PRODUCTS OF TOEPLITZ OPERATORS ON THE FOCK SPACE

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ABSTRACT. Let f and g be functions, not identically zero, in the Fock space F_α^2 of $\mathbb{C}^n$. We show that the product $T_f T_{\bar{g}}$ of Toeplitz operators on F_α^2 is bounded if and only if $f(z) = e^{q(z)}$ and $g(z) = ce^{-q(z)}$, where c is a nonzero constant and q is a linear polynomial.

1. INTRODUCTION

Let $\mathbb{C}^n$ be the complex n -space. For points $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$ we write

$$z \cdot \bar{w} = \sum_{j=1}^n z_j \bar{w}_j, \quad |z| = \sqrt{z \cdot \bar{z}}.$$

Let dv be the ordinary volume measure on $\mathbb{C}^n$. For any positive parameter α we consider the Gaussian measure

$$d\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z).$$

The Fock space F_α^2 is the closed subspace of entire functions in $L^2(\mathbb{C}^n, d\lambda_\alpha)$. The orthogonal projection $P : L^2(\mathbb{C}^n, d\lambda_\alpha) \rightarrow F_\alpha^2$ is given by

$$Pf(z) = \int_{\mathbb{C}^n} K(z, w) f(w) d\lambda_\alpha(w),$$

where $K(z, w) = e^{\alpha z \cdot \bar{w}}$ is the reproducing kernel of F_α^2 .

We say that f satisfies Condition (G) if the function $z \mapsto f(z)e^{\alpha z \cdot \bar{w}}$ belongs to $L^1(\mathbb{C}^n, d\lambda_\alpha)$ for every $w \in \mathbb{C}^n$. Equivalently, f satisfies Condition (G) if every translate of f , $z \mapsto f(z + w)$, belongs to $L^1(\mathbb{C}^n, d\lambda_\alpha)$. If $f \in F_\alpha^2$, then there exists a constant $C > 0$ such that

$$|f(z)| \leq Ce^{\frac{\alpha}{2}|z|^2}, \quad z \in \mathbb{C}^n.$$

This clearly implies that f satisfies Condition (G).

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If f satisfies Condition (G), we can define a linear operator T_f on F_α^2 by $T_f h = P(fh)$, where

$$h(z) = \sum_{k=1}^N c_k K(z, w_k)$$

is any finite linear combination of kernel functions. It is easy to see that the set of all finite linear combinations of kernel functions is dense in F_α^2 . Here $P(fh)$ is to be interpreted as the following integral:

$$T_f h(z) = \int_{\mathbb{C}^n} f(w) h(w) e^{\alpha z \cdot \bar{w}} d\lambda_\alpha(w), \quad z \in \mathbb{C}^n.$$

Therefore, for h in a dense subset of F_α^2 , $T_f h$ is a well-defined entire function (not necessarily in F_α^2 though).

The purpose of this paper is to study the Toeplitz product $T_f T_{\bar{g}}$, where f and g are functions in F_α^2 . Such a product is well defined on the set of finite linear combinations of kernel functions. Our main concern is the following: what conditions on f and g will ensure that the Toeplitz product $T_f T_{\bar{g}}$ extends to a bounded (or compact) operator on F_α^2 ?

This problem was first raised by Sarason in [5] in the context of Hardy and Bergman spaces. It was partially solved for Toeplitz operators on the Hardy space of the unit circle in [9], on the Bergman space of the unit disk in [6], on the Bergman space of the polydisk in [7], and on the Bergman space of the unit ball in [4, 8]. The following condition appears in all these cases:

$$\sup_{z \in \Omega} \widetilde{|f|^{2+\varepsilon}}(z) \widetilde{|g|^{2+\varepsilon}}(z) < \infty,$$

where ε is a positive number greater than a certain “cut-off” and $\widetilde{f}$ denotes the Berezin transform of f . Note that in the Hardy space case, the Berezin transform is nothing but the classical Poisson transform.

