TRUDINGER-MOSER TYPE INEQUALITIES FOR WEIGHTED SOBOLEV SPACES INVOLVING FRACTIONAL DIMENSIONS

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ABSTRACT. We derive sharp Trudinger-Moser inequalities for weighted Sobolev spaces and prove the existence of extremal functions. The inequalities we obtain here extend for fractional dimensions the classical results in the radial case. The main ingredient used in our arguments reveals a new proof of a result due to J. Moser for which we give an improved version.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a smooth domain. The classical Sobolev embedding theorem asserts that the embedding $W_0^{1,n}(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 \leq q < \infty$ is continuous, but easy examples show that $W_0^{1,n}(\Omega) \not\subset L^\infty(\Omega)$ (cf. [2]). When Ω is a *bounded domain*, replacing the target Lebesgue space by an Orlicz space, N. Trudinger [22] proved that there exists $\mu > 0$ such that $W_0^{1,n}(\Omega)$ is embedded in the Orlicz space $L_\phi(\Omega)$ determined by $\phi(t) = e^{\mu|t|^{n/(n-1)}} - 1$. This result was sharpened by J. Moser [19], who found the best exponent μ and proved the following result:

$$(1) \quad \sup_{u \in W_0^{1,n}(\Omega) : \|\nabla u\|_n = 1} \int_{\Omega} e^{\mu|u|^{\frac{n}{n-1}}} dx \quad \begin{cases} \leq C_n |\Omega| & \text{if } \mu \leq \mu_n := n\omega_{n-1}^{1/(n-1)}, \\ = \infty & \text{if } \mu > \mu_n, \end{cases}$$

where $|\Omega|$ denotes the Lebesgue measure of a set Ω in $\mathbb{R}^n$ and ω_{n-1} is the measure of the unit sphere in $\mathbb{R}^n$. Estimates like (1) are now called *Trudinger-Moser type inequalities*. For related works and applications we refer the reader to [8, 9, 18] and the references therein.

To prove (1), J. Moser used *symmetrization*; that is, to every function u is associated a radially symmetric function u^* such that the sublevel-sets of u^* are balls with the same area as the corresponding sublevel-sets of u , i.e. $|\{x \in \mathbb{R}^n : u^*(x) < d\}| = |\{x \in \Omega : u(x) < d\}|$. Then u^* is a positive and non-increasing function defined on $B_R(0)$ with $|B_R| = |\Omega|$. By construction, one has the following property: Let $f \in C(\mathbb{R})$; then $\int_{B_R} f(u^*) dx = \int_{\Omega} f(u) dx$; furthermore, the well-known Pólya-Szegő inequality asserts that

$$(2) \quad \int_{B_R} |\nabla u^*|^n dx \leq \int_{\Omega} |\nabla u|^n dx.$$

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From this we clearly deduce that

$$\sup_{\|\nabla u\|_n \leq 1} \int_{\Omega} e^{\mu|u|^{\frac{n}{n-1}}} dx \leq \sup_{\|\nabla u^*\|_n \leq 1} \int_{B_R} e^{\mu|u^*|^{\frac{n}{n-1}}} dx,$$

and hence it is sufficient to consider symmetric non-negative and non-increasing functions. Next, performing the change of variables $r = |x| = Re^{-t/n}$ and $w(t) = n^{\frac{n-1}{n}} \omega_{n-1}^{\frac{1}{n}} u^*(r)$, J. Moser reduced (1) to prove the following result: If $p \geq 2$, then there exists a constant c_1 (depending on p only) such that

$$(3) \quad \sup_{w \in \mathcal{H}} \int_0^\infty e^{\rho w^{\frac{p}{p-1}}(t)-t} dt \quad \begin{cases} \leq c_1 & \text{if } \rho \leq 1, \\ = \infty & \text{if } \rho > 1, \end{cases}$$

where $\mathcal{H} := \{w \in C^1([0, \infty)) \text{ with } w(0) = 0, w' \geq 0, \int_0^\infty |w'|^p dt \leq 1\}$.

Motivated by this approach we will analyze a more general class of maximizing problem than (3), which allows us to treat a version of (1) to fractional dimensions. To be precise, the purpose of this paper is three-fold: (I) We investigate Trudinger-Moser type inequalities for a class of quasilinear elliptic operators which includes in particular the p -Laplacian and k -Hessian operators in the radial case; (II) we show that the resulting inequalities are sharp, obtaining the best exponents and (III) we also prove the existence of extremal functions. More explicitly, let $\alpha > 0$ and $\beta, \theta \geq 0$ be real numbers. We provide sharp Trudinger-Moser type inequalities for a class of quasilinear elliptic operators in radial form of the type

$$(4) \quad Lu := -r^{-\theta} (r^\alpha |u'(r)|^\beta u'(r))'$$

acting on the weighted Sobolev spaces X_R defined as follows: For $0 < R \leq \infty$, $q \geq 1$ and $\theta \geq 0$, let us denote by $L_\theta^q = L_\theta^q(0, R)$ the weighted Lebesgue space defined as the set of all measurable functions u on $(0, R)$ for which

$$\|u\|_{L_\theta^q} := \begin{cases} \left(\omega_\theta \int_0^R r^\theta |u(r)|^q dr \right)^{\frac{1}{q}} < \infty & \text{if } 1 \leq q < \infty, \\ \operatorname{ess\,sup}_{0 < r < R} |u(r)| < \infty & \text{if } q = \infty, \end{cases}$$

where

$$(5) \quad \omega_\theta = 2\pi^{\frac{\theta+1}{2}} / \Gamma(\frac{\theta+1}{2}), \text{ for all } \theta \geq 0,$$

with $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ the Euler gamma function. If $\theta + 1 = n$ is an integer number, then ω_θ is the measure of the unit sphere in $\mathbb{R}^n$. Denote by X_R the space of all locally absolutely continuous functions $u : (0, R] \rightarrow \mathbb{R}$ for which $u(R) = 0$, $u \in L_\theta^{\beta+2}$ and $u' \in L_\alpha^{\beta+2}$. When $R = \infty$, $u(R) = 0$ means $\lim_{r \rightarrow \infty} u(r) = 0$. We recall that X_R is a Banach space under the norm $\|u\|_{X_R} = \left(\|u\|_{L_\theta^{\beta+2}}^{\beta+2} + \|u'\|_{L_\alpha^{\beta+2}}^{\beta+2} \right)^{\frac{1}{\beta+2}}$.

If $0 < R < \infty$, we have

$$\left(\int_0^R r^\theta |u(r)|^q dr \right)^{\frac{1}{q}} \leq C \left(\int_0^R r^\alpha |u'(r)|^{\beta+2} dr \right)^{\frac{1}{\beta+2}}, \quad u \in X_R,$$

provided that one of the following two conditions is fulfilled (cf. [15, page 69]):

- (i) For $1 \leq \beta+2 \leq q < \infty$; then, either (a) $\alpha > \beta+1$ and $\theta \geq \alpha \frac{q}{\beta+2} - q \frac{\beta+1}{\beta+2} - 1$; or (b) $\alpha \leq \beta+1$ and $\theta \geq 0$,
- (ii) For $1 \leq q < \beta+2$; then, either (c) $\alpha > \beta+1$ and $\theta > \alpha \frac{q}{\beta+2} - q \frac{\beta+1}{\beta+2} - 1$; or (d) $\alpha \leq \beta+1$ and $\theta \geq 0$.

