ON COMMUTING AND SEMI-COMMUTING POSITIVE OPERATORS

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ABSTRACT. Let K be a positive compact operator on a Banach lattice. We prove that if either $[K]$ or $\langle K \rangle$ is ideal irreducible, then $[K] = \langle K \rangle = L_+(X) \cap \{K\}'$. We also establish the Perron-Frobenius Theorem for such operators K . Finally, we apply our results to answer questions posed by Abramovich and Aliprantis (2002) and Bračič et al. (2010).

1. INTRODUCTION

Throughout this paper, X always denotes a real Banach lattice with $\dim X > 1$. Recall that a collection $\mathcal{C}$ of positive operators on X is said to be **ideal irreducible** if there exist no non-trivial (i.e. different from $\{0\}$ and X) closed ideals invariant under each member of $\mathcal{C}$. In particular, a positive operator T is called ideal irreducible if $\{T\}$ is ideal irreducible. The classical Perron-Frobenius theory deals with the peripheral spectrum of ideal irreducible operators on finite-dimensional spaces; cf. [4, Chapter 8]. It has been extended to ideal irreducible operators on infinite-dimensional Banach lattices by various authors; see [5, 21, 22, 24, 26, 30, 31], etc. In particular, one has the following.

Theorem 1.1 ([11, 26]; cf. Theorems 5.2 and 5.4, Chapter V of [32]). *Let $K > 0$ be compact and ideal irreducible.*

- (i) *The spectral radius $r(K) > 0$, $\ker(r(K) - K) = \text{Span}\{x_0\}$ for some quasi-interior point $x_0 > 0$ and $\ker(r(K) - K^*) = \text{Span}\{x_0^*\}$ for some strictly positive functional x_0^* ;*
- (ii) *The peripheral spectrum $\sigma_{\text{per}}(K) = r(K)G$, where G is the set of all k -th roots of unity for some $k \geq 1$, and each point in $\sigma_{\text{per}}(K)$ is a simple pole of the resolvent $R(\cdot, K)$ with one-dimensional eigenspace.*

The fact that $r(K) > 0$ is a result known as de Pagter's theorem; cf. [11]. It was extended by Abramovich, Aliprantis and Burkinshaw [1–3] to commuting and semi-commuting positive operators that possess ideal irreducibility and compactness in one sense or another (cf. Chapters 9 and 10 of [4]). We write $T \leftrightarrow S$ if T commutes with S . The following is Corollary 9.28 of [4].

Theorem 1.2 ([4]). *Let $T, S, K > 0$ be such that $T \leftrightarrow S \leftrightarrow K$, T is ideal irreducible and K is compact. Then $r(S) > 0$.*

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The following can be found in Theorems 10.25 and 10.26 of [4].

Theorem 1.3 ([4]). *Let $T, K > 0$ be such that K is compact. Then T has a non-trivial invariant closed ideal if any of the following are satisfied:*

- (i) $TK \geq KT$ and $\liminf_n \|K^n x\|^{\frac{1}{n}} = 0$ for some $x > 0$;
- (ii) $TK \leq KT$ and K is quasi-nilpotent.

Similar results have also been established for collections of positive operators; see [9, 13, 20], etc. As in [4], for a positive operator T , we define the **super right-commutant** of T by $[T] := \{S \geq 0 : ST \geq TS\}$ and the **super left-commutant** of T by $\langle T \rangle := \{S \geq 0 : ST \leq TS\}$. The following can be deduced from Theorem 4.3 of [13].

Theorem 1.4 ([13]). *Let $K > 0$ be compact. If $\lim_n \|K^n x\|^{\frac{1}{n}} = 0$ for some $x > 0$, then $[K]$ has a common non-trivial invariant closed ideal.*

The aim of this paper is to study properties of compact positive operators K such that either $[K]$ or $\langle K \rangle$ is ideal irreducible. In Section 3, we establish an analogous version of Theorem 1.1 for such operators K . We also prove that, in this case, $[K] = \langle K \rangle = L_+(X) \cap \{K\}'$ and $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K) > 0$ for all $x > 0$. We also prove that the operators S in the following three chains have positive eigenvectors: $T \leftrightarrow K \leftrightarrow S$, $S \leftrightarrow T \leftrightarrow K$ and $T \leftrightarrow S \leftrightarrow K$, where $T > 0$ is ideal irreducible, $K > 0$ is compact and $S > 0$.

In Section 4, we provide some applications of the results from Section 3. In particular, we prove that, for a compact operator $K > 0$, if $TK \leq KT$ and $\liminf_n \|K^n x\|^{\frac{1}{n}} = 0$ for some $x > 0$, then T has a non-trivial invariant closed ideal. This improves Theorem 1.3(ii) and answers a question asked in [4]. As another application, we prove that, for two positive operators T and K with K being compact, if they semi-commute, then their commutator is quasi-nilpotent. This answers a question proposed in [10].

2. PRELIMINARIES

The notation and terminology used in this paper are standard. We refer to [4] for unexplained terms. For $T \in L(X)$, we write $\sigma(T)$ for the spectrum of T , $r(T)$ for the spectral radius of T and $\sigma_{\text{per}}(T) := \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$ for the peripheral spectrum of T . If T is a non-zero positive operator, we write $T > 0$. Furthermore, T is **strictly positive** if $T > 0$ and it does not vanish on non-zero positive vectors. We say that two operators **semi-commute** if their commutator is either positive or negative. For $A \subset X$, we write I_A for the ideal generated by A in X . A vector $x > 0$ is called a **quasi-interior point** if its generated ideal I_x is norm dense in X , or equivalently, $\hat{x}$ acts as a strictly positive functional on X^* .

