FUSION PROCEDURE FOR COXETER GROUPS OF TYPE B AND COMPLEX REFLECTION GROUPS $G(m, 1, n)$

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ABSTRACT. A complete system of primitive pairwise orthogonal idempotents for the Coxeter groups of type B and, more generally, for the complex reflection groups $G(m, 1, n)$ is constructed by a sequence of evaluations of a rational function in several variables with values in the group ring. The evaluations correspond to the eigenvalues of the two arrays of Jucys–Murphy elements.

1. INTRODUCTION

A. Jucys [13] gave a construction of a complete system of pairwise orthogonal primitive idempotents of the group ring of the symmetric group. The construction, now called the *fusion procedure*, involves a rational function in several variables, and the idempotents are obtained by taking certain limiting values of this function. We refer to [1, 3, 5, 10, 16, 17, 19] for different aspects and applications of the fusion procedure for the symmetric groups. There are analogues of the fusion procedure, in the spirit of [1], for the Coxeter group of type B [2], the Hecke algebra of type A [4, 18] and the spinor extension of the symmetric group [11, 15] (see [12] for its q -analogue).

A version of the fusion procedure for the symmetric group was given by A. Molev in [14]. Here the idempotents are obtained by consecutive evaluations of the rational function. An analogue of this fusion procedure was developed for the Hecke algebra [7], the Brauer algebra [6, 8] and the Birman–Murakami–Wenzl algebra [9].

The aim of this paper is to give a fusion procedure, in the spirit of [14], for the complex reflection groups $G(m, 1, n)$. As in [14], and later [6–9], we use the Jucys–Murphy elements. They were introduced for $G(m, 1, n)$ independently in [22] and [24]. The Jucys–Murphy elements of $G(m, 1, n)$ form two arrays, j_i and $\tilde{j}_i$, $i = 1, \dots, n$, and their union is the maximal commutative set in $\mathbb{C}G(m, 1, n)$; see [21, 22]. An irreducible representation of the group $G(m, 1, n)$ is coded by an m -tuple of partitions, and its basis vectors correspond to standard m -tuples of tableaux. The eigenvalues of j_i carry information about the “position” - the place of a tableau in the m -tuple - while the eigenvalues of $\tilde{j}_i$ are related to the classical contents of nodes. In the work [21] both sets appeared as classical limits of simple expressions involving the single set of the Jucys–Murphy elements of the cyclotomic Hecke algebra, the flat deformation of $\mathbb{C}G(m, 1, n)$. By the maximality,

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all diagonal matrix units of $\mathbb{C}G(m, 1, n)$ can be expressed in terms of the Jucys–Murphy elements j_i and $\tilde{j}_i$, $i = 1, \dots, n$. Then we translate this expression as a fusion procedure: any diagonal matrix unit can be obtained by a sequence of evaluations of a certain rational function with values in $\mathbb{C}G(m, 1, n)$. The arrays j_i and $\tilde{j}_i$ play different roles: the positions can be evaluated simultaneously, while the contents should then be evaluated consequently from 1 to n .

The group $G(1, 1, n)$ is isomorphic to the symmetric group S_n and our fusion procedure for $m = 1$ reproduces the fusion procedure of [14].

The group $G(2, 1, n)$ is isomorphic to the hyperoctahedral group B_n , the Coxeter group of type B . Thus, in particular, we obtain a fusion procedure for the Coxeter group of type B and a description of a complete set of pairwise orthogonal primitive idempotents of B_n in terms of a single rational function with values in the group algebra $\mathbb{C}B_n$. A version of the fusion procedure for type B was developed in [2]. Our fusion procedure is different from the one in [2] in that it uses another way to Baxterize the generators of the symmetric group.

For clarity of the exposition, we first describe the fusion procedure for the Coxeter group B_n . This is done not only for aesthetic reasons: the rational function with values in $\mathbb{C}B_n$ leading to the complete set of idempotents can be viewed as a word for the longest element of B_n in which certain entries are “Baxterized”, similarly to the rational function for type A . Although $G(m, 1, n)$ is not a Coxeter group for $m > 2$ and the notion of length of an element is not defined, there is an analogue of the longest element: it is longest with respect to the normal form of [21] (the classical limit of the normal form for the cyclotomic Hecke algebra $H(m, 1, n)$ [20]); the rational function with values in $\mathbb{C}G(m, 1, n)$ leading to the complete set of idempotents can again be viewed as a word for the longest element with certain entries Baxterized.

We stress two features of our construction. First, the groups $G(m, 1, n)$ admit a fusion procedure for any positive integer m , and our construction is uniform for all m . Second, the new phenomenon - compared to the symmetric group - appears: for $m > 1$ only some well-chosen generators (appearing in the expression for the analogue of the longest element) are Baxterized.

The paper is organized as follows. Section 2 contains necessary definitions and notation about the groups B_n and their representations. The Jucys–Murphy elements for B_n were defined in [22–24]; we describe the diagonal matrix units in terms of them. In Section 3 we prove Theorem 3.2 which gives the fusion procedure for the groups B_n . In Section 4, in Theorem 4.2, we extend the results to the general case of the groups $G(m, 1, n)$. As the proofs work mainly along the same lines as for B_n , we only indicate the necessary modifications.

2. IDEMPOTENTS AND JUCYS–MURPHY ELEMENTS OF THE GROUP B_n

2.1. Definitions. The Coxeter group S_{n+1} of type A (the symmetric group on $n + 1$ letters) is generated by the elements $s_1, \dots, s_n$ with the defining relations:

$$\begin{aligned}
 (2.1) \quad & s_i^2 = 1 && \text{for } i = 1, \dots, n, \\
 & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} && \text{for } i = 1, \dots, n-1, \\
 & s_i s_j = s_j s_i && \text{for } i, j = 1, \dots, n \text{ such that } |i - j| > 1.
 \end{aligned}$$

The Coxeter group B_{n+1} of type B (also called the hyperoctahedral group) is generated by the elements $s_1, \dots, s_n$ and t with the defining relations (2.1);

$$(2.2) \quad \begin{aligned} ts_1ts_1 &= s_1ts_1t, \\ s_it &= ts_i \quad \text{for } i = 2, \dots, n; \end{aligned}$$

and

$$(2.3) \quad t^2 = 1.$$

For any $i = 1, \dots, n$, set

$$(2.4) \quad s_i(p, p', a, a') := s_i + \frac{\delta_{p,p'}}{a - a'},$$

where $\delta_{p,p'}$ is the Kronecker delta. For $p = p'$ the elements (2.4) are called Baxterized elements; the parameters a and a' are referred to as spectral parameters. We also define

$$(2.5) \quad \mathbf{t}(p) := \frac{1}{2}(1 + pt).$$

The following relation is satisfied and will be used later:

$$(2.6) \quad s_i(p, p', a, a')s_i(p', p, a', a) = \frac{(a - a')^2 - \delta_{p,p'}}{(a - a')^2} \quad \text{for } i = 1, \dots, n.$$