By (i) the norm $\|u\|_{X_R}$ is equivalent to the norm $\|u'\|_{L_\alpha^{\beta+2}}$ for all $\alpha > 0$ and $\beta, \theta \geq 0$. Also, combining (a) and (c) in the Sobolev case, that is, under the condition

$$\boxed{\alpha - \beta - 1 > 0},$$

we have the continuous embedding $X_R \hookrightarrow L_\theta^q$ for all $q \in (1, q^*]$, where $q^* := \frac{(\theta+1)(\beta+2)}{\alpha-\beta-1}$ is the Sobolev exponent for this class of spaces (cf. [6]). In this paper, we are interested in the limiting embedding for this class of weighted Sobolev spaces, known as the *Trudinger-Moser case*:

$$(6) \quad \boxed{\alpha - \beta - 1 = 0}.$$

In this case, from (b) and (d), we have the continuous embedding $X_R \hookrightarrow L_\theta^q$ for all $q \in [1, \infty)$, but one can see by taking $u(r) = \ln(\ln(eR/r))$ that $X_R \not\hookrightarrow L_\theta^\infty$. Instead, we prove here that X_R is embedded in the weighted Orlicz space L_ϕ determined by $\phi(t) = e^{\mu t^{(\beta+2)/(\theta+1)}} - 1$. Let us denote $\mu_{\alpha,\theta} := (\theta+1)\omega_\alpha^{1/\alpha}$, where ω_α was defined in (5). Our first result reads as follows.

Theorem 1.1. *Let $\alpha \geq 1$ and $\beta, \theta \geq 0$ be real numbers and assume that (6) holds. Then there exists $c_{\alpha,\theta}$ depending only on α, θ and R such that*

$$(7) \quad C_{\alpha,\theta,R}(\mu) := \sup_{u \in X_R: \|u'\|_{L_\alpha^{\beta+2}} \leq 1} \omega_\theta \int_0^R e^{\mu|u|^{\frac{\beta+2}{\theta+1}}} r^\theta dr \quad \begin{cases} \leq c_{\alpha,\theta} & \text{if } \mu \leq \mu_{\alpha,\theta}, \\ = \infty & \text{if } \mu > \mu_{\alpha,\theta}. \end{cases}$$

In the proof of Theorem 1.1 it is enough to consider $u \in C^1$, $u \geq 0$ and $u(R) = 0$. Performing the change of variables $r = Re^{-\frac{t}{\theta+1}}$ and $w(t) = \omega_\alpha^{\frac{1}{\alpha+1}} (\theta+1)^{\frac{\alpha}{\alpha+1}} u(r)$, and using (6) it is easy to check that

$$(8) \quad \omega_\alpha \int_0^R r^\alpha |u'(r)|^{\beta+2} dr = \int_0^\infty |w'(t)|^{\beta+2} dt$$

and

$$(9) \quad \omega_\theta \int_0^R e^{\mu|u|^{\frac{\beta+2}{\theta+1}}} r^\theta dr = \nu_\theta \int_0^\infty e^{\frac{\mu}{\mu_{\alpha,\theta}} |w|^{\frac{\beta+2}{\theta+1}} - t} dt,$$

where $\nu_\theta = \omega_\theta R^{\theta+1}/(\theta+1)$. At this point we emphasize that, since an analogous of the Pólya-Szegő inequality (see (2)) for X_R is not available, we cannot follow the arguments due to J. Moser to reduce (7) to the class of non-increasing functions. However, the complete proof of Theorem 1.1 follows from the next sharpened version of J. Moser's result in (3).

Theorem 1.2. *Let $p \geq 2$ be a real number. Then there exists a constant c_1 depending only on p such that*

$$(10) \quad \sup_{w \in \mathcal{K}} \int_0^\infty e^{\rho w^{\frac{p}{p-1}}(t) - t} dt \quad \begin{cases} \leq c_1 & \text{if } \rho \leq 1, \\ = \infty & \text{if } \rho > 1, \end{cases}$$

where $\mathcal{K} := \{w \in C^1([0, \infty)) \text{ with } w(0) = 0, \quad w \geq 0, \quad \int_0^\infty |w'|^p dt \leq 1\}$.

We point out that to prove Theorem 1.2, we use a different approach from the one used by J. Moser to analyze this class of maximizing problems. Furthermore, following this approach we establish a new estimate (see Lemma 2.1) allowing us to prove the existence of extremal functions for inequality (7) or, equivalently, (10).

More precisely,

Theorem 1.3. *Under the assumptions of Theorem 1.1, there are extremal functions for $C_{\alpha,\theta,R}(\mu)$ when $\mu \leq \mu_{\alpha,\theta}$; that is, there exists $u \in X_R$ such that*

$$C_{\alpha,\theta,R}(\mu) = \omega_\theta \int_0^R e^{\mu|u|^{\frac{\beta+2}{\beta+1}}} r^\theta dr.$$

The first result on this subject is due to L. Carleson and A. Chang [5], who proved in 1986 that there are extremals for classical Trudinger-Moser inequality (1), when Ω is a ball in any dimension. In [21] M. Struwe studied the existence of extremal functions for a non-symmetric domain. In the case that $n = 2$, he obtained a sufficient condition for the existence of an extremal function, using the blow-up analysis. In 1992, M. Flucher [11] introduced another method, the conformal rearrangement, and derived an isoperimetric inequality, which implies the existence of extremal functions when Ω is any smooth bounded domain in $\mathbb{R}^2$. Latter, K. Lin [17] generalized the existence of an extremal function to any smooth bounded domain in $\mathbb{R}^n$ ($n \geq 2$). See also [18] for related results. Recently, Y. Li [16] obtained the existence of an extremal function for certain Trudinger-Moser inequalities on compact Riemannian manifolds.

Next, we state a Trudinger-Moser type inequality for the unbounded case X_∞ . For that we introduce the following notation:

$$p = \beta + 2 \quad \text{and} \quad A_p(\zeta) = \exp(\zeta) - \sum_{m=0}^{\lfloor p \rfloor} \frac{\zeta^m}{m!},$$

where $\lfloor p \rfloor$ is the largest integer not greater than p .

Theorem 1.4. *Let $\alpha \geq 1$ and $\beta, \theta \geq 0$ be real numbers and assume that (6) holds. Then for any $\mu \in (0, \mu_{\alpha,\theta})$ there is a constant C depending only on μ, p and θ such that*

$$(11) \quad \int_0^\infty A_p \left(\mu \left(\frac{|u(r)|}{\|u'\|_{L_\alpha^p}} \right)^{p/(p-1)} \right) r^\theta dr \leq C \left(\frac{\|u\|_{L_\theta^p}}{\|u'\|_{L_\alpha^p}} \right)^p \quad \text{for all } u \in X_\infty \setminus \{0\}.$$

We remark that the restriction $\mu < \mu_{\alpha,\theta}$ in (11) is optimal, which excludes $\mu = \mu_{\alpha,\theta}$ in contrast with (7).