We will need some technical lemmas. An operator $T \in L(X)$ is called **peripherally Riesz** if $r(T) > 0$ and $\sigma_{\text{per}}(T)$ is a spectral set with finite-dimensional spectral subspace. The following fact can be deduced by applying Lemma 1 of [27] to the restriction of T to its spectral subspace for $\sigma_{\text{per}}(T)$.

Lemma 2.1 ([19]). *Let $T \in L(X)$ be peripherally Riesz and $r(T) = 1$. Then either there exists $n_j \uparrow \infty$ such that T^{n_j} converges to the (finite-rank) spectral projection of T for $\sigma_{\text{per}}(T)$ or there exist $n_j \uparrow \infty$ and positive reals $c_j \downarrow 0$ such that $c_j T^{n_j}$ converges to a non-zero finite-rank nilpotent operator.*

The following simple lemma is taken from [17]; we provide the proof for the convenience of the reader.

Lemma 2.2 ([17]). *Let $U, V \in L(X)$ be semi-commuting. Suppose $Ux_0 = \lambda x_0$ for some quasi-interior point $x_0 > 0$, and $U^*x_0^* = \lambda x_0^*$ for some strictly positive functional x_0^* . Then $UV = VU$.*

Proof. Note that $x_0^*((UV - VU)x_0) = (U^*x_0^*)(Vx_0) - x_0^*(VUx_0) = \lambda x_0^*(Vx_0) - x_0^*(V\lambda x_0) = 0$. Since U and V semi-commute and x_0^* is strictly positive, we have $(UV - VU)x_0 = 0$. Thus x_0 being quasi-interior yields $UV - VU = 0$. $\square$

Lemma 2.3. *Suppose $T > 0$.*

- (i) *If $Tx_0 = \lambda x_0$ for some quasi-interior $x_0 > 0$, then $\liminf_n \|T^{n*}x^*\|^{\frac{1}{n}} \geq \lambda$ for any $x^* > 0$. If, in addition, $r(T)$ is an eigenvalue of T^* , then $\lambda = r(T)$.*
- (ii) *If $T^*x_0^* = \lambda x_0^*$ for some strictly positive $x_0^* > 0$, then $\liminf_n \|T^n x\|^{\frac{1}{n}} \geq \lambda$ for any $x > 0$. If, in addition, $r(T)$ is an eigenvalue of T , then $\lambda = r(T)$.*

Proof. We only prove (i); the proof of (ii) is similar. Suppose that $x^* > 0$. Then x_0 being quasi-interior implies $x^*(x_0) > 0$. Note that $\lambda^n x^*(x_0) = x^*(T^n x_0) = T^{n*}x^*(x_0) \leq \|T^{n*}x^*\| \|x_0\|$. Thus $\|T^{n*}x^*\|^{\frac{1}{n}} \geq \lambda \sqrt[n]{x^*(x_0)/\|x_0\|}$. Letting $n \rightarrow \infty$, we have $\liminf_n \|T^{n*}x^*\|^{\frac{1}{n}} \geq \lambda$.

Now suppose $T^*x^* = r(T)x^*$ for some $x^* \neq 0$. Then $\lambda|x^*| \leq r(T)|x^*| \leq T^*|x^*|$. Note also that $(T^*|x^*| - \lambda|x^*|)(x_0) = |x^*|(Tx_0) - \lambda|x^*|(x_0) = 0$. Hence x_0 being quasi-interior yields $T^*|x^*| = \lambda|x^*|$. It follows that $\lambda|x^*| = r(T)|x^*| = T^*|x^*|$. In particular, $\lambda = r(T)$. $\square$

Note that, for a positive operator T , both $[T]$ and $\langle T \rangle$ are multiplicative semi-groups containing the identity. We also need the following fact. Suppose that T and S are two semi-commuting positive operators. If $TS \leq ST$, then $(TS)^n \leq S^n T^n$ for all n ; if $ST \leq TS$, then $(ST)^n \leq T^n S^n$ for all n . Thus, in either case, we have $r(TS) = r(ST) \leq r(T)r(S)$.

Lemma 2.4. *Let $K > 0$ be compact such that either $[K]$ or $\langle K \rangle$ is ideal irreducible. Then $r(K) > 0$.*

Proof. Suppose first that $[K]$ is ideal irreducible and $r(K) = 0$. Let $\mathcal{K}$ be the semigroup ideal generated by K in the semigroup $[K]$. Then each member in $\mathcal{K}$ is of the form $S_1 K S_2$ for some $S_i \in [K]$. Clearly, $S_1 K S_2 \leq S_1 S_2 K$. By the remarks preceding this lemma, $S_1 S_2 \in [K]$ and $r(S_1 S_2 K) \leq r(K)r(S_1 S_2) = 0$. Thus $r(S_1 K S_2) = 0$. It follows that $\mathcal{K}$ consists of quasi-nilpotent compact operators. Therefore, $\mathcal{K}$ is ideal reducible by Drnovšek's Theorem (see Corollary 10.47 of [4]), and thus so is $[K]$, by Proposition 2.1 of [15]. This contradicts our assumption. Similar arguments work for the other case. $\square$

3. MAIN RESULTS

For a compact operator $K > 0$ such that either $[K]$ or $\langle K \rangle$ is ideal irreducible, we have $r(K) > 0$ by Lemma 2.4. We will usually scale it so that $r(K) = 1$.

