ON LEFT KÖTHE RINGS AND A GENERALIZATION OF A KÖTHE-COHEN-KAPLANSKY THEOREM

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ABSTRACT. In this paper, we obtain a partial solution to the following question of Köthe: For which rings R is it true that every left (or both left and right) R -module is a direct sum of cyclic modules? Let R be a ring in which all idempotents are central. We prove that if R is a left Köthe ring (i.e., every left R -module is a direct sum of cyclic modules), then R is an Artinian principal right ideal ring. Consequently, R is a Köthe ring (i.e., each left and each right R -module is a direct sum of cyclic modules) if and only if R is an Artinian principal ideal ring. This is a generalization of a Köthe-Cohen-Kaplansky theorem.

1. INTRODUCTION

All rings have identity elements and all modules are unital. A ring R is *local* in case R has a unique left maximal ideal. An Artinian (resp. Noetherian) ring is a ring which is both a left and right Artinian (resp. Noetherian). A *principal ideal ring* is a ring which is both a left and a right principal ideal ring. Also, a ring whose lattice of left ideals is linearly ordered under inclusion is called a *left uniserial ring*. A *uniserial ring* is a ring which is both left and right uniserial. Note that left and right uniserial rings are in particular local rings, and commutative uniserial rings are also known as valuation rings.

In [9] Köthe proved the following result.

Result 1.1 (Köthe, [9]). *Over an Artinian principal ideal ring, each module is a direct sum of cyclic modules. Furthermore, if a commutative Artinian ring has the property that all its modules are direct sums of cyclic modules, then it is necessarily a principal ideal ring.*

Later Cohen and Kaplansky [3] obtained the following result.

Result 1.2 (Cohen and Kaplansky, [3]). *If R is a commutative ring such that each R -module is a direct sum of cyclic modules, then R must be an Artinian principal ideal ring.*

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A *left Köthe ring* is a ring R such that each left R -module is a direct sum of cyclic submodules. A *right Köthe ring* is defined similarly by replacing the word left with right above. A ring R is called a *Köthe ring* if it is both a left and right Köthe ring. Thus by combining results above one obtains:

Result 1.3 (Köthe-Cohen-Kaplansky, [3, 9]). *A commutative ring R is a Köthe ring if and only if R is an Artinian principal ideal ring.*

The corresponding problem in the non-commutative case is still open (see [13, Appendix B, Problem 2.48] and the recent survey paper by Jain and Srivastava [8, page 40, Problem 1]). Clearly, the class of left Köthe rings is contained in the class of rings whose left modules are direct sums of finitely generated modules. In the commutative case, these two classes coincide (see Griffith [6, Theorem 4.3]). According to Chase [2, Theorem 4.4], every ring enjoying the property that “*left modules are direct sums of finitely generated modules*” is left Artinian and every left module possesses an indecomposable decomposition. The converse was established by Warfield [15, Theorem 3] for commutative rings, and Zimmermann-Huisgen [17, Corollary 2] generalized Warfield’s result for arbitrary rings. Fuller [5] proved that the rings with the above property coincide with those whose left modules have decompositions that complement direct summands (see Anderson and Fuller [1]).

Thus any left Köthe ring is left Artinian. But a left Artinian principal left ideal ring R need not be a left Köthe ring, even if R is a local ring (see Faith [4, page 212, Remark (2)]). On the other hand, Nakayama [11, page 289] gave an example of a right Köthe ring R which is not a principal right ideal ring.

In this paper we shall present a partial solution to the following problems of Köthe [9]:

- (i) (Köthe, reported by Tuganbaev [14]) *Describe non-commutative Köthe rings.*
- (ii) *Describe non-commutative left Köthe rings.*

Let R be a ring in which all idempotents are central. We prove that if R is a left Köthe ring, then R is an Artinian principal right ideal ring. Consequently, R is a Köthe ring if and only if R is an Artinian principal ideal ring (see Corollary 3.3). We note that this is a generalization of the Köthe-Cohen-Kaplansky theorem (see Result 1.3).

Finally, in Theorem 3.4, we prove that if $R = \prod_{i=1}^n R_i$ is a finite product of rings R_i such that for each i , $R_i R_i$ is uniform, then the following statements are equivalent:

- (1) R is a Köthe ring.
- (2) R is a left Köthe ring.
- (3) R is an Artinian principal ideal ring.
- (4) R is isomorphic to a finite product of Artinian uniserial rings.