Theorem 1.5. *For any $\mu \geq \mu_{\alpha,\theta}$ there is a sequence $(u_j) \subset X_\infty$ such that $\|u'_j\|_{L_\alpha^p} = 1$ and*

$$\frac{1}{\|u_j\|_{L_\theta^p}^p} \int_0^\infty A_p \left(\mu (|u_j(r)|)^{p/(p-1)} \right) r^\theta dr \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

Remark 1.6. (1) Extensions of the Trudinger-Moser inequality for unbounded domains have been considered by D. M. Cao [4] in $\mathbb{R}^2$ and for any dimension by J. M. do Ó [10]. See also S. Adachi and K. Tanaka [1] for a related result. Here we generalize these results for operator (4) on X_∞ .

(2) Our results improve and complement those in [1, 5, 6, 14, 17, 19]. Using new arguments and some ideas inspired in [5] and [8], we obtain a new class of Trudinger-Moser inequalities for a more general class of operators.

(3) In view of (7), we can apply variational methods to study the existence and multiplicity of solutions for the following class of quasilinear elliptic

equations of the form $Lu + f(u) = 0$, $u > 0$, in $(0, R)$, $u(R) = 0$, involving critical growth of Trudinger-Moser type, that is, when $f(s)$ behaves like $e^{\mu s^{(\beta+2)/(\beta+1)}}$ at ∞ .

- (4) For articles involving problems for this class of operator and critical growth of Sobolev type, the reader is referred to the works of P. Clément et al. [6] and D. de Figueiredo et al. [7]. See also the works of J. Jacobsen and K. Schmitt [13, 14], where various results are presented for this class of operator under different conditions on parameters α , β and θ .
- (5) The following operators, when acting on radially symmetric functions, are simply special cases of the more general class defined by (4):

Operator	α	β	θ
Laplacian	$n - 1$	0	$n - 1$
p -Laplacian ($p \geq 2$)	$n - 1$	$p - 2$	$n - 1$
k -Hessian	$n - k$	$k - 1$	$n - 1$

In the *Trudinger-Moser case* (6) we have n -Laplacian, $n \geq 2$ and k -Hessian with $1 \leq k = n/2$.

2. SOME PRELIMINARY RESULTS

For each $\delta > 0$ and $p \geq 2$, we define $\mathcal{H}_\delta$ by

$$\mathcal{H}_\delta = \left\{ \psi : [0, \infty) \rightarrow \mathbb{R} \text{ measurable such that } \int_0^\infty |\psi(s)|^p ds \leq \delta \right\}.$$

Also, for all $c > 0$ fixed, we denote $L_{c,\delta}$ by

$$L_{c,\delta} = \sup_{\psi \in \mathcal{H}_\delta} \int_0^\infty \exp \left\{ c \int_0^t \psi(s) ds - t \right\} dt.$$

Let $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, $x > 0$, be the gamma Euler function and let $\Psi(x) = \Gamma'(x)/\Gamma(x)$ be the psi-function. We point out the following properties of the psi-function which can be found in [3, Theorem 1.2.5]:

(12)

$$\Psi(1) = -\gamma, \quad \Psi(x) - \Psi(1) = \sum_{k=0}^\infty \left(\frac{1}{k+1} - \frac{1}{x+k} \right) \quad \text{and} \quad \Psi'(x) = \sum_{k=0}^\infty \frac{1}{(x+k)^2},$$

where $\gamma := \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{1}{j} - \log(n) \right)$ is the Euler constant. In this way we obtain

Lemma 2.1. *For each $c > 0$, the following properties hold for the supremum $L_{c,\delta}$:*

- $L_{c,\delta}$ is finite.
- $L_{c,\delta}$ is attained by a non-negative function $\psi_0 \in \mathcal{H}_\delta$ with $\psi_0 \not\equiv 0$.
- $L_{c,\delta} \leq \exp \left\{ \left(\frac{p-1}{p} \right)^{p-1} \frac{c^p \delta}{p} + \Psi(p) + \gamma \right\}$.

Proof. a) By the definition of $L_{c,\delta}$, it is obvious that we may assume $\psi \geq 0$ since we can replace ψ by $|\psi|$ without decreasing the integral. For all $\psi \in \mathcal{H}_\delta$ and $A \geq 0$, for $t \geq A$ we have

$$\int_0^t \psi(s) ds \leq \delta^{1/p} A^{(p-1)/p} + \delta^{1/p} (t - A)^{(p-1)/p}.$$

Since $\int_0^t \psi(s) ds \leq \delta^{1/p} t^{(p-1)/p}$ and $\lim_{x \rightarrow \infty} x^{(p-1)/p}/x = 0$, choosing $N \geq 2^p c^p \delta + A$ we have that

$$c \int_0^t \psi(s) ds \leq c \delta^{1/p} A^{(p-1)/p} + (t - A)/2, \quad \text{for all } t \geq N.$$

Hence, for $\epsilon > 0$, choosing $N \geq \max\{2^p c^p \delta + A, -2 \log(\epsilon/2) - A + 2c \delta^{1/p} A^{(p-1)/p}\}$ we have

$$(13) \quad \int_N^\infty e^{c \int_0^t \psi(s) ds - t} dt \leq 2e^{c \delta^{1/p} A^{(p-1)/p} - A/2} e^{-N/2} \leq \epsilon \quad \text{for any } \psi \in \mathcal{H}_\delta.$$

On the other hand, for any $\psi \in \mathcal{H}_\delta$ it holds that $\int_0^t \psi(s) ds \leq \delta^{1/p} N^{(p-1)/p}$ if $t \leq N$. Therefore $L_{c,\delta}$ is finite.

b) For $\psi \in \mathcal{H}_\delta$, denote

$$I(\psi) = \int_0^\infty \exp \left\{ c \int_0^t \psi(s) ds - t \right\} dt \quad \text{and} \quad I_N(\psi) = \int_0^N \exp \left\{ c \int_0^t \psi(s) ds - t \right\} dt.$$

