_{|\xi - \eta_0| \geq \delta} e^{-\frac{(\xi - \eta_0)^2}{2b^2}} d\xi \\ &\leq e^{-\frac{\delta^2}{4b^2}} \frac{1}{b^2} \int_{|\xi - \eta_0| \geq \delta} e^{-\frac{(\xi - \eta_0)^2}{4b^2}} d\xi \leq C_0 b^{-1} e^{-\frac{\delta^2}{4b^2}} \leq \epsilon, \end{aligned}$$

with $C_0 > 0$ independent of $b > 0$. We fix $b > 0$ with the above property and we choose a function $\varphi \in C_c^\infty(\mathbb{R}^+)$ such that $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ for $1 \leq t \leq 2t_0 - 1$, and $\varphi(t) = 0$ for $t \leq 1/2$ and for $t \geq 2t_0 - 1/2$. We suppose that $|\varphi^{(k)}(t)| \leq c_1$, $k = 1, 2, \forall t \in \mathbb{R}$. Set $G(t) = (\varphi(t) - 1)g(t)$. We will show that

$$(3.2) \quad |(1 + \xi^2)\hat{G}(\xi)| \leq \sqrt{\frac{C_2}{4\pi}} \epsilon$$

for t_0 large enough with $C_2 > 0$ independent of t_0 . On the support of $(\varphi - 1)$ we have $|t - t_0| > t_0 - 1$, and integrating by parts in $\int_{\mathbb{R}} (1 + \xi^2)G(t)e^{-it\xi} dt$ we must estimate the integral

$$\begin{aligned} &\int_{|t - t_0| \geq t_0 - 1} e^{-\frac{b^2(t - t_0)^2}{2}} (1 + |t - t_0| + (t - t_0)^2) dt \\ &\leq \left(\int_{-\infty}^{1 - t_0} (1 + |y| + y^2) e^{-b^2 y^2 / 2} dy + \int_{t_0 - 1}^{\infty} (1 + y + y^2) e^{-b^2 y^2 / 2} dy \right). \end{aligned}$$

Choosing t_0 large enough we arrange (3.2).

We set $F = \varphi g \in C_c^\infty(\mathbb{R}^+)$ and we obtain

$$\begin{aligned} \int_{\mathbb{R} \setminus V} |\tilde{F}(\xi)|^2 d\xi &\leq 2 \int_{\mathbb{R} \setminus V} |\hat{g}(\xi)|^2 d\xi + 2 \int_{\mathbb{R} \setminus V} |\hat{G}(\xi)|^2 d\xi \\ &\leq 2\epsilon + \frac{C_2 \epsilon}{2\pi} \int_{\mathbb{R}} (1 + \xi^2)^{-2} d\xi \leq (2 + C_2)\epsilon. \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}} |(\mu_a(\xi) - \lambda)\tilde{F}(\xi)|^2 d\xi &\leq \int_{\mathbb{R} \setminus V} |(\mu_a(\xi) - \lambda)\tilde{F}(\xi)|^2 d\xi + \int_V |(\mu_a(\xi) - \lambda)\tilde{F}(\xi)|^2 d\xi \\ &\leq C_1(2 + C_2)\epsilon + (2\pi)^2 \|F\|_{L^2}^2 \epsilon. \end{aligned}$$

Now assume (3.1) fulfilled. Therefore

$$(2\pi)^2 \|F\|_{L^2}^2 \leq C^2 \|(\mu_a(\xi) - \lambda)\hat{F}(\xi)\|_{L^2}^2 \leq C^2 C_1(2 + C_2)\epsilon + (2\pi C)^2 \|F\|_{L^2}^2 \epsilon,$$

and since $C^2 \epsilon < \frac{1}{2}$, we conclude that

$$\|F\|_{L^2}^2 \leq \frac{C^2 C_1}{2\pi^2} (2 + C_2)\epsilon.$$

On the other hand,

$$\|F\|_{L^2}^2 \geq \frac{1}{2} \|g\|_{L^2}^2 - \|(\varphi - 1)g\|_{L^2}^2 \geq \frac{1}{2} \|g\|_{L^2}^2 - (2\pi)^{-2} \frac{C_2}{2} \epsilon$$

and

$$\int_{\mathbb{R}} |g(t)|^2 dt \geq \int_{|t - t_0| \leq \frac{1}{b}} e^{-b^2(t - t_0)^2} dt \geq \frac{2e^{-1}}{b} \geq 2e^{-1}.$$

For small ϵ we obtain a contradiction, since C_2 is independent of ϵ . This completes the proof. $\square$

4. SPECTRA OF $(R_t)_{t \in \mathbb{R}^+}$, $(L_{-t})_{t \in \mathbb{R}^+}$ AND BOUNDED OPERATORS COMMUTING WITH AT LEAST ONE OF THESE SEMIGROUPS