Theorem 3.1. *Suppose $K > 0$ is compact and either $[K]$ or $\langle K \rangle$ is ideal irreducible. Assume $r(K) = 1$. Let P be the spectral projection of K for $\sigma_{\text{per}}(K)$.*

- (i) *There exist disjoint positive vectors $\{x_i\}_1^r$ and disjoint positive functionals $\{x_i^*\}_1^r$, where $r = \text{rank}(P)$, such that*

$$P = \sum_1^r x_i^* \otimes x_i; \quad x_i^*(x_j) = \delta_{ij}, \quad \forall i, j.$$

- (ii) *$K|_{PX}$ is a permutation on $\{x_i\}_1^r$ and $K^*|_{P^*X^*}$ is a permutation on $\{x_i^*\}_1^r$. In particular, there exists $m \geq 1$ such that $P = \lim_n K^{nm}$.*
- (iii) *There exist a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that*

$$Kx_0 = x_0 \quad \text{and} \quad K^*x_0^* = x_0^*.$$

- (iv) *For any $x > 0$ and $x^* > 0$,*

$$0 < \inf_n \|K^n x\| \leq \sup_n \|K^n x\| < \infty,$$

$$0 < \inf_n \|K^{n*} x^*\| \leq \sup_n \|K^{n*} x^*\| < \infty.$$

In particular, $\lim_n \|K^n x\|^{\frac{1}{n}} = 1$ for all $x > 0$ and $\lim_n \|K^{n} x^*\|^{\frac{1}{n}} = 1$ for all $x^* > 0$.*

- (v) *Every operator semi-commuting with K commutes with K . In particular,*

$$[K] = \langle K \rangle = L_+(X) \cap \{K\}'.$$

- (vi) *For any $0 < S \leftrightarrow K$, there exist $\lambda_S \geq 0$, $x > 0$ and $x^* > 0$ such that*

$$Sx = \lambda_S x \quad \text{and} \quad S^*x^* = \lambda_S x^*.$$

Proof. Suppose first that $[K]$ is ideal irreducible. We apply Lemma 2.1 to K . We claim that the nilpotent case in Lemma 2.1 is impossible. Indeed, otherwise, $c_j K^{n_j}$ converges to a non-zero finite-rank nilpotent operator N for some $n_j \uparrow \infty$ and positive reals $c_j \downarrow 0$. Clearly, N is positive and compact. Thus Lemma 2.4 implies that $[N]$ is ideal reducible. It is easy to verify that $[K] \subset [N]$. Hence $[K]$ is also ideal reducible, a contradiction. This proves the claim. Using Lemma 2.1 again, we have $P = \lim_j K^{n_j}$ for some $n_j \uparrow \infty$. In particular, $P > 0$. It follows that the range PX is a lattice subspace of X with lattice operations $x \wedge y = P(x \wedge y)$ and $x \vee y = P(x \vee y)$ for any $x, y \in PX$; see Proposition 11.5 on p. 214 of [32]. Being a finite-dimensional Archimedean vector lattice, it is lattice isomorphic to $\mathbb{R}^r$ with the standard order; see Corollary 1 on p. 70 of [32]. Therefore, we can take positive $*$ -disjoint vectors $x_i \in PX$ ($i = 1, \dots, r$) and positive $y_i^* \in (PX)^*$ such that $y_i^*(x_j) = \delta_{ij}$. Put $x_i^* = y_i^* \circ P$. Then $x_i^* \in X_+^*$, $P = \sum_1^r x_i^* \otimes x_i$ and $x_i^*(x_j) = \delta_{ij}$.

Being a spectral subspace, PX is invariant under K . Note that $K|_{PX}$ is positive on the lattice subspace PX . Moreover, it follows from $P = \lim_j K^{n_j}$ that $I|_{PX} = \lim_j (K|_{PX})^{n_j}$. Thus $(K|_{PX})^{-1} = \lim_j (K|_{PX})^{n_j-1}$ is also positive on PX . It is well known that a positive operator on $\mathbb{R}^r$ has a positive inverse if and only if it is a weighted permutation on the standard basis with positive weights, if and only if it is a direct sum of weighted cyclic permutations with positive weights. Since $\sigma(K|_{PX}) = \sigma_{\text{per}}(K) \subset \{z \in \mathbb{C} : |z| = 1\}$, it is easily seen that after appropriately scaling the basis vectors x_i , $K|_{PX}$ is a permutation on x_i . We accordingly scale x_i^*

so that we still have $x_i^*(x_j) = \delta_{ij}$ and $P = \sum_1^r x_i^* \otimes x_j$. Note that

$$Kx_j = KPx_j = PKx_j = \sum_{i=1}^k x_i^*(Kx_j)x_i$$

and that

$$K^*x_j^* = K^*P^*x_j^* = P^*K^*x_j^* = \sum_{i=1}^k x_j^*(Kx_i)x_i^*.$$

Hence the matrix of $K^*|_{P^*X^*}$ relative to $\{x_i^*\}$ is the transpose of the matrix of $K|_{PX}$ relative to $\{x_i\}$, and thus $K^*|_{P^*X^*}$ is a permutation on x_i^* . Put $x_0 = \sum_1^r x_i$ and $x_0^* = \sum_1^r x_i^*$. Then it is clear that $Kx_0 = x_0$ and $K^*x_0^* = x_0^*$.