Define the elements j_i , $i = 1, \dots, n + 1$, and $\tilde{j}_i$, $i = 1, \dots, n + 1$, of the group algebra $\mathbb{C}B_{n+1}$ by the following initial conditions and recursions:

$$(2.7) \quad j_1 = t, \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0, \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{2}(s_i + j_i s_i j_i).$$

The elements j_i and $\tilde{j}_i$ are analogues, for the group B_{n+1} , of the Jucys–Murphy elements. The elements j_i , $i = 1, \dots, n + 1$, and $\tilde{j}_i$, $i = 1, \dots, n + 1$, form a maximal commutative set in $\mathbb{C}B_{n+1}$ (see [21–23]); in addition, j_i and $\tilde{j}_i$ commute with all generators s_k , except s_i and s_{i-1} :

$$(2.8) \quad j_i s_k = s_k j_i \quad \text{and} \quad \tilde{j}_i s_k = s_k \tilde{j}_i \quad \text{if } k \neq i - 1, i.$$

Let $\lambda \vdash n + 1$ be a partition of $n + 1$, that is, $\lambda = (\lambda_1, \dots, \lambda_k)$, where λ_j , $j = 1, \dots, k$, are positive integers, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $n + 1 = \lambda_1 + \dots + \lambda_k$. We identify partitions with their Young diagrams: the Young diagram of λ is a left-justified array of rows of nodes containing λ_j nodes in the j -th row, $j = 1, \dots, k$; the rows are numbered from top to bottom.

A 2-partition, or a Young 2-diagram, of $n + 1$ is a pair of partitions such that the total number of boxes in the associated Young diagrams is equal to $n + 1$; e.g. the Young 2-diagram $(\square\square, \square)$ represents the 2-partition $((2), (1))$ of 3.

A 2-node $\alpha^{(2)}$ is a pair (α, k) consisting of a usual node α and an integer $k = 1, 2$. The integer k will be called the *position* of the 2-node. A set of 2-nodes can be equivalently described by an ordered pair of sets of nodes (the integer k of a 2-node (α, k) indicates to which set the node α belongs). A 2-partition $\lambda^{(2)}$ is a set of 2-nodes such that the subset consisting of the 2-nodes having position p is a usual partition, $p = 1, 2$.

For a 2-node $\alpha^{(2)} = (\alpha, k)$ lying in the line x and the column y of the k -th diagram, we denote by $c(\alpha^{(2)})$ the classical content of the node α , $c(\alpha^{(2)}) := c(\alpha) = y - x$. Let $\{\xi_1, \xi_2\}$ be the set of distinct square roots of unity, ordered arbitrarily; we also define $p(\alpha^{(2)}) := \xi_k$.

For a 2-partition $\lambda^{(2)}$, a 2-node $\alpha^{(2)}$ of $\lambda^{(2)}$ is called *removable* if the set of 2-nodes obtained from $\lambda^{(2)}$ by removing $\alpha^{(2)}$ is still a 2-partition. A 2-node $\beta^{(2)}$ not in $\lambda^{(2)}$ is called *addable* if the set of 2-nodes obtained from $\lambda^{(2)}$ by adding $\beta^{(2)}$ is still a 2-partition. For a 2-partition $\lambda^{(2)}$, we denote by $\mathcal{E}_-(\lambda^{(2)})$ the set of removable 2-nodes of $\lambda^{(2)}$ and by $\mathcal{E}_+(\lambda^{(2)})$ the set of addable 2-nodes of $\lambda^{(2)}$. For example, the removable/addable 2-nodes (marked with $-/+$) for the 2-partition $(\square\square, \square)$ are

$$\left(\begin{array}{|c|c|c|} \hline \square & - & + \\ \hline + & & \\ \hline \end{array}, \begin{array}{|c|c|} \hline - & + \\ \hline + & \\ \hline \end{array} \right).$$

Let $\lambda^{(2)}$ be a 2-partition of $n+1$. A standard 2-tableau of shape $\lambda^{(2)}$ is obtained by placing the numbers $1, \dots, n+1$ in the 2-nodes of the diagrams of $\lambda^{(2)}$ in such a way that the numbers in the nodes ascend along rows and columns in every diagram. For a standard 2-tableau $\mathcal{T}$ of shape $\lambda^{(2)}$ let $\alpha_i^{(2)}$ be the 2-node of $\mathcal{T}$ with number i , $i = 1, \dots, n+1$; we set $c(\mathcal{T}|i) := c(\alpha_i^{(2)})$ and $p(\mathcal{T}|i) := p(\alpha_i^{(2)})$. For example, take the standard 2-tableau $\mathcal{T} = (\begin{array}{|c|c|} \hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|} \hline 2 \\ \hline \end{array})$; then we have

$$c(\mathcal{T}|1) = 0, \quad c(\mathcal{T}|2) = 0, \quad c(\mathcal{T}|3) = 1 \quad \text{and} \quad p(\mathcal{T}|1) = \xi_1, \quad p(\mathcal{T}|2) = \xi_2, \quad p(\mathcal{T}|3) = \xi_1.$$

The hook of a node α of a partition ν is the set of nodes of ν consisting of the node α and the nodes which lie either under α in the same column or to the right of α in the same row; the hook length $h_\nu(\alpha)$ of α is the cardinality of the hook of α . We extend this definition to 2-nodes. For a 2-node $\alpha^{(2)} = (\alpha, k)$ of a 2-partition $\nu^{(2)}$, the hook length of $\alpha^{(2)}$ in $\nu^{(2)}$, which we denote by $h_{\nu^{(2)}}(\alpha^{(2)})$, is the hook length of the node α in the k -th partition of $\nu^{(2)}$. Let

$$(2.9) \quad f_{\nu^{(2)}} := \left(\prod_{\alpha^{(2)} \in \nu^{(2)}} h_{\nu^{(2)}}(\alpha^{(2)}) \right)^{-1}.$$