Let $(\psi_j) \subset \mathcal{H}_\delta$ be a sequence such that $I(\psi_j)$ converges to $L_{c,\delta}$. In particular, (ψ_j) is bounded in $L^p[0, \infty)$; thus up to a subsequence, we have $\psi_j \rightharpoonup \psi_0$ weakly in $L^p[0, \infty)$, and hence, by the weak lower semicontinuity of norm, we have $\psi_0 \in \mathcal{H}_\delta$. Consequently, for $N > 0$ we have $\psi_j|_{[0,N]} \rightharpoonup \psi_0|_{[0,N]}$ weakly in $L^p[0, N]$, and this implies $\lim_{j \rightarrow \infty} \int_0^t \psi_j(s) ds = \int_0^t \psi_0(s) ds$ for all t in $[0, N]$ (cf. [20, Theorem 11]). Given $\epsilon > 0$, choose N satisfying (13). Since $\int_0^t \psi_j(s) ds \leq \delta^{1/p} N^{(p-1)/p}$ on $[0, N]$, using Lebesgue's Convergence Theorem, we obtain $\lim_{j \rightarrow \infty} I_N(\psi_j) = I_N(\psi_0) \geq I_N(\psi_j) - \epsilon$, for j sufficiently large, which together with (13) implies $I(\psi_0) \geq I_N(\psi_0) \geq I_N(\psi_j) - \epsilon \geq I(\psi_j) - 2\epsilon$. Letting $j \rightarrow \infty$, we get $I(\psi_0) \geq L_{c,\delta} - 2\epsilon$ for all $\epsilon > 0$, which implies that $I(\psi_0) = L_{c,\delta}$. Taking $a > 0$ and $\psi_a : [0, \infty) \rightarrow \mathbb{R}$ given by $\psi_a(t) = a^{1/p}$ if $0 \leq t \leq t_a$, $\psi_a(t) = (a - d_a(t - t_a))^{1/p}$ if $t_a \leq t \leq t_a + a/d_a$ and $\psi_a(t) = 0$ if $t \geq t_a + a/d_a$, one can choose a , t_a and d_a such that $\psi_a \in \mathcal{H}_\delta$ and $I(\psi_a) > 1$. Therefore our extremal function ψ_0 is non-trivial since $I(0) = 1$.

c) The proof of this fact is analogous to [5, Lemma 1]. Here, we include the outline of the proof for completeness. First, note that

$$L_{c,\delta} = \sup_{\psi \in \mathcal{S}_\delta} \int_0^\infty e^{c \int_0^t \psi(s) ds - t} dt,$$

where $\mathcal{S}_\delta = \{\psi \in L^p[0, \infty) : \int_0^\infty |\psi|^p dt = \delta\}$. Let $\psi_0 \in \mathcal{H}_\delta$ be an extremal function ensured in b). We can suppose $\psi_0 \geq 0$ and $\psi_0 \not\equiv 0$. By the Lagrange multipliers theorem there exists a constant $\bar{\lambda} > 0$ such that, for all $\varphi \in L^p(0, \infty)$,

$$\int_0^\infty e^{c \int_0^t \psi_0(s) ds - t} \left(\int_0^t \varphi(s) ds \right) dt = \bar{\lambda} \int_0^\infty |\psi_0(t)|^{p-2} \psi_0(t) \varphi(t) dt.$$

In particular, for all $h \in C_0^1(0, \infty)$,

$$(14) \quad \int_0^\infty e^{c \int_0^t \psi_0(s) ds - t} h(t) dt = \bar{\lambda} \int_0^\infty |\psi_0(t)|^{p-2} \psi_0(t) h'(t) dt.$$

Since $e^{c \int_0^t \psi_0(s) ds - t} \in C[0, \infty)$, its indefinite integral $F(t) = \int_0^t e^{c \int_0^s \psi_0(\tau) d\tau - s} ds$ is a C^1 function in $[0, \infty)$ and $F'(t) = e^{c \int_0^t \psi_0(s) ds - t}$. By (14), since $h \in C_0^1(0, \infty)$,

$$\int_0^\infty (F + \bar{\lambda} |\psi_0|^{p-2} \psi_0) h' dt = \int_0^\infty (F h' + F' h) dt = \int_0^\infty \frac{d}{dt} (F(t) h(t)) dt = 0.$$

Thus, from the Dubois-Reymond Lemma we have $F + \bar{\lambda}|\psi_0|^{p-2}\psi_0 = C$ a.e. in $[0, \infty)$ and it follows that $\varphi = C - F$ is a C^1 function and satisfies $\varphi = \bar{\lambda}|\psi_0|^{p-2}\psi_0$ a.e. in $[0, \infty)$. Also, $\varphi' = -F' < 0$ on $[0, \infty)$. Since $\psi_0 \geq 0$ we have also that $\varphi \geq 0$ a.e. in $[0, \infty)$. If $\varphi(t_0) = 0$ for some $t_0 > 0$, then $\varphi(t) = \int_{t_0}^t \varphi'(s) ds < 0$ for all $t > t_0$ which contradicts $\varphi \geq 0$ a.e. in $[0, \infty)$. Therefore, $\varphi > 0$ and $\psi(t) = (\varphi/\bar{\lambda})^{\frac{1}{p-1}}(t)$ is a C^1 function with $\psi = \psi_0$ a.e. in $[0, \infty)$. Thus, we have $I(\psi) = I(\psi_0)$. Hence, we can take a C^1 extremal function ψ for $L_{c,\delta}$. Moreover, since $\varphi' = -F'$ we obtain

$$e^c \int_0^t \psi(s) ds - t = \lambda_0 \psi'(t) (\psi(t))^{p-2}$$

for some constant λ_0 . Setting $v(t) = c \int_0^t \psi(s) ds - t$, the above equation becomes

$$(15) \quad e^{v(t)} = \lambda v''(t) (1 + v'(t))^{p-2},$$

where $\lambda = \lambda_0 c^{1-p}$. Moreover, v satisfies $\int_0^\infty (1 + v'(t))^p dt = c^p \delta$ and

$$(16) \quad v(0) = 0, \quad v'(\infty) = -1 \quad \text{and} \quad v(\infty) = -\infty.$$

From (15), using integration by parts, we have

$$(17) \quad e^{v(t)} = \frac{\lambda}{p} (1 + v'(t))^p - \frac{\lambda}{p-1} (1 + v'(t))^{p-1} + C.$$

Using (16) and letting $t \rightarrow \infty$, we get $C = 0$. Hence combining the above equation with (15), we obtain

$$v''(t) = \frac{1}{p} (1 + v'(t))^2 - \frac{1}{p-1} (1 + v'(t)).$$

Integrating this last equation gives

$$(18) \quad 1 + v'(t) = \frac{p}{p-1} \left(1 + B e^{t/(p-1)} \right)^{-1},$$

where $B \geq 0$ is chosen such that

$$(19) \quad c^p \delta = \int_0^\infty (1 + v'(t))^p dt.$$

Taking into account (18) and (19), by performing a change-of-variables $u = 1 + B e^{t/(p-1)}$, we obtain

$$(20) \quad \begin{aligned} c^p \delta &= \left(\frac{p}{p-1} \right)^p (p-1) \int_{B+1}^\infty \frac{du}{(u-1)u^p} = \left(\frac{p}{p-1} \right)^p (p-1) \sum_{k=0}^\infty \frac{1}{(k+p)(B+1)^{k+p}} \\ &\geq \left(\frac{p}{p-1} \right)^p (p-1) [\log(1 + 1/B) - (\Psi(p) + \gamma)], \end{aligned}$$

where in this last estimate we use the properties of the psi-function pointed out in (12). On the other hand, taking $t = 0$ in (17) and using (16) and (18), we can write

$$(21) \quad 1 = \frac{\lambda}{p} \left(\frac{p}{p-1} \cdot \frac{1}{1+B} \right)^p - \frac{\lambda}{p-1} \left(\frac{p}{p-1} \cdot \frac{1}{1+B} \right)^{p-1}.$$