Throughout this paper, $(R, \mathcal{M})$ will be a local ring with maximal ideal $\mathcal{M}$. For a ring R we denote by $J(R)$ the Jacobson radical of R and for a left R -module M , we denote by $E(RM)$ and $\text{soc}(RM)$ the injective hull of M and the socle of M , respectively. Also, for a subset S of ${}_R M$, we denote by $l.\text{Ann}_R(S)$ the left annihilator of S in R . A left R -module M which has a composition series is called a module of finite length. The length of a composition series of ${}_R M$ is said to be the length of ${}_R M$ and denoted by $\text{length}({}_R M)$.

Of course, the analogous statements also hold if we replace the word “left” by the word “right” in the above. In general, in the sequel, if we have proved certain results for rings or modules “on the left”, then we shall use such results freely also “on the right”, provided that these results can indeed be proved by the same arguments applied “to the other side”.

2. PRELIMINARIES

Modifying Bass’s definition, we say a ring R is left *perfect* in case R satisfies the descending chain condition (dcc) on principal right ideals. The ring R is left perfect if and only if $R/J(R)$ is left semisimple and $J(R)$ is right t-nilpotent (see [16, page 390, Proposition 43.9]). Thus any left (or right) Artinian ring R is left and right perfect since in this case $J(R)$ is nilpotent and hence left and right t-nilpotent.

Lemma 2.1 (Faith [4, Lemma 2]). *Let R be a right perfect ring. If the injective hull $E({}_R R)$ is cyclic, then R is left self-injective.*

A ring R is called a *semiperfect* ring if $R/J(R)$ is left (or right) semisimple and idempotents in $R/J(R)$ can be lifted to R . If I is a nil ideal in the ring R , then idempotents in R/I can be lifted under $R \rightarrow R/I$. Thus every left (or right) Artinian ring is semiperfect.

Lemma 2.2 (See [16, page 564, 56.3]). *A ring R is semiperfect, and every finitely generated left ideal in R is cyclic if and only if R is isomorphic to a finite product of matrix rings over left uniserial rings.*

By Lemma 2.2, we have the following proposition, which determines the structure of left Artinian principal left ideal rings.

Proposition 2.3. *A ring R is a left Artinian principal left ideal ring if and only if R is isomorphic to a finite product of matrix rings over left Artinian left uniserial rings.*

Lemma 2.4 (See [16, page 539, 55.1]). *Let M be a non-zero left uniserial R -module. Then M is uniform and finitely generated submodules of M are cyclic.*

We call a ring R a *quasi-Frobenius* ring (or a *QF-ring*) if R is right Noetherian and right self-injective.

Lemma 2.5 (See [10, Theorem 15.1]). *For any ring R , the following statements are equivalent:*

- (1) R is a quasi-Frobenius ring.
- (2) R is left Noetherian and right self-injective.
- (3) R is right Noetherian and satisfies the following double annihilator conditions:

$$(3a) \quad r.\text{Ann}_R(l.\text{Ann}_R(A)) = A \text{ for any proper right ideal } A \text{ of } R.$$

$$(3b) \quad l.\text{Ann}_R(r.\text{Ann}_R(B)) = B \text{ for any proper left ideal } B \text{ of } R.$$

- (4) R is Artinian and satisfies (3a) and (3b).

Lemma 2.6 (Faith [4, Section 4, Theorem 1]). *Let R be a ring. Then R is a principal left ideal and is quasi-Frobenius if and only if R is an Artinian principal ideal ring.*

Lemma 2.7 ([7, Proposition 3]). *Let R be a left Artinian ring. Then R is a finite product of local rings if and only if all the idempotents of R are central.*

Also, by a standard application of Nakayama's lemma, we obtain the following lemma.

Lemma 2.8. *Let $(R, \mathcal{M})$ be a local ring such that ${}_R\mathcal{M}$ is finitely generated. Then ${}_R\mathcal{M}$ is generated by $\{x_1, \dots, x_n\}$ if and only if $\mathcal{M}/\mathcal{M}^2$ is generated by the set $\{x_1 + \mathcal{M}^2, \dots, x_n + \mathcal{M}^2\}$ as a left $R/\mathcal{M}$ -module.*

Lemma 2.9 (See Nicholson and Sánchez-Campos [12, Theorem 9]). *For any ring R , the following statements are equivalent:*

- (1) *R is local, $J(R) = Rx$ for some $x \in R$ and $x^k = 0$ for some $k \in \mathbb{N}$.*
- (2) *There exist $x \in R$ and $k \in \mathbb{N}$ such that $x^{k-1} \neq 0$ and $R \supset Rx \supset \dots \supset Rx^k = (0)$ are the only left ideals of R .*
- (3) *R is left uniserial of finite composition length.*

We conclude this section with the following proposition which is crucial in our investigation.