2.2. Idempotents of the group B_{n+1} . The representation theory of the Coxeter groups of type B was developed by A. Young [25]; see also [21–23]. The irreducible representations of B_{n+1} are in bijection with 2-partitions of $n+1$. The basis vectors of the irreducible representation of B_{n+1} corresponding to the 2-partition $\lambda^{(2)}$ are parameterized by standard 2-tableaux of shape $\lambda^{(2)}$. The Jucys–Murphy elements are diagonal in this basis. For a standard 2-tableau $\mathcal{T}$, we denote by $E_{\mathcal{T}}$ the primitive idempotent of B_{n+1} corresponding to $\mathcal{T}$. We have, for any $i = 1, \dots, n+1$,

$$(2.10) \quad j_i E_{\mathcal{T}} = E_{\mathcal{T}} j_i = p_i E_{\mathcal{T}} \quad \text{and} \quad \tilde{j}_i E_{\mathcal{T}} = E_{\mathcal{T}} \tilde{j}_i = c_i E_{\mathcal{T}}.$$

Here we set $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$ for all $i = 1, \dots, n+1$ for brevity. Due to the maximality of the commutative set $\{\tilde{j}_i, j_i\}_{i=1, \dots, n+1}$ of Jucys–Murphy elements, the idempotent $E_{\mathcal{T}}$ can be expressed in terms of $j_i, \tilde{j}_i$, $i = 1, \dots, n+1$. Let $\alpha^{(2)}$ be the 2-node of $\mathcal{T}$ with the number $n+1$. As the tableau $\mathcal{T}$ is standard, the 2-node $\alpha^{(2)}$ of $\lambda^{(2)}$ is removable. Let $\mathcal{U}$ be the standard 2-tableau obtained from $\mathcal{T}$ by removing the 2-node $\alpha^{(2)}$ and let $\mu^{(2)}$ be the shape of $\mathcal{U}$. The inductive formula for $E_{\mathcal{T}}$ in terms of the Jucys–Murphy elements reads:

$$(2.11) \quad E_{\mathcal{T}} = E_{\mathcal{U}} \prod_{\substack{\beta^{(2)}: \\ \beta^{(2)} \in \mathcal{E}_+(\mu^{(2)}) \\ c(\beta^{(2)}) \neq c(\alpha^{(2)})}} \frac{\tilde{j}_{n+1} - c(\beta^{(2)})}{c(\alpha^{(2)}) - c(\beta^{(2)})} \prod_{\substack{\beta^{(2)}: \\ \beta^{(2)} \in \mathcal{E}_-(\mu^{(2)}) \\ p(\beta^{(2)}) \neq p(\alpha^{(2)})}} \frac{j_{n+1} - p(\beta^{(2)})}{p(\alpha^{(2)}) - p(\beta^{(2)})}.$$

Note that the second product in the right-hand side of (2.11) contains only one term (this will not be so for the cyclotomic groups $G(m, 1, n)$ with $m > 2$). We have $B_0 \cong \mathbb{C}$ and $E_{\mathcal{U}_0} = 1$ for the (unique) 2-tableau $\mathcal{U}_0$ of 0.

Let $\{\mathcal{T}_1, \dots, \mathcal{T}_k\}$ be the set of pairwise different standard 2-tableaux that can be obtained from $\mathcal{U}$ by adding a 2-node with the number $n + 1$. The formula

$$E_{\mathcal{U}} = \sum_{i=1}^k E_{\mathcal{T}_i},$$

together with (2.10), implies that the rational function

$$E_{\mathcal{U}} \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \frac{v - p_{n+1}}{v - j_{n+1}}$$

is non-singular at $u = c_{n+1}$ and $v = p_{n+1}$, and, moreover,

$$(2.12) \quad E_{\mathcal{U}} \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{\substack{u = c_{n+1} \\ v = p_{n+1}}} = E_{\mathcal{T}}.$$

Since j_{n+1} takes values ± 1 , the rational function $\frac{v - p_{n+1}}{v - j_{n+1}}$ is non-singular at $v = p_{n+1}$ and

$$(2.13) \quad \frac{v - p_{n+1}}{v - j_{n+1}} \Big|_{v = p_{n+1}} = \frac{1}{2}(1 + p_{n+1}j_{n+1}).$$

For the clarity of the calculations in the sequel, we define, generalizing (2.5),

$$(2.14) \quad \mathbf{j}_i(p) := \frac{1}{2}(1 + pj_i) \quad \text{for } i = 1, \dots, n + 1.$$

Combining (2.12) and (2.13), we obtain the following formula for the idempotent $E_{\mathcal{T}}$:

$$(2.15) \quad E_{\mathcal{T}} = E_{\mathcal{U}} \mathbf{j}_{n+1}(p_{n+1}) \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \Big|_{u = c_{n+1}}.$$

3. FUSION FORMULA FOR IDEMPOTENTS OF B_{n+1}

We start with the following lemma which will be useful in the sequel.

Lemma 3.1. *For any integer l , $1 \leq l \leq n$, we have*

$$\begin{aligned} \text{(i)} \quad \tilde{j}_{n+1} &= s_n s_{n-1} \dots s_l \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^n s_n \dots s_{i+1} s_i s_{i+1} \dots s_n (1 + j_{n+1} j_i), \\ \text{(ii)} \quad \mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} \\ &= \mathbf{j}_l(p) \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p) (1 + j_{n+1} j_i). \end{aligned}$$

The product $s_l s_{l+1} \dots s_{i-1}$ in the right-hand side of (ii) is understood to be equal to 1 if $i = l$.

Proof. We prove (i) by induction on $n - l$. The basis of the induction is, for $l = n$, the formula $\tilde{j}_{n+1} = s_n \tilde{j}_n s_n + \frac{1}{2}(s_n + s_n j_{n+1} j_n)$, which follows from the definition (2.7) of the Jucys–Murphy elements, namely from $\tilde{j}_{n+1} = s_n \tilde{j}_n s_n + \frac{1}{2}(s_n + j_n s_n j_n)$ and $j_n s_n = s_n j_{n+1}$. Now assume that

$$\tilde{j}_{n+1} = s_n s_{n-1} \dots s_{l+1} \tilde{j}_{l+1} s_{l+1} \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l+1}^n s_n \dots s_{i+1} s_i s_{i+1} \dots s_n (1 + j_{n+1} j_i),$$

and replace $\tilde{j}_{l+1}$ by $s_l \tilde{j}_l s_l + \frac{1}{2}(s_l + j_l s_l j_l)$. One obtains the assertion (i) using that $j_l s_{l+1} \dots s_{n-1} s_n = s_{l+1} \dots s_{n-1} s_n j_l$ and $j_l s_l s_{l+1} \dots s_{n-1} s_n = s_l s_{l+1} \dots s_{n-1} s_n j_{n+1}$.