Here we obtain a much more explicit characterization for $T_f T_{\bar{g}}$ to be bounded on the Fock space.

Main Theorem. *Let f and g be functions in F_α^2 , not identically zero. Then $T_f T_{\bar{g}}$ is bounded on F_α^2 if and only if $f = e^q$ and $g = ce^{-q}$, where c is a nonzero complex constant and q is a complex linear polynomial.*

Furthermore, our proof reveals that when $T_f T_{\bar{g}}$ is bounded, it must be a constant times a unitary operator. Consequently, $T_f T_{\bar{g}}$ is never compact unless it is the zero operator.

As another by-product of our analysis, we will construct a class of unbounded, densely defined operators on the Fock space whose Berezin transform is bounded. It has been known that such operators exist, but our examples are very simple products of Toeplitz operators.

2. PROOF OF THE MAIN RESULT

For any point $a \in \mathbb{C}^n$ we consider the operator $U_a : F_\alpha^2 \rightarrow F_\alpha^2$ defined by

$$U_a f(z) = f(z - a) k_a(z),$$

where

$$k_a(z) = \frac{K(z, a)}{\sqrt{K(a, a)}} = e^{\alpha z \cdot \bar{a} - \frac{\alpha}{2} |a|^2}$$

is the normalized reproducing kernel of F_α^2 at a . It follows from a change of variables that each U_a is a unitary operator on F_α^2 .

We begin with the very special case of Toeplitz operators induced by kernel functions.

Lemma 1. *Let $a \in \mathbb{C}^n$, $f(z) = e^{\alpha z \cdot \bar{a}}$, and $g(z) = e^{-\alpha z \cdot \bar{a}}$. We have*

$$T_f T_{\bar{g}} = e^{\frac{\alpha}{2}|a|^2} U_a.$$

In particular, $T_f T_{\bar{g}}$ is bounded on F_α^2 .

Proof. To avoid triviality we assume that a is nonzero. The Toeplitz operator T_f is just multiplication by f , as a densely defined unbounded linear operator. So we focus on the operator $T_{\bar{g}}$.

Given any function $h \in F_\alpha^2$, we have

$$\begin{aligned} T_{\bar{g}} h(z) &= \int_{\mathbb{C}^n} \overline{g(w)} h(w) K(z, w) d\lambda_\alpha(w) \\ &= \int_{\mathbb{C}^n} h(w) e^{\alpha(z-a) \cdot \bar{w}} d\lambda_\alpha(w) \\ &= h(z-a). \end{aligned}$$

Therefore, the Toeplitz operator $T_{\bar{g}}$ is an operator of translation, and

$$T_f T_{\bar{g}} h(z) = e^{\alpha z \cdot \bar{a}} h(z-a) = e^{\frac{\alpha}{2}|a|^2} U_a h(z).$$

This proves the desired result. $\square$

An immediate consequence of Lemma 1 is that if $f = C_1 e^q$ and $g = C_2 e^{-q}$, where C_1 and C_2 are complex constants and q is a complex linear polynomial, then there exists a complex constant c and a unitary operator U such that $T_f T_{\bar{g}} = cU$.

To deal with more general symbol functions, we need the following characterization of nonvanishing functions in F_α^2 .

Lemma 2. *If f is a nonvanishing function in F_α^2 , then there exists a complex polynomial q , with $\deg(q) \leq 2$, such that $f = e^q$.*

Proof. In the case when the dimension $n = 1$, the Weierstrass factorization of functions in the Fock space F_α^2 takes the form $f(z) = P(z)e^{q(z)}$, where P is the canonical Weierstrass product associated to the zero sequence of f , and $q(z) = az^2 + bz + c$ is a quadratic polynomial with $|a| < \frac{\alpha}{2}$. In particular, if f is zero-free, then $f = e^q$ for some quadratic polynomial. See [10].