Since $K|_{PX}$ is a permutation on x_i , we can take $m \geq 1$ such that $(K|_{PX})^m = I|_{PX}$. Denote by Q the spectral projection of K for $\sigma(K) \setminus \sigma_{\text{per}}(K)$. Then $r(K|_{QX}) < 1$. Thus $(K|_{QX})^n \rightarrow 0$ as $n \rightarrow \infty$. It follows that $K^{mn} = (K|_{PX})^{mn} \oplus (K|_{QX})^{mn} \rightarrow I|_{PX} \oplus 0 = P$ as $n \rightarrow \infty$.

Since $P = \sum_1^r x_i^* \otimes x_i$, it is easy to see that P is strictly positive if and only if x_0^* is strictly positive, and that P^* is strictly positive if and only if x_0 is quasi-interior. We now prove that both P and P^* are strictly positive. It is easy to verify that $N_P := \{x \in X : P|x| = 0\}$ is a closed ideal invariant under $[P]$. Since $[K] \subset [P]$, we know N_P is also invariant under $[K]$. From this it follows easily that $N_P = \{0\}$. Thus P is strictly positive, and so is x_0^* . Now for any $T \in [K]$, we have $x_0^*((TK - KT)x_0) = x_0^*(TKx_0) - (K^*x_0^*)(Tx_0) = 0$. By strict positivity of x_0^* , we have $KTx_0 = TKx_0 = Tx_0$. Thus $Tx_0 \in \ker(1 - K) \subset PX \subset \overline{I_{x_0}}$. This implies that $\overline{I_{x_0}}$ is invariant under $[K]$, hence $\overline{I_{x_0}} = X$. It follows that x_0 is quasi-interior and thus P^* is strictly positive.

Since P is strictly positive, it follows from $0 = x_i \wedge x_j = P(x_i \wedge x_j)$ that $x_i \perp x_j$ whenever $i \neq j$. We now prove that the x_i^* are disjoint. Indeed, by Riesz-Kantorovich formulas, for $i \neq j$,

$$0 \leq (x_i^* \wedge x_j^*)(x_0) = \inf_{0 \leq u \leq x_0} \{x_i^*(u) + x_j^*(x_0 - u)\} \leq x_i^*(x_j) + x_j^*(x_0 - x_j) = 0.$$

Thus $(x_i^* \wedge x_j^*)(x_0) = 0$, yielding that $x_i^* \wedge x_j^* = 0$ since x_0 is quasi-interior. This proves (i), (ii) and (iii). (v) follows from (iii) and Lemma 2.2.

For (iv), fix any $x > 0$. Since P is strictly positive, we have $Px > 0$. Now $K|_{PX}$ being a permutation implies that

$$0 < \liminf_n \|(K|_{PX})^n Px\| \leq \limsup_n \|(K|_{PX})^n Px\| < \infty.$$

Recall that Q is the spectral projection of K for $\sigma(K) \setminus \sigma_{\text{per}}(K)$ and $(K|_{QX})^n \rightarrow 0$. It follows from $K^n x = (K|_{PX})^n Px + (K|_{QX})^n Qx$ that

$$0 < \liminf_n \|K^n x\| \leq \limsup_n \|K^n x\| < \infty.$$

In particular, K is strictly positive. This in turn implies that

$$0 < \inf_n \|K^n x\| \leq \sup_n \|K^n x\| < \infty.$$

Taking the n -th root, we have $\lim_n \|K^n x\|^{\frac{1}{n}} = 1$ for all $x > 0$. For the dual case, a similar argument works.

For (vi), pick any $0 < S \in \{K\}'$. Then $SP = PS$. Note that the matrix of $S|_{PX}$ relative to $\{x_i\}$ is $(x_i^*(Sx_j))_{i,j}$ and that the matrix of $S^*|_{P^*X^*}$ relative to

$\{x_i^*\}$ is $(x_i^*(Sx_j))_{j,i}$, which is the transpose of $(x_i^*(Sx_j))_{i,j}$. Since both matrices are positive, they have positive eigenvectors for the spectral radius. It follows that there exist $0 < x \in PX$ and $0 < x^* \in P^*X^*$ such that $Sx = r(S|_{PX})x$ and $S^*x^* = r(S|_{PX})x^*$.

Now assume that $\langle K \rangle$ is ideal irreducible. We shall apply similar arguments. In fact, we only need to modify the proof of strict positivity of P and P^* . It is easy to verify that the ideal $I_{PX} \neq \{0\}$ is invariant under $\langle P \rangle$ and thus is invariant under $\langle K \rangle$. Therefore, $\overline{I_{PX}} = X$. On the other hand, we clearly have $I_{PX} = I_{x_0}$. Hence $\overline{I_{x_0}} = X$. It follows that x_0 is quasi-interior and thus P^* is strictly positive. We claim that x_0^* is strictly positive. Suppose, otherwise, $x_0^*(x) = 0$ for some $x > 0$. Then $x_i^*(x) = 0$ for $1 \leq i \leq r$. Note that, for any $T \in \langle K \rangle$, $\langle (K^*T^* - T^*K^*)x_0^*, x_0 \rangle = x_0^*(TKx_0) - x_0^*(KTx_0) = 0$. Hence $K^*T^*x_0^* = T^*K^*x_0^* = T^*x_0^*$. In particular, $T^*x_0^* \in \ker(1 - K^*) \subset P^*X^* = \text{Span}\{x_i^*\}_1^r$. It follows that $x_0^*(Tx) = (T^*x_0^*)(x) = 0$ for any $T \in \langle K \rangle$. By Proposition 2.1 of [15], $\langle K \rangle$ is ideal reducible, a contradiction. It follows that x_0^* and P are both strictly positive. $\square$