Using (i), we find

$$\begin{aligned} \mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} \\ = \mathbf{j}_l(p) \left(\tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n (1 + j_{n+1} j_i) \right), \end{aligned}$$

and (ii) follows since $j_l s_l s_{l+1} \dots s_{i-1} = s_l s_{l+1} \dots s_{i-1} j_i$ and j_i commutes with $s_{i+1} \dots s_{n-1} s_n$. $\square$

Let $\phi_1(v, u) := \mathbf{t}(v)$; for $k = 1, \dots, n$ define

$$\begin{aligned} (3.1) \quad & \phi_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) \\ & := \mathbf{s}_k(v, v_k, u, u_k) \phi_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u) s_k \\ & = \mathbf{s}_k(v, v_k, u, u_k) \mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1) \mathbf{t}(v) s_1 \dots s_{k-1} s_k. \end{aligned}$$

Define the following rational function with values in the group ring of $\mathbf{B}_{n+1}$:

$$(3.2) \quad \Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) := \prod_{k=0, \dots, n}^{\leftarrow} \phi_{k+1}(v_1, \dots, v_k, v_{k+1}, u_1, \dots, u_k, u_{k+1});$$

the arrow over $\prod$ indicates that the (non-commuting) factors are taken in descending order.

Let $\lambda^{(2)}$ be a 2-partition of $n+1$ and $\mathcal{T}$ a standard 2-tableau of shape $\lambda^{(2)}$. For $i = 1, \dots, n+1$, set $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$.

Theorem 3.2. *The idempotent $E_{\mathcal{T}}$ corresponding to the standard 2-tableau $\mathcal{T}$ of shape $\lambda^{(2)}$ can be obtained by the following consecutive evaluations:*

$$(3.3) \quad E_{\mathcal{T}} = f_{\lambda^{(2)}} \Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) \Big|_{v_i=p_i, i=1, \dots, n+1} \Big|_{u_1=c_1} \dots \Big|_{u_n=c_n} \Big|_{u_{n+1}=c_{n+1}}.$$

Proof. Define

$$(3.4) \quad F_{\mathcal{T}}(u) := \frac{u - c_{n+1}}{u} \prod_{i=1}^n \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}}.$$

Let $\mathcal{U}$ be the standard 2-tableau obtained from $\mathcal{T}$ by removing the 2-node with the number $n+1$ and let $\mu^{(2)}$ be the shape of $\mathcal{U}$.

Proposition 3.3. *We have*

$$(3.5) \quad F_{\mathcal{T}}(u) \phi_{n+1}(p_1, \dots, p_n, p_{n+1}, c_1, \dots, c_n, u) E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}}.$$

Proof. We prove (3.5) by induction on n . As $c_1 = 0$ and $\tilde{j}_1 = 0$, the basis of induction for $n = 0$ is the formula $\mathbf{t}(p_1) = \mathbf{j}_1(p_1)$ which is satisfied by definition; see (2.5) and (2.14).

If $p_{n+1} \neq p_i$, $i = 2, \dots, n$, then fix $l = 1$. Otherwise fix l such that $p_{n+1} = p_l$ and $p_{n+1} \neq p_i$, $i = l+1, \dots, n$.

Define $\mathcal{V}$ to be the standard 2-tableau obtained from $\mathcal{U}$ by removing the 2-nodes containing the numbers $l+1, \dots, n$ and $\mathcal{W}$ to be the standard 2-tableau obtained

from $\mathcal{V}$ by removing the 2-node with the number l . We will use that $E_{\mathcal{W}}E_{\mathcal{U}} = E_{\mathcal{U}}$ and that $E_{\mathcal{W}}$ commutes with s_i , for $i = l, l+1, \dots, n$. We rewrite the left-hand side of (3.5) as

$$F_{\mathcal{T}}(u)s_n \dots s_{l+1}s_l(p_{n+1}, p_l, u, c_l) \cdot \phi_l(p_1, \dots, p_{l-1}, p_{n+1}, c_1, \dots, c_{l-1}, u)E_{\mathcal{W}} \cdot s_l \dots s_n E_{\mathcal{U}}.$$

If $l > 1$, then we use the induction hypothesis, namely

$$\phi_l(p_1, \dots, p_{l-1}, p_l, c_1, \dots, c_{l-1}, u)E_{\mathcal{W}} = (F_{\mathcal{V}}(u))^{-1} \frac{u - c_l}{u - \tilde{j}_l} \mathbf{j}_l(p_l)E_{\mathcal{W}},$$

and we notice that $p_{n+1} = p_l$.

If $l = 1$, then $E_{\mathcal{W}} = 1$, $F_{\mathcal{V}}(u) = 1$ and, by definition, $\phi_1(p_{n+1}, u)$ is equal to $\mathbf{j}_1(p_{n+1})$. Thus, in both situations (for $l = 1$ and for $l > 1$), we obtain for the left-hand side of (3.5):

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} s_n \dots s_{l+1}s_l(p_{n+1}, p_l, u, c_l) \frac{u - c_l}{u - \tilde{j}_l} \mathbf{j}_l(p_{n+1})s_l s_{l+1} \dots s_n E_{\mathcal{U}}.$$

Therefore, the equality (3.5) is equivalent to

$$\begin{aligned} (3.6) \quad & F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} (u - c_l) \mathbf{j}_l(p_{n+1})s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1})E_{\mathcal{U}} \\ &= \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l, p_{n+1}}} (u - \tilde{j}_l) s_l(p_l, p_{n+1}, c_l, u) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1})E_{\mathcal{U}}, \end{aligned}$$

where, in moving $s_n \dots s_{l+1}s_l(p_{n+1}, p_l, u, c_l) \frac{1}{u - \tilde{j}_l}$ to the right-hand side and $\frac{1}{u - \tilde{j}_{n+1}}$ to the left-hand side, we have used that $\tilde{j}_{n+1}$ commutes with j_{n+1} and $E_{\mathcal{U}}$, and also the formula (2.6) to take the inverse of $s_l(p_{n+1}, p_l, u, c_l)$.