When $n > 1$, we no longer have such a nice factorization. But the absence of zeros makes a special version of the factorization above still valid. More specifically, if f is any function in $F_\alpha^2 = F_\alpha^2(\mathbb{C}^n)$ and f is nonvanishing, then the function $z_1 \mapsto f(z_1, \dots, z_n)$ is in $F_\alpha^2(\mathbb{C})$, so by the factorization theorem stated in the previous paragraph, we have

$$f(z_1, \dots, z_n) = e^q, \quad q = az_1^2 + bz_1 + c,$$

where a , b , and c are holomorphic functions of $z_2, \dots, z_n$. In particular,

$$\frac{\partial^k q}{\partial z_1^k} = 0, \quad k > 2.$$

Repeat this for every independent variable. We see that

$$\frac{\partial^k q}{\partial z_j^k} = 0, \quad k > 2, 1 \leq j \leq n.$$

It follows that

$$\frac{\partial^m q}{\partial z^m} = 0$$

for every multi-index $m = (m_1, \dots, m_n)$ of nonnegative integers with $|m| = m_1 + \dots + m_n > 2n$ (which implies that at least one of the m_j 's must be greater than 2). This shows that $f = e^q$ for some polynomial q of degree $2n$ or less.

Recall that every function $f \in F_\alpha^2$ satisfies the pointwise estimate

$$|f(z)| \leq C e^{\frac{\alpha}{2}|z|^2}, \quad z \in \mathbb{C}^n.$$

If q is a polynomial of degree N and $N > 2$, then for any fixed $\zeta = (\zeta_1, \dots, \zeta_n)$ on the unit sphere of $\mathbb{C}^n$ with each $\zeta_k \neq 0$, and for $z = r\zeta$, where $r > 0$, we have $q(z) \sim r^N$ as $r \rightarrow \infty$, which shows that the estimate $|f(z)| \leq C e^{\frac{\alpha}{2}|z|^2}$ is impossible to hold. This shows that the degree of q is less than or equal to 2. $\square$

We can now prove the main result, which we restate as follows.

Theorem 3. *Suppose f and g are functions in F_α^2 . Then the Toeplitz product $T_f T_{\bar{g}}$ is bounded on F_α^2 if and only if one of the following two conditions holds:*

- (a) *At least one of f and g is identically zero.*
- (b) *There exists a linear polynomial q and a nonzero constant c such that $f = e^q$ and $g = ce^{-q}$.*

Proof. If condition (a) holds, then the Toeplitz product $T_f T_{\bar{g}}$ is 0. If condition (b) holds, the boundedness of $T_f T_{\bar{g}}$ follows from Lemma 1.

Next assume that $T = T_f T_{\bar{g}}$ is bounded on F_α^2 . Then the Berezin transform $\tilde{T}$ is a bounded function on $\mathbb{C}^n$, where

$$\tilde{T}(z) = \langle T_f T_{\bar{g}} k_z, k_z \rangle, \quad z \in \mathbb{C}^n.$$

It follows from the integral representation of $T_{\bar{g}}$ and the reproducing property of the kernel function $e^{\alpha z \cdot \bar{w}}$ that $T_{\bar{g}} k_z = \overline{g(z)} k_z$. Therefore,

$$\tilde{T}(z) = \overline{g(z)} \langle f k_z, k_z \rangle, \quad z \in \mathbb{C}^n.$$

Write the inner product above as an integral and apply the reproducing property of the kernel function $e^{\alpha z \cdot \bar{w}}$ one more time. We obtain $\tilde{T}(z) = f(z) \overline{g(z)}$. It follows that $|f(z)g(z)| \leq \|T\|$ for all $z \in \mathbb{C}^n$. But fg is entire, so by Liouville's theorem, there is a constant c such that $fg = c$.

If $c = 0$, then at least one of f and g must be identically zero, so condition (a) holds.

If $c \neq 0$, then both f and g are nonvanishing. By Lemma 2, there exists a complex polynomial q , with $\deg(q) \leq 2$, such that $f = e^q$ and $g = ce^{-q}$. It remains for us to show that $\deg(q) \leq 1$.