Observing that the symbol of R_t is $z \longrightarrow e^{-itz}$, an application of Proposition 3.1 to the operator R_t yields

$$(4.1) \quad \{z \in \mathbb{C}, e^{-\alpha_1 t} \leq |z| \leq e^{\alpha_0 t}\} \subset \sigma(R_t).$$

This inclusion describes only a part of the spectrum of R_t . We will show that in our general setting we have (1.3). To prove this, for $t > 0$ assume that $z \in \mathbb{C}$ is such that $0 < |z| < e^{-\alpha_1 t}$. Let $g \in H$ be a function such that $g(x) = 0$ for $x \geq t$ and $g \neq 0$. If the operator $(zI - R_t)$ is surjective on H , then there exists $f \neq 0$ such that $(z - R_t)f = g$. This implies $L_{-t}g = 0$ and hence

$$\left(L_{-t} - \frac{1}{z}I\right)f = 0,$$

which is a contradiction. So every such z is in the spectrum of R_t and we obtain (1.3). Next, it is easy to see that in our setup for the approximative point spectrum $\Pi(R_t)$ of R_t we have the inclusion

$$(4.2) \quad \Pi(R_t) \subset \{z \in \mathbb{C} : e^{-\alpha_1 t} \leq |z| \leq e^{\alpha_0 t}\}.$$

Indeed, for $z \neq 0$, we have the equality

$$R_{-t} - \frac{1}{z}I = \frac{1}{z}R_{-t}(zI - R_t).$$

If for $z \in \mathbb{C}$ with $0 < |z| < e^{-\alpha_1 t}$ there exists a sequence (f_n) such that $\|f_n\| = 1$ and $\|(zI - R_t)f_n\| \rightarrow 0$ as $n \rightarrow \infty$, then

$$\left(L_{-t} - \frac{1}{z}I\right)f_n \rightarrow 0, \quad n \rightarrow \infty,$$

and this leads to $\frac{1}{z} \in \sigma(L_{-t})$ which is a contradiction. Next, if $0 \in \Pi(R_t)$, there exists a sequence $g_n \in H$ such that $R_t g_n \rightarrow 0$, $\|g_n\| = 1$. Then $g_n = L_{-t} R_t g_n$ and we obtain a contradiction.

Since the symbol of L_{-t} is $z \longrightarrow e^{itz}$, by applying Proposition 3.1 we obtain

$$\{z \in \mathbb{C} : e^{-\alpha_0 t} \leq |z| \leq e^{\alpha_1 t}\} \subset \sigma(L_{-t}).$$

Passing to the proof of (1.4), notice that $R_t^*(L_{-t})^* = I$. Then for $0 < |z| < e^{-\alpha_0 t}$ we have

$$(4.3) \quad z\left(\frac{1}{z}I - R_t^*\right) = R_t^*\left((L_{-t})^* - zI\right).$$

It is clear that $0 \in \sigma_r(R_t)$, where $\sigma_r(R_t)$ denotes the residual spectrum of R_t . In fact, if $0 \notin \sigma_r(R_t)$, then 0 is in the approximative point spectrum of R_t and this contradicts (4.2). Since $0 \in \sigma_r(R_t)$, we deduce that 0 is an eigenvalue of R_t^* . Let $R_t^*g = 0$, $g \neq 0$. Assume that $(L_{-t})^* - zI$ is surjective. Therefore, there exists $f \neq 0$ so that $((L_{-t})^* - z)f = g$ and (4.3) yields $(\frac{1}{z} - R_t^*)f = 0$. Consequently, $\frac{1}{|z|} \leq \rho(R_t^*) = \rho(R_t) = e^{\alpha_0 t}$ and we obtain a contradiction. Thus we conclude that $z \in \sigma((L_{-t})^*)$, hence $\bar{z} \in \sigma(L_{-t})$, and the proof of (1.4) is complete.

To study the operators commuting with $(R_t)_{t \in \mathbb{R}^+}$, we need the following:

Lemma 4.1. *Let $\phi \in C_c^\infty(\mathbb{R})$. The operator T_ϕ commutes with R_t , $\forall t > 0$, if and only if the support of ϕ is in $\overline{\mathbb{R}^+}$.*

Proof. First, if $\psi \in L_\omega^2(\mathbb{R}^+)$ has compact support in $\overline{\mathbb{R}^+}$, it is easy to see that T_ψ commutes with R_t , $t \geq 0$. Now consider $\phi \in C_c^\infty(\mathbb{R})$ and suppose that T_ϕ commutes with R_t , $t \geq 0$. We write $\phi = \phi\chi_{\mathbb{R}^-} + \phi\chi_{\overline{\mathbb{R}^+}}$. If T_ϕ commutes with R_t , $t \geq 0$, then the operator $T_{\phi\chi_{\mathbb{R}^-}}$ commutes too. Let the function $\psi = \phi\chi_{\mathbb{R}^-}$ have support in $[-a, 0]$ with $a > 0$. Setting $f = \chi_{[0, a]}$, we get $S_a f = \chi_{[a, 2a]}$. For $x \geq 0$ we have

$$T_\psi R_a r_+ f(x) = \int_{-a}^0 \psi(t) \chi_{\{a \leq x-t \leq 2a\}} dt = \int_{\max(-a, -2a+x)}^{\min(x-a, 0)} \psi(t) dt.$$