From (16) and (18), integrating (15) yields

$$\int_0^\infty e^{v(t)} dt = \frac{\lambda}{p-1} \int_0^\infty \frac{d}{dt} (1 + v'(t))^{p-1} dt = -\frac{\lambda}{p-1} \left(\frac{p}{p-1} \cdot \frac{1}{1+B} \right)^{p-1},$$

which together with (20) and (21) gives the desired estimate. $\square$

Remark 2.2. We point out that Lemma 2.1 improves Lemma 1 in [5], and therefore all of its consequences. For example, the next result is a sharpened version of Lemma 2 in [5]: Here the monotonicity hypothesis under w is not required and we give a new estimate in which the integer number n can be replaced by an arbitrary real number $p \geq 2$. More explicitly,

Lemma 2.3. *Let $p \geq 2$, $a > 0$ and $\delta > 0$ be real numbers. For each w non-negative C^1 function on $[0, \infty)$ satisfying $\int_a^\infty |w'(t)|^p dt \leq \delta$ we have*

$$\int_a^\infty e^{w^q(t)-t} dt \leq e^{w^q(a)-a} \frac{1}{1 - \delta^{1/(p-1)}} \exp \left\{ \left(\frac{p-1}{p} \right)^{p-1} \frac{c^p \gamma_p}{p} + \Psi(p) + \gamma \right\},$$

where $\gamma_p = \delta(1 - \delta^{1/(p-1)})^{1-p}$, $c = qw^{q-1}(a)$ and $q = p/(p-1)$.

Proof. Let w be a function under the above hypotheses. Set $x = t - a$ and $\psi(x) = w(x+a) - w(a)$; then $w(t) = \psi(x) + w(a) \geq 0$ for all $x \geq 0$. Note that if $1 < q \leq 2$ and $d \geq 0$, we have $(s+d)^q \leq d^q + qd^{q-1}s + |s|^q$ for all $s \geq -d$. Thus,

$$(22) \quad w(t)^q = (\psi(x) + w(a))^q \leq w(a)^q + qw(a)^{q-1}\psi(x) + |\psi(x)|^q.$$

Since $\psi(0) = 0$ and $\int_0^\infty |\psi'(x)|^p dx \leq \delta$, by Hölder it follows that $|\psi(x)|^q \leq \delta^{1/(p-1)}x$. Therefore, using estimate (22) we obtain

$$\int_a^\infty e^{w^q(t)-t} dt \leq e^{w^q(a)-a} \frac{1}{1 - \delta^{1/(p-1)}} \int_0^\infty e^{c\varphi(y)-y} dy,$$

where $y = (1 - \delta^{1/(p-1)})x$, $c = qw^{q-1}(a)$ and $\varphi(y) = \psi(x)$. Since

$$\int_0^\infty |\varphi'(y)|^p dy = \left(1 - \delta^{1/(p-1)}\right)^{1-p} \int_0^\infty |\psi'(x)|^p dx \leq \gamma_p,$$

the result follows by Lemma 2.1. □

3. PROOFS OF THEOREMS 1.1 AND 1.2

As we already mentioned, in view of reduction (8)-(9), Theorem 1.1 is an immediate consequence of Theorem 1.2. Let $p \geq 2$ and $q = p/(p-1)$ be the Hölder conjugate of p . First, we prove that for any $\rho > 0$ and $w \in \mathcal{K}$ we have

$$(23) \quad \int_0^\infty e^{\rho w^q(t)-t} dt < \infty.$$

Notice that given $\epsilon > 0$, there exists $T = T(\epsilon)$ such that $\int_T^\infty |w'(t)|^p dt < \epsilon$. Thus, the Hölder inequality implies

$$w(t) \leq w(T) + \epsilon^{1/p}(t-T)^{1/q}, \quad \text{for all } t \geq T.$$

Hence, $\lim_{t \rightarrow \infty} w(t)/t^{1/q} = 0$. Thus, we have $\rho w^q(t) < t/2$, for t sufficiently large, which implies that (23) holds. Next, we show that (23) is uniformly bounded for $w \in \mathcal{K}$ under the condition $\rho \leq 1$. If $\rho < 1$ it is easier. Indeed, in this case, for any $w \in \mathcal{K}$ we have

$$w(t) = \int_0^t w'(s) ds \leq \left(\int_0^t |w'(s)|^p ds \right)^{1/p} t^{1/q} \leq t^{1/q};$$

thus

$$\int_0^\infty e^{\rho w^q(t)-t} dt \leq \int_0^\infty e^{(\rho-1)t} dt \leq 1/(1-\rho).$$

Now, suppose that $\rho = 1$ and $w \in \mathcal{K}$. We have two cases to analyze:

Case (a): $w^q(t) \leq t - 2 \ln^+ t$ for all $t \in [1, \infty)$. In this case we have

$$\int_0^\infty e^{w^q(t)-t} dt \leq \int_0^\infty e^{-2 \ln^+ t} dt \leq 2.$$

Case (b): There exists $a \in [1, \infty)$, the smallest number, such that $w^q(a) = a - 2 \ln^+ a$. We set $\delta = \int_a^\infty |w'(t)|^p dt$. Using Lemma 2.3, we obtain

$$(24) \quad \int_a^\infty e^{w^q(t)-t} dt \leq \frac{1}{1 - \delta^{1/(p-1)}} e^{K + \Psi(p) + \gamma},$$

where $K = w^q(a) \left(1 + \frac{1}{p-1} \frac{\delta}{(1 - \delta^{1/(p-1)})^{p-1}}\right) - a$. Also,

$$w^q(a) \leq \left(\int_0^a |w'(s)|^p ds \right)^{q/p} a \leq (1 - \delta)^{1/(p-1)} a,$$

which implies

$$(25) \quad \delta \leq 1 - \left(1 - \frac{2 \ln^+ a}{a}\right)^{p-1} \leq (p-1) \frac{2 \ln^+ a}{a}$$

because $w^q(a) = a - 2 \ln^+ a$ and $1 - t^d \leq d(1 - t)$, $t \geq 0$ for any $d \geq 1$. Note that $K = a(1 - \delta^{1/(p-1)})^{1-p} F(a)$, where

$$F(a) = \frac{\delta}{p-1} - \frac{\delta}{p-1} \frac{2 \ln^+ a}{a} - \frac{2 \ln^+ a}{a} \left(1 - \delta^{\frac{1}{p-1}}\right)^{p-1}.$$

Since $1 - \left(1 - \delta^{\frac{1}{p-1}}\right)^{p-1} \leq (p-1) \delta^{\frac{1}{p-1}}$, (25) yields

$$F(a) \leq \frac{2 \ln^+ a}{a} \left[1 - \left(1 - \delta^{\frac{1}{p-1}}\right)^{p-1} - \frac{\delta}{p-1}\right] \leq (p-1)^{\frac{p}{p-1}} \left(\frac{2 \ln^+ a}{a}\right)^{\frac{p}{p-1}}.$$

Thus, it follows that

$$(26) \quad K \leq \frac{(p-1)^q}{(1 - \delta^{1/(p-1)})^{p-1}} \cdot a \left(\frac{2 \ln^+ a}{a}\right)^q.$$

Since $e \ln a \leq a$ for all $a > 0$, using (25) we get $\delta \leq 1 - \left(1 - \frac{2}{e}\right)^{p-1} := d_p$; then

$$(27) \quad \frac{1}{1 - \delta^{1/(p-1)}} \leq \frac{1}{1 - d_p^{1/(p-1)}}.$$

Since $g(x) = x \left(\frac{\ln x}{x}\right)^q$ is uniformly bounded on $[1, \infty)$ by some constant depending only on p , by estimates (24), (26) and (27) we conclude that $\int_a^\infty e^{w^q(t)-t} dt \leq c_1$. This estimate and the inequality $\int_0^a e^{w^q(t)-t} dt \leq 2$ shows that (23) is uniformly bounded for $w \in \mathcal{K}$ when $\rho \leq 1$.