To prove the equality (3.6), first notice that we have

$$F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} (u - c_l) = (u - c_{n+1}) \prod_{i=1}^n \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}} \prod_{i=1}^{l-1} \frac{(u - c_i)^2 - \delta_{p_i, p_l}}{(u - c_i)^2},$$

which gives, since $p_i \neq p_{n+1}$ if $i > l$ and $p_l = p_{n+1}$ if $l > 1$,

$$(3.7) \quad F_{\mathcal{T}}(u)(F_{\mathcal{V}}(u))^{-1} (u - c_l) = \frac{(u - c_{n+1})(u - c_l)^2}{(u - c_l)^2 - \delta_{p_l, p_{n+1}}}.$$

So it remains to prove that

$$(3.8) \quad \mathbf{j}_l(p_{n+1})s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1})E_{\mathcal{U}} = (u - \tilde{j}_l) s_l(p_l, p_{n+1}, c_l, u) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1})E_{\mathcal{U}}.$$

Expand $s_l(p_l, p_{n+1}, c_l, u)$ in the right-hand side of (3.8):

$$(3.9) \quad ((u - \tilde{j}_l)s_l - \delta_{p_l, p_{n+1}} \frac{\tilde{j}_l - u}{c_l - u}) s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1})E_{\mathcal{U}}.$$

As $\tilde{j}_l$ commutes with $s_{l+1} \dots s_n$ and j_{n+1} and $\tilde{j}_l E_{\mathcal{U}} = c_l E_{\mathcal{U}}$, we find that the expression (3.9) equals

$$\left((u - \tilde{j}_l)s_l s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1}) \right) E_{\mathcal{U}}.$$

Then using that $s_l s_{l+1} \dots s_n j_{n+1} = j_l s_l s_{l+1} \dots s_n$, we obtain for the right-hand side of (3.8):

$$(3.10) \quad ((u - \tilde{j}_l) \mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_n \mathbf{j}_{n+1}(p_{n+1})) E_{\mathcal{U}}.$$

Using Lemma 3.1(ii), we write the left-hand side of (3.8) in the form

$$(3.11) \quad \left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p_{n+1})(1 + j_{n+1} j_i) \right) E_{\mathcal{U}}.$$

As $j_i E_{\mathcal{U}} = p_i E_{\mathcal{U}}$, $i = 1, \dots, n$, the expression (3.11) is equal to

$$(3.12) \quad \left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{2} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \frac{1}{2} (1 + p_i p_{n+1})(1 + p_i j_{n+1}) \right) E_{\mathcal{U}}.$$

Since $p_k p_{n+1} = -1$, $k = l+1, \dots, n$, we finally obtain for the left-hand side of (3.8):

$$(3.13) \quad \left(\mathbf{j}_l(p_{n+1})(u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_{n-1} s_n \mathbf{j}_{n+1}(p_l) \right) E_{\mathcal{U}}.$$

The comparison of (3.13) and (3.10) proves the equality (3.8). $\square$

Proposition 3.4. *The rational function $F_{\mathcal{T}}(u)$, defined by (3.4), is regular at $u = c_{n+1}$ and, moreover,*

$$(3.14) \quad F_{\mathcal{T}}(c_{n+1}) = f_{\lambda^{(2)}}(f_{\mu^{(2)}})^{-1}.$$

Proof. The proposition will directly follow from the result, used in [14], concerning usual tableaux. Let λ be a partition of $\tilde{n} + 1$, with $\tilde{n} \leq n$. Let S be a subset of $\{1, \dots, n+1\}$ such that S contains the number $n+1$ and has a cardinal equal to $\tilde{n} + 1$. Let $\tilde{\mathcal{T}}$ be a tableau of shape λ filled with numbers belonging to S such that the numbers in the nodes are in strictly ascending order along rows and columns in the right and down directions. Let γ be the node of $\tilde{\mathcal{T}}$ with the number $n+1$ and μ be the shape of the tableau obtained from $\tilde{\mathcal{T}}$ by removing the node γ . Define the following rational function:

$$(3.15) \quad \tilde{F}_{\tilde{\mathcal{T}}}(u) = \frac{u - c(\gamma)}{u} \prod_{\alpha \in \mu} \frac{(u - c(\alpha))^2}{(u - c(\alpha) + 1)(u - c(\alpha) - 1)}.$$

The product in the right-hand side of (3.15) depends only on the shape μ , and one has

$$\prod_{\alpha \in \mu} \frac{(u - c(\alpha))^2}{(u - c(\alpha) + 1)(u - c(\alpha) - 1)} = u \prod_{\beta \in \mathcal{E}_-(\mu)} (u - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu)} (u - c(\alpha))^{-1},$$

where $\mathcal{E}_-(\mu)$ (respectively, $\mathcal{E}_+(\mu)$) is the set of removable (respectively, addable) nodes of μ . Therefore the rational function $\tilde{F}_{\tilde{\mathcal{T}}}(u)$ is regular at $u = c(\gamma)$ and, moreover,

$$(3.16) \quad \tilde{F}_{\tilde{\mathcal{T}}}(c(\gamma)) = \prod_{\beta \in \mathcal{E}_-(\mu)} (c(\gamma) - c(\beta)) \prod_{\alpha \in \mathcal{E}_+(\mu) \setminus \{\gamma\}} (c(\gamma) - c(\alpha))^{-1}.$$

It is known that the right-hand side of (3.16) is equal to

$$(3.17) \quad \prod_{\alpha \in \lambda} (h_{\lambda}(\alpha))^{-1} \prod_{\alpha \in \mu} h_{\mu}(\alpha).$$

Define $\tilde{\mathcal{T}}$ to be the tableau of the standard 2-tableau $\mathcal{T}$ which contains the node with number $n+1$. The assertion of Proposition 3.4 follows; the only observation one has to make is that the 2-nodes (α, k) with $p_k \neq p_{n+1}$ do not contribute to (3.14). $\square$

Theorem 3.2 follows, by induction on n , from formula (2.15), Proposition 3.3 and Proposition 3.4. $\square$

4. FUSION PROCEDURE FOR THE COMPLEX REFLECTION GROUP $G(m, 1, n + 1)$

We extend the results of the previous section to the complex reflection groups $G(m, 1, n + 1)$ for all positive integers m . We skip the proofs when they are completely similar to the proofs in the preceding section; we only indicate modifications.