Remark 3.2. We apply Theorem 3.1 to the operator K in Theorem 1.1. By Lemma 2.4, $r(K) > 0$. We scale K so that $r(K) = 1$. Then by Theorem 3.1, $K|_{PX}$ is a permutation on the disjoint vectors x_i . We claim that it is a cyclic permutation. Suppose, otherwise, $K|_{PX}$ has a cycle of length $m < r$. Without loss of generality, assume that $K|_{PX}$ has a cycle on $x_1, \dots, x_m$. Then the closed ideal $\overline{I_{\{x_1, \dots, x_m\}}}$ is non-zero and invariant under K . Since $\overline{I_{\{x_1, \dots, x_m\}}}$ is disjoint from x_r , it is proper. This contradicts ideal irreducibility of K and proves the claim. Theorem 1.1 now follows immediately.

Example 3.3. Consider $T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ & 1 & 0 \\ & 0 & 1 \end{bmatrix}$. Then $K \leftrightarrow T$ and T is irreducible (in particular, $[K]$ and $\langle K \rangle$ are both irreducible). Note also that the peripheral spectral projection of K is the identity and thus $K|_{PX} = K$. Modifying this example, we can easily see that, for the operator K in Theorem 3.1, $K|_{PX}$ could be an arbitrary permutation.

Example 3.4. We cannot expect $\lambda_S > 0$ in Theorem 3.1(vi). Let K be as in Example 3.3. Put $S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ & 0 & 1 \\ & 0 & 0 \end{bmatrix}$. Then $S \leftrightarrow K$, but S is nilpotent.

Theorem 3.1 immediately yields the following corollary.

Corollary 3.5. Suppose T and K are two non-zero positive semi-commuting operators such that T is ideal irreducible and K is compact. Then $TK = KT$.

We provide an alternative proof of this corollary which is of independent interest. Recall that a positive operator T is called **strongly expanding** if it sends non-zero positive vectors to quasi-interior points. It is known that if T is ideal irreducible, then $\sum_1^\infty \frac{T^n}{\lambda^n}$ is strongly expanding for all $\lambda > r(T)$ (see Corollary 9.14 of [4]). For the alternative proof of Corollary 3.5, we need the following *comparison theorem* which can be deduced from Theorem 4.3 of [25] and de Pagter's theorem [11] (see e.g. [17] for details).

Lemma 3.6. Suppose $A \geq B \geq 0$, $r(A) = r(B)$ and A is ideal irreducible. If A and B are both compact, then $A = B$.

A second proof of Corollary 3.5. Without loss of generality, assume $\|T\| < 1$. Then $\tilde{T} = \sum_1^\infty T^n$ is strongly expanding. We claim that $K\tilde{T}$ and $\tilde{T}K$ do not have a common non-trivial invariant closed ideal. Otherwise, let J be such an ideal. If there exists $0 < x \in J$ such that $Kx > 0$, then $\tilde{T}(Kx)$ is a quasi-interior point. But $\tilde{T}Kx \in \tilde{T}K(J) \subset J$ implies $J = X$, a contradiction. Hence, $J \subset \ker K$. Now pick any $0 < x \in J$. Then $\tilde{T}x > 0$ is a quasi-interior point. From $K(K\tilde{T}x) \in K(K\tilde{T}(J)) \subset K(J) = \{0\}$, it follows that $K^2 = 0$. In particular, $r(K) = 0$, contradicting ideal irreducibility of T by Lemma 2.4. This proves the claim.

Now assume $TK \geq KT$. Then $T^n K \geq KT^n$ for all $n \geq 1$. Thus $\tilde{T}K \geq K\tilde{T} \geq 0$. If $\tilde{T}K$ has a non-trivial invariant closed ideal, then it is also invariant under $K\tilde{T}$, contradicting the preceding claim. Thus $\tilde{T}K$ is ideal irreducible. Note now that $\tilde{T}K$ and $K\tilde{T}$ are both compact and that $r(\tilde{T}K) = r(K\tilde{T})$. Thus $\tilde{T}K = K\tilde{T}$ by Lemma 3.6. This immediately implies that $TK = KT$. For $TK \leq KT$, we have $K\tilde{T} \geq \tilde{T}K$. The same argument yields $K\tilde{T} = \tilde{T}K$ and $TK = KT$. $\square$

We refer to [6, 7, 17, 23], etc., for more comparison theorems. We look at the operator T in Corollary 3.5 more closely.

Proposition 3.7. *Let $T > 0$ be ideal irreducible and $K > 0$ be compact. Suppose that $T \leftrightarrow K$.*

- (i) *There exist $0 < \lambda \leq r(T)$, a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that*

$$Tx_0 = \lambda x_0, \quad T^*x_0^* = \lambda x_0^*; \quad Kx_0 = r(K)x_0, \quad K^*x_0^* = r(K)x_0^*.$$

Moreover, $\lambda = r(T)$ if $r(T)$ is an eigenvalue of either T or T^ .*

- (ii) *$\liminf_n \|T^n x\|^{\frac{1}{n}} \geq \lambda$ for any $x > 0$ and $\liminf_n \|T^{n*} x^*\|^{\frac{1}{n}} \geq \lambda$ for any $x^* > 0$.*
- (iii) *Every operator semi-commuting with T commutes with T . In particular,*

$$\langle T \rangle = \langle T \rangle' = L_+(X) \cap \{T\}'.$$

- (iv) *For any $S \in \{T\}'$, $Sx_0 = \lambda_S x_0$ for some $\lambda_S \in \mathbb{R}$. If, in addition, $S > 0$, then $r(S) > 0$.*

(i) and (ii) have been proved in [19] via semigroup techniques. Here we provide a direct elementary proof.