Let us assume $\deg(q) = 2$, in the hope of reaching a contradiction, and write $q = q_2 + q_1$, where q_1 is linear and q_2 is a homogeneous polynomial of degree 2. By the boundedness of $T = T_f T_{\bar{g}}$ on F_α^2 , the function

$$T(z, w) = \langle T_f T_{\bar{g}} k_z, k_w \rangle, \quad z \in \mathbb{C}^n, w \in \mathbb{C}^n,$$

is bounded on $\mathbb{C}^n \times \mathbb{C}^n$. We proceed to show that this is impossible unless $q_2 = 0$.

Again, by the integral representation for Toeplitz operators and the reproducing property of the kernel function $e^{\alpha z \cdot \bar{w}}$, it is easy to obtain that

$$T(z, w) = f(w) \overline{g(z)} e^{-\frac{\alpha}{2}|z|^2 + \alpha w \cdot \bar{z} - \frac{\alpha}{2}|w|^2}.$$

It follows that

$$|T(z, w)| = |f(w)g(z)| e^{-\frac{\alpha}{2}|z-w|^2}$$

for all $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. Using the explicit form of f and g , we can write

$$|T(z, w)| = |c \exp(q_2(w) - q_2(z) + q_1(w) - q_1(z))| e^{-\frac{\alpha}{2}|z-w|^2}.$$

Since q_1 is linear, it is easy to see that there is a point $a \in \mathbb{C}^n$ such that

$$q_1(w) - q_1(z) = (w - z) \cdot \bar{a}$$

for all z and w .

For the second-degree homogeneous polynomial q_2 we can find a complex matrix $A = A_{n \times n}$, symmetric in the real sense, such that $q_2(z) = \langle Az, z \rangle$, where $\langle \cdot, \cdot \rangle$ is the real inner product. Fix two points u and v in $\mathbb{C}^n$ such that $\operatorname{Re} \langle Au, v \rangle \neq 0$. This is possible as long as $A \neq 0$. Now let $z = ru$ and $w = ru + v$, where r is any real number. We have

$$\begin{aligned} q_2(w) - q_2(z) &= q_2(z + v) - q_2(z) \\ &= \langle A(z + v), z + v \rangle - \langle Az, z \rangle \\ &= \langle Az, v \rangle + \langle Av, z \rangle + \langle Av, v \rangle \\ &= 2r \langle Au, v \rangle + \langle Av, v \rangle. \end{aligned}$$

It follows that there exists a positive constant $M = M(u, v)$ such that

$$|T(z, w)| = M |\exp(2r \langle Au, v \rangle)| = M \exp(2r \operatorname{Re} \langle Au, v \rangle).$$

Since $\operatorname{Re} \langle Au, v \rangle \neq 0$, this shows that $T(z, w)$ cannot be a bounded function on $\mathbb{C}^n \times \mathbb{C}^n$. This contradiction shows that $A = 0$ and the polynomial q must be linear. $\square$

As a consequence of the analysis above, we obtain an interesting class of unbounded operators on F_α^2 whose Berezin transforms are bounded.

Corollary 4. *Suppose $f(z) = e^q$ and $g = e^{-q}$, where q is any second-degree homogeneous polynomial whose coefficients are small enough so that f and g belong to F_α^2 . Then the Toeplitz product $T_f T_{\bar{g}}$ is unbounded on F_α^2 , but its Berezin transform is bounded.*

Proof. By Theorem 3, the operator $T_f T_{\bar{g}}$ is unbounded. On the other hand, by the proof of Theorem 3, the Berezin transform of $T = T_f T_{\bar{g}}$ is given by

$$\tilde{T}(z) = f(z) \overline{g(z)}, \quad z \in \mathbb{C}^n.$$

It follows that $|T(z)| = |f(z)g(z)| = 1$ for all $z \in \mathbb{C}^n$. $\square$

Another consequence of our earlier analysis is the following.

Corollary 5. *If f and g are functions in F_α^2 , then the following conditions are equivalent:*

- (a) $T_f T_{\bar{g}}$ is compact.
- (b) $T_f T_{\bar{g}} = 0$.
- (c) $f = 0$ or $g = 0$.