4.1. Definitions. The complex reflection group $G(m, 1, n + 1)$ is generated by the elements $s_1, \dots, s_n$ and t with the defining relations (2.1), (2.2) and

$$(4.1) \quad t^m = 1.$$

In particular, the group $G(1, 1, n + 1)$ is isomorphic to the symmetric group S_{n+1} and the group $G(2, 1, n + 1)$ to B_{n+1} .

We extend the definition (2.4) to the generators $s_1, \dots, s_n$ of $G(m, 1, n + 1)$:

$$(4.2) \quad s_i(p, p', a, a') := s_i + \frac{\delta_{p,p'}}{a - a'}, \quad i = 1, \dots, n.$$

The Jucys–Murphy elements for the group $G(m, 1, n + 1)$ are the elements j_i , $i = 1, \dots, n + 1$, and $\tilde{j}_i$, $i = 1, \dots, n + 1$, of the group ring defined inductively by the following initial conditions and recursions:

$$(4.3) \quad j_1 = t, \quad j_{i+1} = s_i j_i s_i \quad \text{and} \quad \tilde{j}_1 = 0, \quad \tilde{j}_{i+1} = s_i \tilde{j}_i s_i + \frac{1}{m} \sum_{k=0}^{m-1} j_i^k s_i j_i^{m-k}.$$

For $m = 1$, that is, for S_{n+1} , $j_k = 1$, $k = 1, \dots, n + 1$; the recursion formula for $\tilde{j}_{i+1}$ reduces to $\tilde{j}_{i+1} = s_i \tilde{j}_i s_i + s_i$.

As for $m = 2$, the elements j_i , $i = 1, \dots, n + 1$, and $\tilde{j}_i$, $i = 1, \dots, n + 1$, form a maximal commutative set in $\mathbb{C}G(m, 1, n + 1)$; see [21, 22]. In addition, j_i and $\tilde{j}_i$ commute with all s_k , except s_i and s_{i-1} .

The definitions of a 2-partition, 2-node, standard 2-tableaux and hook length generalize naturally to any m ; for example, an m -partition is an m -tuple of partitions and an m -node $\alpha^{(m)}$ is a pair (α, k) , $k = 1, \dots, m$. For an m -node $\alpha^{(m)} = (\alpha, k)$ of an m -partition $\lambda^{(m)}$ such that the node α lies in the line x and the column y of the k -th diagram, we define $c(\alpha^{(m)}) := c(\alpha) = y - x$. Let $\{\xi_1, \dots, \xi_m\}$ be the set of distinct m -th roots of unity, ordered arbitrarily; we define $p(\alpha^{(m)}) := \xi_k$.

As for $m = 2$, we define

$$(4.4) \quad f_{\nu^{(m)}} := \left(\prod_{\alpha^{(m)} \in \nu^{(m)}} h_{\nu^{(m)}}(\alpha^{(m)}) \right)^{-1},$$

where $h_{\nu^{(m)}}(\alpha^{(m)})$ is the hook length of $\alpha^{(m)}$ calculated in $\nu^{(m)}$.

The irreducible representations of the group $G(m, 1, n + 1)$ are parameterized by m -partitions of $n + 1$. For a given m -partition $\lambda^{(m)}$, the basis vectors of the corresponding representation are indexed by standard m -tableaux of shape $\lambda^{(m)}$. We denote by $E_{\mathcal{T}}$ the idempotent of the group ring corresponding to a standard m -tableau $\mathcal{T}$.

4.2. Fusion formula for idempotents of $G(m, 1, n + 1)$. Let $\lambda^{(m)}$ be an m -partition of $n + 1$; fix a standard m -tableau $\mathcal{T}$ of shape $\lambda^{(m)}$. Let $\mathcal{U}$ be the standard m -tableau obtained by removing from $\mathcal{T}$ the m -node containing $n + 1$ and let $\mu^{(m)}$ be the shape of $\mathcal{U}$. Set, for $i = 1, \dots, n + 1$, $p_i := p(\mathcal{T}|i)$ and $c_i := c(\mathcal{T}|i)$

The same reasoning as in Subsection 2.2 leads to the formula for $E_{\mathcal{T}}$ (cf. (2.12)):

$$(4.5) \quad E_{\mathcal{U}} \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \frac{v - p_{n+1}}{v - \tilde{j}_{n+1}} \Big|_{\substack{u = c_{n+1} \\ v = p_{n+1}}} = E_{\mathcal{T}}.$$

Since $\tilde{j}_{n+1}$ takes values in $\{\xi_1, \dots, \xi_m\}$, the rational function $\frac{v - p_{n+1}}{v - \tilde{j}_{n+1}}$ is non-singular for $v = p_{n+1}$ and

$$(4.6) \quad \frac{v - p_{n+1}}{v - \tilde{j}_{n+1}} \Big|_{v = p_{n+1}} = \frac{1}{m} \sum_{k=0}^{m-1} p_{n+1}^{m-k} j_{n+1}^k.$$

An analogue of (2.14) is

$$(4.7) \quad \mathbf{j}_i(p) := \frac{1}{m} \sum_{k=0}^{m-1} p^{m-k} j_i^k \quad \text{for } i = 1, \dots, n + 1.$$

For $i = 1$ we shall write $\mathbf{t}(p) := \frac{1}{m} \sum_{k=0}^{m-1} p^{m-k} t^k$ instead of $\mathbf{j}_1(p)$. As p is always evaluated at an m -th root of unity, definition (4.7) agrees, for $m = 2$, with (2.5) and (2.14).

Combining (4.5) and (4.6), we obtain for the idempotent $E_{\mathcal{T}}$ (cf. (2.15))

$$(4.8) \quad E_{\mathcal{T}} = E_{\mathcal{U}} \mathbf{j}_{n+1}(p_{n+1}) \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \Big|_{u = c_{n+1}}.$$

We generalize Lemma 3.1 to an arbitrary positive integer m .

Lemma 4.1. *For any integer l , $1 \leq l \leq n$, we have*

$$(i) \quad \tilde{j}_{n+1} = s_n s_{n-1} \dots s_l \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{m} \sum_{i=l}^n s_n \dots s_{i+1} s_i s_{i+1} \dots s_n \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k}.$$

$$(ii) \quad \mathbf{j}_l(p) s_l \dots s_{n-1} s_n \tilde{j}_{n+1} \\ = \mathbf{j}_l(p) \tilde{j}_l s_l \dots s_{n-1} s_n + \frac{1}{m} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p) \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k}.$$

The product $s_l s_{l+1} \dots s_{i-1}$ in the right-hand side of (ii) is understood to be equal to 1 if $i = l$.