Proof. Without loss of generality, assume $\|T\| < 1$. Recall that since T is ideal irreducible, $\sum_1^\infty T^n$ is strongly expanding. Hence so is $\tilde{K} := (\sum_1^\infty T^n)K(\sum_1^\infty T^n)$. In particular, $\tilde{K}$ is an ideal irreducible compact operator. Applying Theorem 1.1(i) to $\tilde{K}$, we obtain a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that

$$\ker(r(\tilde{K}) - \tilde{K}) = \text{Span}\{x_0\} \text{ and } \ker(r(\tilde{K}) - \tilde{K}^*) = \text{Span}\{x_0^*\}.$$

Since $T \leftrightarrow \tilde{K}$, these one-dimensional spaces $\ker(r(\tilde{K}) - \tilde{K})$ and $\ker(r(\tilde{K}) - \tilde{K}^*)$ are invariant under T and T^* , respectively. Thus there exist $\lambda, \delta \in \mathbb{R}$ such that

$$Tx_0 = \lambda x_0 \text{ and } T^*x_0^* = \delta x_0^*.$$

Note that $\delta x_0^*(x_0) = T^*x_0^*(x_0) = x_0^*(Tx_0) = \lambda x_0^*(x_0)$. Hence $\delta = \lambda$. Since $T > 0$ cannot vanish on quasi-interior points, we have $\lambda > 0$. The “moreover” part in (i) as well as (ii) follows from Lemma 2.3.

Since $K \leftrightarrow \tilde{K}$, a similar argument yields $0 < \mu \leq r(K)$ such that

$$Kx_0 = \mu x_0 \text{ and } K^*x_0^* = \mu x_0^*.$$

By the Krein-Rutman Theorem, $r(K)$ is an eigenvalue of K . Hence Lemma 2.3 implies $\mu = r(K)$.

(iii) follows from Lemma 2.2.

(iv) Using the same argument as in Theorem 5.2, p. 329, of [32], one can easily see that $\ker(\lambda - T) = \text{Span}\{x_0\}$. Since $S \leftrightarrow T$, we know $Sx_0 \in \ker(\lambda - T)$, and thus $Sx_0 = \lambda_S x_0$ for some $\lambda_S \in \mathbb{R}$. If $S > 0$, then $Sx_0 > 0$; thus $r(S) \geq \lambda_S > 0$. $\square$

Example 3.8. In Proposition 3.7(i), x_0 and x_0^* depend on T . Let T and K be as in Example 3.3. It is easy to see that the positive eigenvectors of T are exactly all the positive scalar multiples of $(1, 1, \dots, 1)^t$. Let $R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & \frac{5}{2} & \frac{5}{2} \\ 1 & 1 & \frac{5}{2} & \frac{5}{2} \end{bmatrix}$. Then R is also irreducible and commutes with K . But the positive eigenvectors of R are positive scalar multiples of $(1, 1, 2, 2)^t$.

Remark 3.9. Sirotkin [29] proved a Lomonosov-type theorem for positive operators on real Banach lattices, which implies that if $T > 0$ is non-scalar, $K > 0$ is compact and $S \leftrightarrow T \leftrightarrow K$, then S has a non-trivial invariant closed subspace; cf. Corollary 2.4 of [29]. Proposition 3.7(iv) implies that, in such a chain, if S is also non-scalar, then either T has a non-trivial invariant closed ideal or S has a non-trivial hyperinvariant closed subspace (namely, the eigenspace of S for λ_S).

We end this section with the following proposition.

Proposition 3.10. *Suppose $T > 0$ is ideal irreducible and $K > 0$ is compact.*

- (i) *There exist a quasi-interior point $x_0 > 0$ and a strictly positive functional x_0^* such that for any $S \in L(X)$ with $T \leftrightarrow S \leftrightarrow K$,*

$$Sx_0 = \lambda_S x_0, \quad S^*x_0^* = \lambda_S x_0^*,$$

for some $\lambda_S \in \mathbb{R}$.

- (ii) *If, in addition, $S > 0$, then $r(S) \geq \lambda_S > 0$, $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda_S$ for any $x > 0$, and $\liminf_n \|S^{n*} x^*\|^{\frac{1}{n}} \geq \lambda_S$ for any $x^* > 0$. Moreover, $\lambda_S = r(S)$ if $r(S)$ is an eigenvalue of either S or S^* .*
- (iii) *Every operator semi-commuting with S commutes with S .*

Proof. Without loss of generality, assume $\|T\| < 1$. As before, it is easily seen that $\tilde{K} := (\sum_1^\infty T^n) K (\sum_1^\infty T^n)$ is a compact ideal irreducible operator. Thus by Theorem 1.1(i), there exist a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that

$$\ker(r(\tilde{K}) - \tilde{K}) = \text{Span}\{x_0\} \text{ and } \ker(r(\tilde{K}) - \tilde{K}^*) = \text{Span}\{x_0^*\}.$$

Since $S \leftrightarrow \tilde{K}$, these one-dimensional spaces are invariant under S and S^* , respectively. From this (i) follows easily.

(ii) follows from Lemma 2.3. (iii) follows from Lemma 2.2. $\square$

Clearly, (ii) improves Theorem 1.2.