Finally we verify that $\mu_{\alpha, \theta}$ in Theorem 1.1 is optimal or, equivalently, $\rho = 1$ is optimal in Theorem 1.2. For this, we consider Moser's functions:

$$w_j(t) = \begin{cases} \frac{t}{j^{1/p}} & \text{if } 0 \leq t \leq j, \\ j^{1/q} & \text{if } t \geq j. \end{cases}$$

It is clear that $\int_0^\infty |w'_j(t)|^p dt = 1$ and $\int_0^\infty e^{\rho w_j^q(t)-t} dt \geq e^{(\rho-1)j} \rightarrow \infty$ if $\rho > 1$.

4. PROOF OF THEOREM 1.3

As we already mentioned, because of the reduction (8)-(9), it is enough to show that there exists an extremal for (10) when $\rho \leq 1$. For $\rho < 1$, we can use similar arguments to those in (i) below. Indeed, assertion (28) below is almost immediate and it implies the existence of extremals. Assume $\rho = 1$ and let $q = p/(p-1)$ be the Hölder conjugate of $p \geq 2$. We set

$$J(w) = \int_0^\infty e^{w^q(t)-t} dt, \quad \forall w \in \mathcal{K} \quad \text{and} \quad \mathcal{I}_p := \sup_{w \in \mathcal{K}} J(w),$$

where $\mathcal{K}$ was defined in Theorem 1.2. In order to prove the existence of extremal (10) we will show that $\mathcal{I}_p$ is attained by some $w_0 \in \mathcal{K}$. For this, let γ and $\Psi(p)$ be as in (12). Here, we use the two-step strategy of Carleson and Chang (cf. [5]):

- Step 1: Assuming has no extremal for the supreme in $\mathcal{I}_p$, then we prove that $\mathcal{I}_p \leq 1 + e^{\Psi(p)+\gamma}$.
- Step 2: For each $p \geq 2$, we find a specific function $f_p \in \mathcal{K}$ such that $J(f_p) > 1 + e^{\Psi(p)+\gamma}$, which is a contraction.

To verify Step 1, we use Lemma 2.3. Let $(w_j) \subset \mathcal{K}$ be a sequence such that $J(w_j)$ converges to $\mathcal{I}_p$. Assuming that there is no extremal for $\mathcal{I}_p$, we claim that, up to subsequence, (w_j) satisfies:

- (i) For each $A > 0$, $\int_0^A |w'_j(t)|^p dt \rightarrow 0$ as $j \rightarrow \infty$.
- (ii) There exists a_j in $[1, \infty)$ such that $w_j^q(a_j) = a_j - 2 \ln a_j$ and $\lim_{j \rightarrow \infty} a_j = \infty$.
- (iii) $\limsup_j \int_{a_j}^\infty e^{w_j^q(t)-t} dt \leq e^{\Psi(p)+\gamma}$.
- (iv) $\lim_j \int_0^{a_j} e^{w_j^q(t)-t} dt = 1$.

It is clear that the Step 1 follows by (iii) and (iv).

(i). We argue by contradiction. Suppose that (i) is false. In other words, there exist constants $A > 0$ and $0 < \delta < 1$ such that

$$\int_0^A |w'_j(t)|^p dt \geq \delta \quad \text{for} \quad j \geq j_0.$$

Then $\int_A^\infty |w'_j(t)|^p dt \leq 1 - \delta$, which implies $w_j(t) - w_j(A) \leq (1 - \delta)^{1/p}(t - A)^{1/q}$, $t \geq A$. By elementary calculus, for each $a \geq 0$, $1 < p \leq 2$ and $0 < \kappa < 1$, we have

$$(a + x)^p \leq a^p + pa^{p-1}x + x^p, \quad x \geq 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{x^\kappa}{x} = 0.$$

Thus, for $\epsilon > 0$ we can choose $N > A$ such that $w_j(t)^q \leq (1 - \delta/2)^{1/(p-1)}(t - A) + qA$, $t > N$, which implies

$$(28) \quad \int_N^\infty e^{w_j^q(t)-t} dt \leq e^{qA} \frac{1}{1 - (1 - \delta/2)^{1/(p-1)}} e^{((1 - \delta/2)^{1/(p-1)} - 1)N} \leq \epsilon$$

for N is sufficiently large and $j \geq j_0$. As such, as in item b) of Lemma 2.1 and using [20, Theorem 11], up to a subsequence, we can assume that w_j converges pointwise to $w_0 \in \mathcal{K}$. For $0 \leq t \leq N$ one has $w_j(t) \leq N^{1/q}$. Then by Lebesgue's Convergence Theorem

$$\mathcal{J}_N(w_0) := \int_0^N e^{w_0^q(t)-t} dt = \lim_j \int_0^N e^{w_j^q(t)-t} dt = \lim_j \mathcal{J}_N(w_j) \geq \mathcal{J}_N(w_j) - \epsilon$$

when j is large. By (28) it follows that $J(w_0) \geq \mathcal{J}_N(w_0) \geq \mathcal{J}_N(w_j) - \epsilon \geq J(w_j) - 2\epsilon$. For $j \rightarrow \infty$ we get $J(w_0) \geq \mathcal{I}_p - 2\epsilon$ for all $\epsilon > 0$. Then we obtain $J(w_0) = \mathcal{I}_p$. But this leads to a contradiction with the assumption of Step 1, and so (i) holds.

(ii). We first claim that, for j sufficiently large, there exists $a_j \in [1, \infty)$ satisfying $w_j^q(a_j) = a_j - 2 \ln^+ a_j$. Since $w_j^q(t) \leq t - 2 \ln^+ t = t$ for all $t \in [0, 1]$, if we assume that our claim is false, we have that $w_j^q(t) < t - 2 \ln^+ t$ for all $t \in [1, \infty)$. Thus,

$$J(w_j) = \int_0^\infty e^{w_j^q(t)-t} dt \leq 1 + \int_1^\infty e^{-2 \ln^+ t} dt = 2,$$

which implies that $\mathcal{I}_p \leq \limsup_j J(w_j) \leq 2$. This contradicts Step 2.