Proposition 2.10. *Let $(R, \mathcal{M})$ be a local ring such that $\mathcal{M}^k = 0$ for some k . Then:*

- (i) *R is a principal left ideal ring if and only if $\mathcal{M}$ is a principal left ideal ring.*
- (ii) *R is a left Artinian principal left ideal ring if and only if the ring $R/\mathcal{M}^2$ is a principal left ideal ring.*

Proof. Without loss of generality, we can assume that $\mathcal{M}^{k-1} \neq (0)$. The first statement follows immediately from Lemma 2.9. For the second statement, we assume that $R/\mathcal{M}^2$ is a principal left ideal ring and $\mathcal{M}/\mathcal{M}^2 = R(x + \mathcal{M}^2)$ where $x \in \mathcal{M}$. By induction on i , one can easily see that $\mathcal{M}^i/\mathcal{M}^{i+1} = R(x^i + \mathcal{M}^{i+1})$ for every $i \geq 1$. It follows that for each i the left R -module $\mathcal{M}^i/\mathcal{M}^{i+1}$ is simple, since $\mathcal{M} = l.\text{Ann}_R(x^i + \mathcal{M}^{i+1})$. Thus $R \supset \mathcal{M} \supset \mathcal{M}^2 \supset \dots \supset \mathcal{M}^{k-1} \supset \mathcal{M}^k = (0)$ is a composition series for ${}_R\mathcal{M}$, and hence R is a left Artinian ring. Thus $\mathcal{M}$ is finitely generated, and hence by Lemma 2.8, $\mathcal{M}$ is cyclic. Now by Lemma 2.9, R is a principal left ideal ring. The converse is clear. $\square$

3. MAIN RESULTS

We are now in a position to prove our main results.

Theorem 3.1. *Let R be a ring in which all idempotents are central. If R is a left Köthe ring, then R is an Artinian principal right ideal ring.*

Proof. Assume that R is a left Köthe ring. Then by Chase [2, Theorem 4.4], R is left Artinian. Thus by Lemma 2.7, R is a finite product of local rings. It would clearly have been sufficient to assume that R is a local ring. Let $\mathcal{M}$ be the maximal ideal of R . Then $\mathcal{M}^k = (0)$ for some $k \geq 1$, and hence by Proposition 2.10 (ii) we may assume that $\mathcal{M}^2 = (0)$. Now we proceed by cases.

Case 1. R is a principal left ideal ring. Then by Proposition 2.3, R is a left uniserial ring. By Lemma 2.4, ${}_R R$ is uniform. Thus $E({}_R R)$ is indecomposable. Since $E({}_R R)$ is a direct sum of cyclic R -modules, $E({}_R R)$ is cyclic, and so by Lemma 2.1, R is left self-injective. Thus by Lemma 2.5, R is a quasi-Frobenius ring. Now by Lemma 2.6, R is an Artinian principal ideal ring.

Case 2. R is not a principal left ideal ring. Thus by Proposition 2.10 (i), $\mathcal{M}$ is not cyclic as a left ideal. Since R is left Artinian and $\mathcal{M}^2 = (0)$, by [1, Proposition 15.17], $\text{soc}_l(R) = \text{soc}_r(R) = \mathcal{M}$. Thus ${}_R\mathcal{M} = Ry_1 \oplus \dots \oplus Ry_t$ such that $t \geq 2$ and each Ry_i is a simple left R -module. It follows that $\text{length}({}_R R) = t + 1$.

We claim that $\mathcal{M}_R = xR$, for if not, we can assume that $\mathcal{M}_R = \bigoplus_{i \in I} x_i R$, where $|I| \geq 2$ and each $x_i R$ is a simple right R -module. We put ${}_R G = (R \oplus R)/R(x_1, x_2)$. Since $x_1, x_2 \in \mathcal{M}$ and $\mathcal{M}^2 = (0)$, we conclude that $l.\text{Ann}_R((x_1, x_2)) = \mathcal{M}$. Thus the cyclic R -module $R(x_1, x_2)$ is simple, and hence

$$\text{length}({}_R G) = 2 \times \text{length}({}_R R) - \text{length}({}_R R(x_1, x_2)) = 2(t + 1) - 1.$$