Proof. The proof is completely similar to the proof of Lemma 3.1. $\square$

Let $\phi_1(v, u) := \mathbf{t}(v)$; for $k = 1, \dots, n$ define

$$(4.9) \quad \begin{aligned} & \phi_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) \\ &:= \mathbf{s}_k(v, v_k, u, u_k) \phi_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u) s_k \\ &= \mathbf{s}_k(v, v_k, u, u_k) \mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1) \mathbf{t}(v) s_1 \dots s_{k-1} s_k. \end{aligned}$$

Formula (4.9) reads the same way for any m ; only the definition of $\mathbf{t}(v)$ depends on m .

Define the following rational function with values in the group ring of $G(m, 1, n)$:

$$(4.10) \quad \Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) := \prod_{k=0, \dots, n}^{\leftarrow} \phi_{k+1}(v_1, \dots, v_k, v_{k+1}, u_1, \dots, u_k, u_{k+1}) ;$$

the arrow over $\prod$ indicates that the (non-commuting) factors are taken in descending order.

Theorem 4.2. *The idempotent $E_{\mathcal{T}}$ corresponding to the standard m -tableau $\mathcal{T}$ of shape $\lambda^{(m)}$ can be obtained by the following consecutive evaluations:*

$$(4.11) \quad E_{\mathcal{T}} = f_{\lambda^{(m)}} \Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1}) \Big|_{v_i=p_i, \ i=1, \dots, n+1} \Big|_{u_1=c_1} \cdots \Big|_{u_n=c_n} \Big|_{u_{n+1}=c_{n+1}} .$$

Proof. Define

$$(4.12) \quad F_{\mathcal{T}}(u) := \frac{u - c_{n+1}}{u} \prod_{i=1}^n \frac{(u - c_i)^2}{(u - c_i)^2 - \delta_{p_i, p_{n+1}}} .$$

Proposition 4.3. *We have*

$$(4.13) \quad F_{\mathcal{T}}(u) \phi_{n+1}(p_1, \dots, p_n, p_{n+1}, c_1, \dots, c_n, u) E_{\mathcal{U}} = \frac{u - c_{n+1}}{u - \tilde{j}_{n+1}} \mathbf{j}_{n+1}(p_{n+1}) E_{\mathcal{U}} .$$

Proof. The proof follows the same lines as the proof of Proposition 3.3. Actually it is exactly the same until the calculation of $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ just after formula (3.10). We give the modified end of the proof.

Here, we rewrite $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ using Lemma 4.1(ii):

$$(4.14) \quad \left(\mathbf{j}_l(p_{n+1}) (u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{m} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \mathbf{j}_i(p_{n+1}) \sum_{k=0}^{m-1} j_{n+1}^k j_i^{m-k} \right) E_{\mathcal{U}} .$$

As $j_i E_{\mathcal{U}} = p_i E_{\mathcal{U}}$ for $i = 1, \dots, n$, the expression (4.14) is equal to

$$(4.15) \quad \left(\mathbf{j}_l(p_{n+1}) (u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \frac{1}{m} \sum_{i=l}^n s_l s_{l+1} \dots s_{i-1} \cdot s_{i+1} \dots s_{n-1} s_n \sum_{k=0}^{m-1} p_{n+1}^k p_i^{m-k} \mathbf{j}_{n+1}(p_i) \right) E_{\mathcal{U}} .$$

Since $\frac{1}{m} \sum_{k=0}^{m-1} p_{n+1}^k p_i^{m-k} = \delta_{p_i, p_{n+1}}$ we obtain that $\mathbf{j}_l(p_{n+1}) s_l s_{l+1} \dots s_n (u - \tilde{j}_{n+1}) E_{\mathcal{U}}$ equals

$$(4.16) \quad \left(\mathbf{j}_l(p_{n+1}) (u - \tilde{j}_l) s_l s_{l+1} \dots s_n - \delta_{p_l, p_{n+1}} s_{l+1} \dots s_{n-1} s_n \mathbf{j}_{n+1}(p_l) \right) E_{\mathcal{U}} .$$

This concludes the proof. $\square$

The analogue of Proposition 3.4 holds as well.

Proposition 4.4. *The rational function $F_{\mathcal{T}}(u)$, defined by (4.12), is regular at $u = c_{n+1}$ and, moreover,*

$$(4.17) \quad F_{\mathcal{T}}(c_{n+1}) = f_{\lambda^{(m)}}(f_{\mu^{(m)}})^{-1} .$$

Proof. The proof is completely similar to the proof of Proposition 3.4. $\square$

Similarly to Theorem 3.2, Theorem 4.2 follows, using induction on n , from formula (4.8), Proposition 4.3 and Proposition 4.4. $\square$

For calculations, it is sometimes useful to write the function Φ in a slightly different form. Namely, let $\tilde{\phi}_1(v, u) := 1$ and define

$$\begin{aligned} & \tilde{\phi}_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u) \\ (4.18) \quad &:= \mathbf{s}_k(v, v_k, u, u_k) \tilde{\phi}_k(v_1, \dots, v_{k-1}, v, u_1, \dots, u_{k-1}, u) s_k \\ &= \mathbf{s}_k(v, v_k, u, u_k) \mathbf{s}_{k-1}(v, v_{k-1}, u, u_{k-1}) \dots \mathbf{s}_1(v, v_1, u, u_1) s_1 \dots s_{k-1} s_k, \end{aligned}$$

for $k = 1, \dots, n$. The elements $\tilde{\phi}_{k+1}(v_1, \dots, v_k, v, u_1, \dots, u_k, u)$ do not involve the generator t , and $\Phi(v_1, \dots, v_{n+1}, u_1, \dots, u_{n+1})$, defined in (4.10), equals

$$(4.19) \quad \prod_{k=0, \dots, n}^{\leftarrow} \tilde{\phi}_{k+1}(v_1, \dots, v_{k+1}, u_1, \dots, u_{k+1}) \cdot \mathbf{j}_1(v_1) \mathbf{j}_2(v_2) \dots \mathbf{j}_{n+1}(v_{n+1}).$$

For example, let $m = 2$; choose the order $\{1, -1\}$ on the set of square roots of 1. The primitive idempotent, corresponding to the standard 2-tableau $\left(\begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}\right)$, reads $s_2(1 + s_1)s_2\mathbf{j}_1(1)\mathbf{j}_2(-1)\mathbf{j}_3(1)/2$.