Now, by (i), given $A > 0$, we can choose $j_0 = j_0(A)$ such that

$$\left(\int_0^A |w_j'(s)|^p ds \right)^{1/(p-1)} < \frac{e-2}{e} \quad \text{for all } j \geq j_0.$$

Thus, by Hölder's inequality and since $\frac{e-2}{e}t \leq t - 2 \ln^+ t$ for all $t \in [0, \infty)$,

$$w_j^q(t) \leq \left(\int_0^A |w_j'(s)|^p ds \right)^{1/(p-1)} t < \frac{e-2}{e}t \leq t - 2 \ln^+ t,$$

for all $0 \leq t \leq A$ and $j \geq j_0$, which implies that $a_j \geq A$ for all $j \geq j_0$. Therefore, $\lim_{j \rightarrow \infty} a_j = \infty$.

(iii). This follows from Lemma 2.3. Indeed, choosing $w = w_j$, $a = a_j$, $\delta = \delta_j = \int_{a_j}^\infty |w_j'(t)|^p dt$ and applying Lemma 2.3, we obtain

$$\int_{a_j}^\infty e^{w_j^q(t)-t} dt \leq \frac{1}{1 - \delta_j^{1/(p-1)}} e^{K_j + \Psi(p) + \gamma},$$

where

$$K_j = w_j^q(a_j) \left[1 + \frac{\delta_j}{(p-1)(1 - \delta_j^{1/(p-1)})^{p-1}} \right] - a_j.$$

Thus, the proof of (iii) is completed by showing that $\delta_j \rightarrow 0$ and $K_j \rightarrow 0$ as $j \rightarrow \infty$. Since $w_j \in \mathcal{K}$ we have $w_j^q(a_j) \leq (1 - \delta_j)^{1/(p-1)} a_j$, which implies $\delta_j \rightarrow 0$, because

$$\delta_j \leq 1 - \left(1 - \frac{2 \ln^+ a_j}{a_j} \right)^{p-1} \leq (p-1) \frac{2 \ln^+ a_j}{a_j} \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

since $a_j \rightarrow \infty$ by (ii). We also have, for all $j \geq j_0$,

$$K_j \leq w_j^q(a_j) \left[1 + \frac{\delta_j}{p-1} + \frac{2p\delta_j^{p/(p-1)}}{p-1} \right] - a_j \leq 2(p-1)^q a_j \left(\frac{2 \ln^+ a_j}{a_j} \right)^q,$$

which implies $K_j \rightarrow 0$ as $j \rightarrow \infty$.

(iv). It follows from (i) that $w_j \rightarrow 0$ uniformly on compact sets. Thus, given $\epsilon > 0$ and $A > 0$, we have $w_j(t)^q \leq \epsilon$ for all $0 \leq t \leq A$ and j is sufficiently large.

Since a_j is the smallest number such that $w_j(t)^q \geq t - 2 \ln^+ t$, we obtain

$$\int_0^{a_j} e^{w_j(t)^q - t} dt = \int_0^A e^{w_j(t)^q - t} dt + \int_A^{a_j} e^{w_j(t)^q - t} dt \leq e^\epsilon \left(1 - \frac{1}{e^A}\right) + \left(\frac{1}{A} - \frac{1}{a_j}\right) \leq 1,$$

for ϵ sufficiently small and A sufficiently large. On the other hand,

$$\int_0^{a_j} e^{w_j(t)^q - t} dt \geq \int_0^{a_j} e^{-t} dt = 1 - e^{-a_j} \rightarrow 1 \quad \text{as } j \rightarrow \infty,$$

which completes the proof of (iv) and Step 1 of the theorem.

The proof of the Step 2 consists in the construction of a function $f_p \in \mathcal{K}$ such that

$$f_p(t) = \begin{cases} \frac{1}{q}(p-1)^{-1/p}t & \text{if } 0 \leq t \leq p, \\ (t-1)^{1/q} & \text{if } p \leq t \leq N_p, \\ (N_p-1)^{1/q} & \text{if } t \geq N_p, \end{cases}$$

where N_p was chosen such that $\int_0^\infty f_p'(t)^p dt = 1$. Explicitly, $N_p = (p-1)e^{q^p-q} + 1$. It is easy to see that

$$J(f_p) = \int_0^p \exp\left\{q^{-q}(p-1)^{-1/(p-1)}t^q - t\right\} dt + (p-1)e^{q^p-q-1} + \frac{2-p}{e}.$$

Considering the change of variable $v = t/p$, we have

$$J(f_p) = p \int_0^1 e^{h(v)} dv + (p-1)e^{q^p-q-1} + \frac{2-p}{e},$$

where $h(v) = (p-1)v^q - pv$. Clearly, $h(v) > -pv$ on $(0, 1/p]$ and $h(v) \geq -1$ on $[1/p, 1]$. Thus,

$$p \int_0^1 e^{h(v)} dv > p \int_0^{1/p} e^{-pv} dv + \frac{e}{p} \int_{1/p}^1 dv = 1 - \frac{2-p}{e}.$$

Hence $J(f_p) > 1 + (p-1)e^{q^p-q-1}$. The proof is completed by showing that $(p-1)e^{q^p-q-1} \geq e^{\Psi(p)+\gamma}$, $p \geq 2$. To see this fact consider the auxiliary function $\xi : [2, \infty) \rightarrow \mathbb{R}$ defined by $\xi(p) = \Psi(p) + \gamma - \log(p-1) - q^p + q + 1$. Note that $\xi(2) = 0$ and $\xi'(p) \leq 0$, for all $p \in (2, \infty)$. Indeed, from (12),

$$\Psi'(p) = \sum_{k=0}^{\infty} \frac{1}{(k+p)^2} \leq \sum_{k=0}^{\infty} \frac{1}{(k+p-\frac{1}{2})(k+p+\frac{1}{2})} = \frac{1}{(p-\frac{1}{2})}$$

and $\frac{d}{dp}(q^p - q) = q^p(\log(q) - q + 1) + (q-1)^2$. Finally, after subtle estimates of the terms q^p and $\log q$ we get $1/(p-1/2) \leq q^p(\log(q) - q + 1) + (q-1)^2 + (q-1)$, which completes the proof of Theorem 1.3.

5. PROOF OF THEOREM 1.4

Let $u \in X_\infty$ be a C^1 function, and set $r = e^{-\frac{t}{\theta+1}}$ and $w(t) = \omega_{p-1}^{\frac{1}{p}}(\theta+1)^{\frac{1}{q}}u(r)$ with $q = p/(p-1)$. Then $w(t)$ is defined on the interval $(-\infty, \infty)$ and satisfies

$$(29) \quad \lim_{t \rightarrow -\infty} w(t) = 0.$$

Taking into account that $p = \beta + 2$ and (6), we obtain

$$(30) \quad \omega_\alpha \int_0^\infty r^\alpha |u'(r)|^p dr = \int_{-\infty}^\infty |w'(t)|^p dt,$$

$$(31) \quad \omega_\theta \int_0^\infty r^\theta |u(r)|^p dr = \frac{\omega_\theta}{\omega_\alpha(1+\theta)^p} \int_{-\infty}^\infty |w(t)|^p e^{-t} dt$$

and

$$(32) \quad \int_0^\infty A_p(\mu |u(r)|^q) r^\theta dr = \frac{1}{\theta+1} \int_{-\infty}^\infty A_p\left(\frac{\mu}{\mu_{\alpha,\theta}} |w(t)|^q\right) e^{-t} dt.$$