We claim that every non-zero cyclic submodule Rz of G has length 1 or $t + 1$. If $\mathcal{M}z = 0$, then $\text{length}(Rz) = 1$ since $Rz \simeq R/\mathcal{M}$. Suppose that $\mathcal{M}z \neq 0$; then there exist $c_1, c_2 \in R$ such that $z = (c_1, c_2) + R(x_1, x_2)$. If $c_1, c_2 \in \mathcal{M}$, then $\mathcal{M}z = 0$, since $\mathcal{M}^2 = 0$. Thus without loss of generality, we can assume that $z = (1, c_2) + R(x_1, x_2)$ (since if $c_1 \notin \mathcal{M}$, then c_1 is a unit). Now let $r \in l.\text{Ann}_R(z)$; then $r(1, c_2) = t(x_1, x_2)$ for some $t \in R$. It follows that $r = tx_1$ and $rc_2 = tx_2$. Thus $tx_2 = tx_1 c_2$. If $t \notin \mathcal{M}$, then t is a unit, and so $x_2 = x_1 c_2$, which is a contradiction (since $x_1 R \cap x_2 R = (0)$). Thus $t \in \mathcal{M}$, and so $r = tx_1 = 0$. Therefore, $l.\text{Ann}_R(z) = 0$, and so $Rz \cong R$. It follows that $\text{length}(Rz) = t + 1$. Now since R is a left Köthe ring, we have

$$G = Rw_1 \oplus \dots \oplus Rw_k \oplus Rv_1 \oplus \dots \oplus Rv_l,$$

where $l, k \geq 0$, and each Rw_i is of length $t + 1$ and each Rv_j is of length 1. Clearly $\mathcal{M} \oplus \mathcal{M}$ is not a simple left R -module. Since $R(x_1, x_2)$ is simple, $\mathcal{M}G = (\mathcal{M} \oplus \mathcal{M})/R(x_1, x_2) \neq 0$. It follows that $k \geq 1$. Also, $\text{length}({}_R G) = 2(t + 1) - 1 = k(t + 1) + l$, and this implies that $k = 1$ and $l = t$. Since $\mathcal{M}v_i = 0$ for each i , $\mathcal{M}G = \mathcal{M}w_1$, and hence

$$G/\mathcal{M}G \simeq Rw_1/\mathcal{M}w_1 \oplus Rv_1 \oplus \dots \oplus Rv_t.$$

It follows that $\text{length}({}_R G/\mathcal{M}G) = 1 + t$. On the other hand, we have

$$G/\mathcal{M}G \cong R/\mathcal{M} \oplus R/\mathcal{M},$$

and so $\text{length}({}_R G/\mathcal{M}G) = 2$ and thus $t = 1$, a contradiction.

Thus $\mathcal{M}$ is simple as a right R -module. Thus R is right Artinian, and so by Proposition 2.10 (i), R is a principal right ideal ring. Since R is also left Artinian, R is an Artinian principal right ideal ring. $\square$

Remark 3.2. We note that the idempotents in a duo-ring (i.e., every one-sided ideal is two-sided) are central (see [7, Lemma 2]). Also, if a ring R is local or ${}_R R$ is uniform, then R has no non-trivial idempotent elements. Therefore, if R is a left Köthe ring and is a finite product of rings R_i such that each R_i is duo, local or ${}_{R_i} R_i$ is uniform, then by Theorem 3.1, R is an Artinian principal right ideal ring.

As a corollary, we have the following result, which is a generalization of the Köthe-Cohen-Kaplansky theorem (see Result 1.3).

Corollary 3.3. *Let R be a ring in which all idempotents are central. Then the following statements are equivalent:*

- (1) R is a Köthe ring.
- (2) R is an Artinian principal ideal ring.
- (3) R is isomorphic to a finite product of Artinian uniserial rings.

Proof. Follows from Theorem 3.1 and Proposition 2.3. $\square$

Finally, we conclude this article with the following characterization of a left Köthe ring R when $R = \prod_{i=1}^n R_i$ is a finite product of rings R_i , where for each i , R_i is a uniform left R_i -module.

Theorem 3.4. *Let $R = \prod_{i=1}^n R_i$ be a finite product of rings R_i where each $R_i R_i$ is uniform. Then the following statements are equivalent:*

- (1) *R is a left Köthe ring.*
- (2) *R is a Köthe ring.*
- (3) *R is an Artinian principal ideal ring.*
- (4) *R is isomorphic to a finite product of Artinian uniserial rings.*

Proof. By Corollary 3.3 and Result 1.1, we only need to show that (1) $\Rightarrow$ (3).

Without loss of generality, we can assume that ${}_R R$ is uniform. Since R is a left Köthe ring, by Remark 3.2 R is an Artinian principal right ideal ring. Since ${}_R R$ is uniform, $E({}_R R)$ is indecomposable. Since $E({}_R R)$ is a direct sum of cyclic R -modules, $E({}_R R)$ is cyclic, and so by Lemma 2.1, R is left self-injective. Thus by Lemma 2.5, R is a quasi-Frobenius ring. Now by Lemma 2.6, R is an Artinian principal ideal ring. $\square$

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