Remark 3.11. Note that the operator K in Theorem 3.1 can be replaced with a peripheral Riesz operator $R > 0$. The same proof goes along. In Propositions 3.7 and 3.10, the operator K can also be replaced with a peripheral Riesz operator $R > 0$. Simply note that Lemma 2.1 yields a compact operator to take the position of R .

4. APPLICATIONS

Using the results in the previous section, we can establish a few interesting criteria on ideal reducibility. The following is immediate by Lemma 2.4 and Theorem 3.1.

Proposition 4.1. *Let $K > 0$ be compact. Then $[K]$ and $\langle K \rangle$ both have non-trivial invariant closed ideals if any of the following are satisfied:*

- (i) $r(K) = 0$ or $\liminf_n \|K^n x\|^{\frac{1}{n}} < r(K)$ for some $x > 0$ or $\liminf_n \|K^{n*} x^*\|^{\frac{1}{n}} < r(K)$ for some $x^* > 0$;
- (ii) there exists $S \in L(X)$ such that either $SK > KS$ or $SK < KS$.

Proposition 4.1(i) clearly improves and extends Theorem 1.4.

Recall that if $T > 0$ and $S > 0$ semi-commute, then $r(TS) \leq r(T)r(S)$.

Proposition 4.2. *Suppose T and K are two non-zero semi-commuting positive operators such that K is compact. Then T has non-trivial invariant closed ideals if any of the following are satisfied:*

- (i) $r(K) = 0$ or $\liminf_n \|K^n x\|^{\frac{1}{n}} < r(K)$ for some $x > 0$ or $\liminf_n \|K^{n*} x^*\|^{\frac{1}{n}} < r(K)$ for some $x^* > 0$;
- (ii) $r(TK) = 0$ or $\liminf_n \|T^n x\|^{\frac{1}{n}} < \frac{r(TK)}{r(K)}$ for some $x > 0$ or $\liminf_n \|T^{n*} x^*\|^{\frac{1}{n}} < \frac{r(TK)}{r(K)}$ for some $x^* > 0$;
- (iii) there exists a quasi-nilpotent operator $S > 0$ semi-commuting with T ;
- (iv) there exists $S \in L(X)$ such that $ST < TS$ or $ST > TS$.

Proof. We prove by way of contradiction. Assume that T is ideal irreducible. (i) is clear by Proposition 4.1. By Corollary 3.5, $TK = KT$. It is easy to verify that $TK > 0$. Since $TK \leftrightarrow T$, we have $r(TK) > 0$ by Proposition 3.7(iv). Replacing K with TK in Proposition 3.7, we have

$$Tx_0 = \lambda x_0, \quad T^* x_0^* = \lambda x_0^*, \quad TKx_0 = r(TK)x_0, \quad (TK)^* x_0^* = r(TK)x_0^*.$$

Applying Lemma 2.3 to TK , we have

$$r(TK) = \lim_n \|(TK)^n x\|^{\frac{1}{n}} \leq r(K) \liminf_n \|T^n x\|^{\frac{1}{n}}$$

for any $x > 0$. The dual case can be proved similarly. This proves (ii). (iii) follows from Lemma 2.2 and Proposition 3.7(iv). (iv) also follows from Lemma 2.2. $\square$

Proposition 4.2(i) clearly improves Theorem 1.3.

In [4], it is proved that Theorem 1.3 remains true if K is merely assumed to dominate a non-zero AM-compact operator; see Theorems 10.25 and 10.26 of [4]. The authors were motivated to ask the following natural question. If K dominates a non-zero AM-compact operator, then in case (ii) of Theorem 1.3, can we replace the quasi-nilpotency condition on K by the local quasi-nilpotency at a positive vector? The following example shows that it fails in general even when K itself is a non-zero positive AM-compact operator. Let $X = \ell_2$. Let L and R be the left and

right shifts on X , respectively. Let $T = L + R$. Since the order intervals in ℓ_2 are compact, both T and L are AM-compact. It is a straightforward verification that $TL < LT$, $Le_1 = 0$ where $e_1 = (1, 0, 0, \dots)$, but T is ideal irreducible. Surprisingly, this question has an affirmative answer when K is a compact operator; this follows from our Proposition 4.2(i).

We now turn to an application of Corollary 3.5. In [10], it is proved that if two operators $T > 0$ and $S > 0$ semi-commute and are both compact, then their commutator $TS - ST$ is quasi-nilpotent. It is also shown there that there exist two semi-commuting operators $T > 0$ and $S > 0$, neither of which is compact, such that $TS - ST$ is not quasi-nilpotent. So the authors asked the following question.

Question. Suppose T and K are two semi-commuting positive operators such that K is compact. Is $TK - KT$ necessarily quasi-nilpotent?

Theorem 3.6 in [16] gave a partial solution of this question by proving that the commutator is indeed quasi-nilpotent provided that, in addition, it semi-commutes with K .

We now answer this question and prove that it is generally true. To this end, we need to recall some necessary notions. A collection $\mathcal{C}$ of closed subspaces of X is called a chain if it is totally ordered under inclusion. For any $M \in \mathcal{C}$, the predecessor M_- of M in $\mathcal{C}$ is defined to be the closed linear span of all proper closed subspaces of M that belong to $\mathcal{C}$. The following lemma is straightforward to verify.