Proof. Combining Lemma 1 and Theorem 3, we see that whenever $T_f T_{\bar{g}}$ is compact on F_α^2 , we must have $f = 0$ or $g = 0$. This clearly gives the desired result. $\square$

3. FURTHER REMARKS

For any $0 < p \leq \infty$ let F_α^p denote the Fock space consisting of entire functions f such that the function $f(z)e^{-\frac{\alpha}{2}|z|^2}$ belongs to $L^p(\mathbb{C}^n, dv)$. When $0 < p < \infty$, the norm in F_α^p is defined by

$$\|f\|_{p,\alpha} = \left[\left(\frac{p\alpha}{2\pi} \right)^n \int_{\mathbb{C}^n} |f(z)e^{-\frac{\alpha}{2}|z|^2}|^p dv(z) \right]^{\frac{1}{p}}.$$

For $p = \infty$, the norm in F_α^∞ is defined by

$$\|f\|_{\infty,\alpha} = \sup_{z \in \mathbb{C}^n} |f(z)|e^{-\frac{\alpha}{2}|z|^2}.$$

It is easy to check that the normalized reproducing kernel

$$k_a(z) = e^{\alpha z \cdot \bar{a} - \frac{\alpha}{2}|a|^2}$$

is a unit vector in each F_α^p , where $0 < p \leq \infty$. Also, it can be shown that the set of functions of the form

$$f(z) = \sum_{k=1}^N c_k K(z, a_k) = \sum_{k=1}^N c_k e^{\alpha z \cdot \bar{a}_k}$$

is dense in each F_α^p , where $0 < p < \infty$. See [10].

Therefore, if $0 < p < \infty$ and f satisfies Condition (G), we can consider the action of the Toeplitz operator T_f (defined using the integral representation in Section 1) on F_α^p . Also, if $f \in F_\alpha^p$, then it satisfies the pointwise estimate $|f(z)| \leq C e^{\frac{\alpha}{2}|z|^2}$, which implies that f satisfies Condition (G).

When $1 < p < \infty$ and $1/p + 1/q = 1$, the dual space of F_α^p can be identified with F_α^q under the integral pairing

$$\langle f, g \rangle_\alpha = \int_{\mathbb{C}^n} f(z) \overline{g(z)} d\lambda_\alpha(z).$$

When $0 < p \leq 1$, the dual space of F_α^p can be identified with F_α^∞ under the same integral pairing above. See [3, 10].

Thus for functions f and g in F_α^p , if the Toeplitz product $T = T_f T_{\bar{g}}$ is bounded on F_α^p , we can still consider the function

$$T(z, w) = \langle T_f T_{\bar{g}} k_z, k_w \rangle_\alpha$$

on $\mathbb{C}^n \times \mathbb{C}^n$. Exactly the same arguments from Section 2 will yield the following result.

Theorem 6. *Suppose $0 < p < \infty$. If f and g are functions in F_α^p , not identically zero, then the Toeplitz product $T_f T_{\bar{g}}$ is bounded on F_α^p if and only if $f = e^q$ and $g = ce^{-q}$, where c is a nonzero complex constant and q is a complex linear polynomial.*

It would be nice to extend our results here to more general Fock-type spaces. In particular, a generalization to the Fock-Sobolev spaces studied in [1, 2] should be possible.

It would also be interesting to take a second look at the original Hardy space setting. More specifically, if f and g are functions in the Hardy space H^2 (of the

unit disk, for example), then the boundedness of the Toeplitz product $T_f T_{\bar{g}}$ on H^2 implies that the product function fg is in H^∞ . Is it possible to derive more detailed information about f and g , say in terms of inner and outer functions? A more explicit condition on f and g (as opposed to the condition $\widetilde{|f|^{2+\varepsilon}} \widetilde{|g|^{2+\varepsilon}} \in L^\infty$) would certainly be more desirable.

We hope that this paper will generate some further interest in this subject.

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