REFERENCES

- [1] I. V. Cherednik, *Special bases of irreducible representations of a degenerate affine Hecke algebra* (Russian), Funktsional. Anal. i Prilozhen. **20** (1986), no. 1, 87–88. MR831062 (87m:22031)
- [2] I. V. Cherednik, *Calculation of the monodromy of some W -invariant local systems of type B, C and D* (Russian), Funktsional. Anal. i Prilozhen. **24** (1990), no. 1, 88–89, DOI 10.1007/BF01077930; English transl., Funct. Anal. Appl. **24** (1990), no. 1, 78–79. MR1052280 (91i:17019)
- [3] James Grime, *The hook fusion procedure*, Electron. J. Combin. **12** (2005), Research Paper 26, 14 pp. (electronic). MR2156680 (2006m:05255)
- [4] James Grime, *The hook fusion procedure for Hecke algebras*, J. Algebra **309** (2007), no. 2, 744–759, DOI 10.1016/j.jalgebra.2006.06.024. MR2303204 (2008a:05270)
- [5] Valentina Guizzi and Paolo Papi, *A combinatorial approach to the fusion process for the symmetric group*, European J. Combin. **19** (1998), no. 7, 835–845, DOI 10.1006/eujc.1998.0242. MR1649970 (2000g:05149)
- [6] A. P. Isaev and A. I. Molev, *Fusion procedure for the Brauer algebra*, Algebra i Analiz **22** (2010), no. 3, 142–154. English translation in St. Petersburg Math. J. **22** (2011), no. 3, 437–446. MR2729943 (2011i:16043)
- [7] A. P. Isaev, A. I. Molev, and A. F. Os'kin, *On the idempotents of Hecke algebras*, Lett. Math. Phys. **85** (2008), no. 1, 79–90, DOI 10.1007/s11005-008-0254-7. MR2425663 (2009e:20010)
- [8] A. P. Isaev, A. I. Molev, and O. V. Ogievetsky, *A new fusion procedure for the Brauer algebra and evaluation homomorphisms*, Int. Math. Res. Not. IMRN 2012, no. 11, 2571–2606. MR2926990
- [9] A. Isaev, A. Molev and O. Ogievetsky, *Idempotents for Birman-Murakami-Wenzl algebras and reflection equation*, arXiv:1111.2502
- [10] M. Jimbo, A. Kuniba, T. Miwa, and M. Okado, *The $A_n^{(1)}$ face models*, Comm. Math. Phys. **119** (1988), no. 4, 543–565. MR973016 (90h:17044)
- [11] Andrew R. Jones, *The structure of the Young symmetrizers for spin representations of the symmetric group. I*, J. Algebra **205** (1998), no. 2, 626–660, DOI 10.1006/jabr.1997.7400. MR1632785 (99j:20017)
- [12] A. R. Jones and M. L. Nazarov, *Affine Sergeev algebra and q -analogues of the Young symmetrizers for projective representations of the symmetric group*, Proc. London Math. Soc. (3) **78** (1999), no. 3, 481–512, DOI 10.1112/S002461159900177X. MR1674836 (2000a:20021)
- [13] A. A. Jucis, *On the Young operators of symmetric groups* (Russian, with Lithuanian and English summaries), Litovsk. Fiz. Sb. **6** (1966), 163–180. MR0202866 (34 #2725)

- [14] A. I. Molev, *On the fusion procedure for the symmetric group*, Rep. Math. Phys. **61** (2008), no. 2, 181–188, DOI 10.1016/S0034-4877(08)80005-5. MR2424084 (2009f:20012)
- [15] Maxim Nazarov, *Young’s symmetrizers for projective representations of the symmetric group*, Adv. Math. **127** (1997), no. 2, 190–257, DOI 10.1006/aima.1997.1621. MR1448714 (98m:20019)
- [16] Maxim Nazarov, *Yangians and Capelli identities*, Kirillov’s seminar on representation theory, Amer. Math. Soc. Transl. Ser. 2, vol. 181, Amer. Math. Soc., Providence, RI, 1998, pp. 139–163. MR1618751 (99g:17033)
- [17] Maxim Nazarov, *Mixed hook-length formula for degenerate affine Hecke algebras*, Asymptotic combinatorics with applications to mathematical physics (St. Petersburg, 2001), Lecture Notes in Math., vol. 1815, Springer, Berlin, 2003, pp. 223–236, DOI 10.1007/3-540-44890-X_10. MR2009842 (2004k:20012)
- [18] Maxim Nazarov, *A mixed hook-length formula for affine Hecke algebras*, European J. Combin. **25** (2004), no. 8, 1345–1376, DOI 10.1016/j.ejc.2003.10.010. MR2095485 (2005g:20010)
- [19] Maxim Nazarov and Vitaly Tarasov, *On irreducibility of tensor products of Yangian modules associated with skew Young diagrams*, Duke Math. J. **112** (2002), no. 2, 343–378, DOI 10.1215/S0012-9074-02-11225-3. MR1894364 (2003b:17021)
- [20] O. V. Ogievetsky and L. Poulain d’Andecy, *On representations of cyclotomic Hecke algebras*, Modern Phys. Lett. A **26** (2011), no. 11, 795–803, DOI 10.1142/S0217732311035377. MR2795629 (2012g:20011)
- [21] O. V. Ogievetsky and L. Poulain d’Andecy, *Cyclotomic Hecke algebras: Jucys-Murphy elements, representations, classical limit*, to appear.
- [22] I. A. Pushkarev, *On the theory of representations of the wreath products of finite groups and symmetric groups* (Russian, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **240** (1997), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 2, 229–244, 294–295, DOI 10.1007/BF02175835; English transl., J. Math. Sci. (New York) **96** (1999), no. 5, 3590–3599. MR1691647 (2000c:20026)
- [23] Arun Ram, *Seminormal representations of Weyl groups and Iwahori-Hecke algebras*, Proc. London Math. Soc. (3) **75** (1997), no. 1, 99–133, DOI 10.1112/S0024611597000282. MR1444315 (98d:20007)
- [24] Weiqiang Wang, *Vertex algebras and the class algebras of wreath products*, Proc. London Math. Soc. (3) **88** (2004), no. 2, 381–404, DOI 10.1112/S0024611503014382. MR2032512 (2005a:17029)
- [25] Alfred Young, *On Quantitative Substitutional Analysis*, Proc. London Math. Soc. **S2-31**, no. 1, 273, DOI 10.1112/plms/s2-31.1.273. MR1577466

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