Lemma 4.3. *Let $\mathcal{C}$ be a chain of closed ideals of X , $M \in \mathcal{C}$. Then M_- is a closed ideal of X , $M_- \subset M$ and $\mathcal{C} \cup \{M_-\}$ is a chain.*

Lemma 4.4 ([12]). *Let $\mathcal{C}$ be a chain of closed ideals of X . Then it is maximal as a chain of closed subspaces of X if and only if it is maximal as a chain of closed ideals of X .*

Recall that a collection $\mathcal{S}$ of positive operators is called **ideal triangularizable** if there exists a chain of closed ideals of X such that each member in the chain is invariant under $\mathcal{S}$ and the chain itself is maximal as a chain of closed subspaces of X (cf. Lemma 4.4). Such a chain is called an ideal triangularizing chain for $\mathcal{S}$.

Theorem 4.5. *Suppose T and K are two non-zero positive semi-commuting operators such that K is compact. Then $S := TK - KT$ is quasi-nilpotent.*

Proof. Since replacing T with $T + K$ does not change the commutator, we can assume $T \geq K > 0$. Let $\mathcal{C}$ be a maximal chain of invariant closed ideals of T (existence of such a chain follows from Zorn's lemma). Take any $M \in \mathcal{C}$. It is easily seen that M_- is invariant under T . Hence, by Lemma 4.3, $M_- \in \mathcal{C}$. We claim that the induced quotient operator $\tilde{T}$ on M/M_- is ideal irreducible. Suppose that, otherwise, J is a non-trivial closed ideal of M/M_- invariant under $\tilde{T}$. We consider $\pi^{-1}(J) = \{x \in M : [x] \in J\}$, where π is the canonical quotient mapping from M onto M_- . By Proposition 1.3, p. 156, of [32], $\pi^{-1}(J)$ is a closed ideal of M , and thus is a closed ideal of X . It is clearly invariant under T , properly contains M_- and is properly contained in M . Thus it is easily seen that $\pi^{-1}(J)$ is comparable with members of $\mathcal{C}$. But $\pi^{-1}(J) \notin \mathcal{C}$, contradicting maximality of $\mathcal{C}$.

It follows that $\tilde{T}$ is ideal irreducible on M/M_- . Since $T \geq K \geq 0$, both M and M_- are invariant under K ; hence the quotient operator $\tilde{K}$ is well defined on M/M_- . Corollary 3.5 implies $\tilde{S} = \tilde{T}\tilde{K} - \tilde{K}\tilde{T} = 0$.

For each $M \in \mathcal{C}$, let $\widetilde{\mathcal{C}}_M$ be a maximal chain of closed ideals of M/M_- (existence of such a chain follows from Zorn's lemma). Put $\mathcal{C}_M = \{\pi^{-1}(J) : J \in \widetilde{\mathcal{C}}_M\}$. Then $\mathcal{C}_M$ consists of closed ideals of X , each of which contains M_- and is contained in M . Since $\widetilde{S} = 0$ on M/M_- , each member of $\mathcal{C}_M$ is invariant under S . Thus it is easily seen that $\mathcal{C}_1 = \mathcal{C} \cup_{M \in \mathcal{C}} \mathcal{C}_M$ is a **chain** of closed ideals of X , each of which is invariant under S .

We claim that $\mathcal{C}_1$ is an ideal triangularizing chain for S . It remains to prove $\mathcal{C}_1$ is maximal as a chain of closed subspaces of X . Suppose, otherwise, there exists a closed subspace $Y \notin \mathcal{C}_1$ such that $\mathcal{C}_1 \cup \{Y\}$ is still a chain. Consider $M := \bigcap_{J \in \mathcal{C}, J \supset Y} J$ and $N := \bigcup_{J \in \mathcal{C}, J \subset Y} J$. They are well defined since $\{0\}, X \in \mathcal{C}$. It is easily seen that they are closed ideals of X invariant under T . Since each member of $\mathcal{C}$ is comparable with Y , it is easy to see that each $J \in \mathcal{C}$ either is contained in N or contains M . Note also that $N \subset Y \subset M$. Hence, by maximality of $\mathcal{C}$, we have $M, N \in \mathcal{C}$. It also follows that $N = M$ or M_- , the predecessor of M in $\mathcal{C}$. The first case is impossible, since it forces $Y = N = M \in \mathcal{C} \subset \mathcal{C}_1$. Hence, $M_- = N \subsetneq Y \subsetneq M$. Note that Y/M_- is a closed subspace of M/M_- . Clearly, $Y/M_- \notin \widetilde{\mathcal{C}}_M$. Since Y is comparable with each member of $\mathcal{C}_1$, Y/M_- is comparable with each member of $\widetilde{\mathcal{C}}_M$, contradicting maximality of $\widetilde{\mathcal{C}}_M$, by Lemma 4.4. This proves the claim.

Now note that for any $N \in \mathcal{C}_1$, we can find $M \in \mathcal{C}$ such that $M_- \subset N_- \subset N \subset M$, where M_- is the predecessor of M in $\mathcal{C}$ and N_- is the predecessor of N in $\mathcal{C}_1$. Since $\widetilde{S} = 0$ on M/M_- , $\widetilde{S} = 0$ on N/N_- . Hence, by Ringrose's theorem (Theorem 7.2.3, p. 156, of [28]), $\sigma(S) = 0$, i.e. S is quasi-nilpotent. $\square$

A recent preprint [14] provides an independent proof of Theorem 4.5.

The results of this paper remain valid if ideal irreducibility is replaced with band irreducibility provided that the operators involved are order continuous. The proofs are similar except that in the proof of Lemma 2.4, instead of Drnovšek's Theorem, we use its variant for band irreducible semigroups consisting of order continuous operators [19]. See [18] for details.

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