Therefore, the proof of Theorem 1.4 is completed by showing that, for each $0 < \rho < 1$, there exists a constant $C_\rho > 0$ such that

$$(33) \quad \int_{-\infty}^\infty A_p(\rho |w(t)|^q) e^{-t} dt \leq C_\rho \int_{-\infty}^\infty |w(t)|^p e^{-t} dt,$$

for all C^1 functions $w(t)$ satisfying (29) and such that $\int_{-\infty}^\infty |w'(t)|^p dt = 1$. To this end, consider

$$A = \{t \in \mathbb{R} : |w(t)| \leq 1\} \quad \text{and} \quad B = \{t \in \mathbb{R} : |w(t)| > 1\}.$$

Clearly A and B are disjoint sets with $A \cup B = (-\infty, \infty)$, and by (29) we have $A \neq \emptyset$. Since $A_p(\zeta) \leq m_p \zeta^{[p]}$ when $0 \leq \zeta \leq 1$, for some $m_p > 0$, we have

$$(34) \quad \int_A A_p(\rho |w(t)|^q) e^{-t} dt \leq m_p \int_{-\infty}^\infty |w(t)|^p e^{-t} dt.$$

By (29), the set B has a lower bound and we can write $B = \bigcup_{i \in \mathbb{N}} I_i$ with $I_i \cap I_j = \emptyset$ if $i \neq j$, $I_i = (a_i, b_i)$, $a_i \in A$ and $w(a_i) = 1$. Thus, for each $i \in \mathbb{N}$, we obtain

$$(35) \quad |w(t)| \leq 1 + \int_{a_i}^t |w'(s)| ds \leq 1 + (t - a_i)^{1/q} \quad \text{for all } t \in I_i.$$

Note that given $\varepsilon > 0$, there exists C_ε such that $1 + s^{1/q} \leq ((1 + \varepsilon)s + C_\varepsilon)^{1/q}$ for all $s \geq 0$, which together with (35) implies $|w(t)|^q \leq (1 + \varepsilon)(t - a_i) + C_\varepsilon$ for all $t \in I_i$. Since $0 < \rho < 1$, taking $\varepsilon > 0$ so that $\rho(1 + \varepsilon) < 1$, we can estimate

$$\begin{aligned} \int_{a_i}^{b_i} A_p(\rho |w(t)|^q) e^{-t} dt &\leq \int_{a_i}^{b_i} e^{\rho |w(t)|^q - t} dt \leq \int_{a_i}^{b_i} \exp\{(\rho(1 + \varepsilon) - 1)(t - a_i) + \rho C_\varepsilon - a_i\} dt \\ &= \frac{e^{\rho C_\varepsilon}}{1 - (\rho(1 + \varepsilon))} (e^{-a_i} - e^{-b_i}) f(b_i - a_i), \end{aligned}$$

where $f(x) = \frac{e^x - e^{\rho(1 + \varepsilon)x}}{e^x - 1}$, $x \in \mathbb{R}$ and $f(\infty) = 1$ if $b_i = \infty$. Since $|f(x)| \leq 1$ for all $x \in \mathbb{R}$, we have

$$(36) \quad \int_{a_i}^{b_i} A_p(\rho |w(t)|^q) e^{-t} dt \leq \frac{e^{\rho C_\varepsilon}}{1 - (\rho(1 + \varepsilon))} (e^{-a_i} - e^{-b_i}).$$

Now, $|w| > 1$ on B , which implies $\int_{a_i}^{b_i} |w(t)|^p e^{-t} dt \geq \int_{a_i}^{b_i} e^{-t} dt = (e^{-a_i} - e^{-b_i})$. Hence, by (36) we get

$$\sum_{i=1}^\infty \int_{a_i}^{b_i} A_p(\rho |w(t)|^q) e^{-t} dt \leq \frac{e^{\rho C_\varepsilon}}{1 - (\rho(1 + \varepsilon))} \sum_{i=1}^\infty \int_{a_i}^{b_i} |w(t)|^p e^{-t} dt.$$

Therefore, by the monotone convergence theorem,

$$(37) \quad \int_B A_p(\rho|w(t)|^q) e^{-t} dt \leq \frac{e^{\rho C_\varepsilon}}{1 - (\rho(1+\varepsilon))} \int_B |w(t)|^p e^{-t} dt \leq \frac{e^{\rho C_\varepsilon}}{1 - (\rho(1+\varepsilon))} \int_{-\infty}^{\infty} |w(t)|^p e^{-t} dt.$$

Thus, combining (34) and (37), we have (33) for $C_\rho \geq m_p + e^{\rho C_\varepsilon} / (1 - \rho(1+\varepsilon))$, which completes the proof.

6. PROOF OF THEOREM 1.5

We consider Moser's sequence $w_j(t)$ given by $w_j(t) = 0$ if $t \leq 0$, $w_j(t) = j^{-1/p}t$ if $0 \leq t \leq j$ and $w_j(t) = j^{1/q}$ if $t \geq j$. It is clear that

$$(38) \quad \lim_{t \rightarrow -\infty} w_j(t) = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} |w'_j(t)|^p dt = 1.$$

We also have

$$(39) \quad \int_{-\infty}^{\infty} |w_j(t)|^p e^{-t} dt \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.$$

Indeed,

$$\int_{-\infty}^{\infty} |w_j(t)|^p e^{-t} dt = \frac{1}{j} \int_0^j t^p e^{-t} dt + j^{p-1} e^{-j} \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty.$$

A straightforward calculation yields

$$\begin{aligned} \int_{-\infty}^{\infty} A_p(|w_j(t)|^q) e^{-t} dt &= \int_0^j A_p\left(j\left(\frac{t}{j}\right)^q\right) e^{-t} dt + \int_j^{\infty} A_p(j) e^{-t} dt \\ &= \int_0^j \exp\left(j\left(\frac{t}{j}\right)^q - t\right) dt - \sum_{m=0}^{\lfloor p \rfloor} \frac{1}{m!} j^{-\frac{m}{p-1}} \int_0^j t^{mq} e^{-t} dt \\ &\quad + \left(e^j - \sum_{m=0}^{\lfloor p \rfloor} \frac{j^m}{m!}\right) e^{-j} \\ &\geq \int_0^j e^{-t} dt - \sum_{m=0}^{\lfloor p \rfloor} \frac{j^{-\frac{m}{p-1}}}{m!} \int_0^j t^{mq} e^{-t} dt + \left(e^j - \sum_{m=0}^{\lfloor p \rfloor} \frac{j^m}{m!}\right) e^{-j}. \end{aligned}$$

Thus, for j enough large,

$$(40) \quad \int_{-\infty}^{\infty} A_p(|w_j(t)|^q) e^{-t} dt \geq \frac{1}{2}.$$

The proof of Theorem 1.5 follows by using an argument similar to the one in the proof of Theorem 1.4 combined with the estimates (38)-(40).

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