Since $T_\psi R_a = R_a T_\psi$, we deduce that $\int_{-a}^{x-a} \psi(t) dt = 0$, $\forall x \in [0, a]$. This implies that $\psi(t) = 0$, for $t \in [-a, 0]$ and $\text{supp}(\phi) \subset \overline{\mathbb{R}^+}$. $\square$

Lemma 4.2. *Let λ be such that $e^\lambda \in \sigma(R)$. Then there exists a sequence $(n_k)_{k \in \mathbb{N}}$ of integers and a sequence $(f_{m_k})_{k \in \mathbb{N}}$ of functions of H such that*

$$(4.4) \quad \lim_{k \rightarrow \infty} \left\langle \left(R_t - e^{(\lambda + 2\pi i n_k)t} \right) f_{m_k}, f_{m_k} \right\rangle = 0, \quad \forall t \in \mathbb{R}^+, \quad \|f_{m_k}\| = 1, \quad \forall k \in \mathbb{N}.$$

Proof. If $\lambda \notin \sigma(A)$, we repeat the argument of the proof of Lemma 2.1. Denote by $\sigma_r(A)$ the residual spectrum of A . If $\lambda \in \sigma(A) \setminus \sigma_r(A)$, we deduce that λ is in the approximative point spectrum of A and we can apply the argument of Lemma 2.1. Finally, if $\lambda \in \sigma_r(A)$, then there exists $f \in H$ such that $A^* f = \bar{\lambda} f$ and $\|f\| = 1$. We set $f_{m_k} = f$, $n_k = 0$, for $k \in \mathbb{N}$ and we use the fact that $(R_t)^* f = e^{\bar{\lambda} t} f$. $\square$

Lemma 4.3. *For all $\phi \in C_c^\infty(\overline{\mathbb{R}^+})$ and λ such that $e^\lambda \in \sigma(R)$, we have*

$$(4.5) \quad |\hat{\phi}(i\lambda)| \leq \|T_\phi\|.$$

The proof is based on the equality

$$\begin{aligned} \hat{\phi}(i\lambda) &= \int_{\mathbb{R}^+} \phi(t) e^{\lambda t} dt = \int_{\mathbb{R}^+} \langle \phi(t) e^{(\lambda + 2\pi i n_k)t} f_{m_k}, e^{2\pi i n_k t} f_{m_k} \rangle dt \\ &= \int_{\mathbb{R}^+} \langle \phi(t) (e^{(\lambda + 2\pi i n_k)t} I - R_t) f_{m_k}, e^{2\pi i n_k t} f_{m_k} \rangle dt + \int_{\mathbb{R}^+} \langle \phi(t) R_t f_{m_k}, e^{2\pi i n_k t} f_{m_k} \rangle dt. \end{aligned}$$

We apply Lemma 4.2 and we repeat the argument of the proof of Lemma 2.3. Notice that here the integration is over $\mathbb{R}^+$ and we do not need to examine the integral for $t < 0$.

Following [12], the operator T is a limit of a sequence of operators T_{ϕ_n} , where $\phi_n \in C_c^\infty(\mathbb{R})$ and $\|T_{\phi_n}\| \leq C\|T\|$. The sequence $(T_{\phi_n})_{n \geq 0}$ has been constructed in [12], and it follows from its construction that if T commutes with R_t , $t > 0$, then T_{ϕ_n} has the same property for all $n \in \mathbb{N}$. Therefore, Lemma 4.1 implies that $\phi_n \in C_c^\infty(\overline{\mathbb{R}^+})$, and to obtain Theorem 1.5 for bounded operators commuting with $(R_t)_{t > 0}$, we apply Lemma 4.3 and the same arguments as in the proof of Theorem 1.2. Finally, applying Theorem 1.5 and the arguments of the proof of Proposition 3.1, we establish (1.5), and this completes the proof of (iii) in Theorem 1.6. Next we prove the following.

Lemma 4.4. *Let $\phi \in C_c^\infty(\mathbb{R})$. Then T_ϕ commutes with L_{-t} , $\forall t > 0$, if and only if $\text{supp}(\phi) \subset \overline{\mathbb{R}^-}$.*

The proof of Lemma 4.4 is essentially the same as that of Lemma 4.1. By using Lemma 4.4, we obtain an analogue of Lemma 4.3 and Theorem 1.5 for bounded operators commuting with $(L_{-t})_{t>0}$, and applying these results we establish (iv) in Theorem 1.6.

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