COMPLEX SYMMETRY OF COMPOSITION OPERATORS INDUCED BY INVOLUTIVE BALL AUTOMORPHISMS

S. WALEED NOOR

(Communicated by Pamela B. Gorkin)

ABSTRACT. Suppose $\mathcal{H}$ is a weighted Hardy space of analytic functions on the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ such that the composition operator C_ψ defined by $C_\psi f = f \circ \psi$ is bounded on $\mathcal{H}$ whenever ψ is a linear fractional self-map of $\mathbb{B}_n$. If φ is an involutive Moebius automorphism of $\mathbb{B}_n$, we find a conjugation operator $\mathcal{J}$ on $\mathcal{H}$ such that $C_\varphi = \mathcal{J}C_\psi^* \mathcal{J}$. The case $n = 1$ answers a question of Garcia and Hammond.

1. INTRODUCTION

Let $\mathbb{B}_n$ denote the open unit ball of $\mathbb{C}^n$. A linear fractional self-map ψ of $\mathbb{B}_n$ is a map of the form

$$(1.1) \quad \psi(z) = \frac{Az + B}{\langle z, C \rangle + d}$$

where A is a linear operator on $\mathbb{C}^n$, with vectors $B, C \in \mathbb{B}_n$ and d a complex number. Fix a vector $a \in \mathbb{B}_n$. Let P_a be the orthogonal projection of $\mathbb{C}^n$ onto the complex line generated by a and let $Q_a = I - P_a$. Setting $s_a = (1 - |a|^2)^{1/2}$, we denote by $\varphi_a : \mathbb{B}_n \rightarrow \mathbb{B}_n$ the linear fractional map

$$(1.2) \quad \varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}$$

which by Section 2.2.1 of Rudin [9] is the involutive Moebius automorphism of $\mathbb{B}_n$ that interchanges 0 and a .

Suppose $\mathcal{H}$ is a weighted Hardy space on $\mathbb{B}_n$ as introduced in [2]; that is, a Hilbert space of analytic functions on $\mathbb{B}_n$ such that the monomials $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ form an orthogonal basis for $\mathcal{H}$. Additionally, the monomials z^α satisfy $\frac{\|z^\alpha\|}{\|z^\alpha\|_{H^2}} = \frac{\|z^\gamma\|}{\|z^\gamma\|_{H^2}}$ whenever $|\alpha| = |\gamma|$ for $\gamma \in \mathbb{N}^n$, where $|\gamma| = \gamma_1 + \dots + \gamma_n$ and $\|\cdot\|$ and $\|\cdot\|_{H^2}$ are the norms in $\mathcal{H}$ and the classical Hardy space $H^2(\mathbb{B}_n)$ respectively. The sequence $\beta_m = \frac{\|z^\alpha\|}{\|z^\alpha\|_{H^2}}$ for $m = |\alpha|$ determines $\mathcal{H}$: the inner product on $\mathcal{H}$ is given by

$$(1.3) \quad \langle f, g \rangle = \sum_{m=0}^{\infty} \langle f_m, g_m \rangle_{H^2} \beta_m^2,$$

Received by the editors August 31, 2012 and, in revised form, September 17, 2012.

2010 *Mathematics Subject Classification*. Primary 47B33, 47B32, 47B99; Secondary 47B35.

Key words and phrases. Complex symmetric operator, conjugation, Moebius automorphism, composition operator, normal operator.

where f_m, g_m are the homogeneous polynomials of degree m in the homogeneous expansions $f = \sum f_m$ and $g = \sum g_m$ on $\mathbb{B}_n$ respectively. If ψ is an analytic self-map of $\mathbb{B}_n$, then the composition operator C_ψ is defined by $C_\psi f = f \circ \psi$ for $f \in \mathcal{H}$. When $\mathcal{H}$ is the classical Hardy space or a weighted Bergman space on $\mathbb{D} = \mathbb{B}_1$, then the boundedness of C_ψ on $\mathcal{H}$ follows from the Littlewood Subordination Theorem. But for $n \geq 2$, there exist unbounded composition operators, even on the classical spaces [2]. For our purpose, it is sufficient to assume that $\mathcal{H}$ is a weighted Hardy space such that C_ψ is bounded on $\mathcal{H}$ whenever ψ is a linear fractional self-map of $\mathbb{B}_n$. From here onwards $\mathcal{H}$ shall always denote such a space. Many of the classical Hilbert spaces of analytic functions on $\mathbb{B}_n$ satisfy this requirement: For any real $s > 0$, let H_s be the Hilbert space of analytic functions on $\mathbb{B}_n$ with reproducing kernel function

$$K_w^s(z) = \frac{1}{(1 - \langle w, z \rangle)^s}$$

for $w, z \in \mathbb{B}_n$. These spaces are called generalized weighted Bergman spaces by Zhao and Zhu [10]. Recently Le [8] showed that C_ψ is bounded on H_s for $s > 0$ whenever ψ is a linear fractional self-map of $\mathbb{B}_n$. The space H_n is the classical Hardy space $H^2(\mathbb{B}_n)$ and H_s is the weighted Bergman space $A_{s-n-1}^2(\mathbb{B}_n)$ for $s > n$. When $n \geq 2$, then H_1 is the Drury-Arveson space from multi-variable operator theory [1].

A bounded operator T on $\mathcal{H}$ is called *complex symmetric* if T has a self-transpose matrix representation with respect to some orthonormal basis of $\mathcal{H}$. This is equivalent to the existence of a *conjugation* (i.e., a conjugate-linear, isometric involution) C such that $T = CT^*C$, and T is called C -symmetric. If $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, then the termwise complex conjugation $\bar{z} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ is a conjugation operator on $\mathbb{C}^n$. The general study of complex symmetric operators on Hilbert spaces was initiated by Garcia, Putinar and Wogen ([4], [5], [6], [7]).

The complex symmetry of weighted composition operators ($M_h C_\psi$; where M_h is multiplication by an analytic function h on $\mathbb{B}_n$) was recently studied by Garcia and Hammond [3] for $n = 1$. A complete description of complex symmetric weighted composition operators remains an open problem (even in the case $h \equiv 1$). Normal operators are the most obvious examples of complex symmetric operators, and the characterization of normal composition operators on H_s for $s > 0$ follows from Theorem 8.2 of [2]:

Proposition 1.1. *Let ψ be an analytic self-map of $\mathbb{B}_n$. Then C_ψ is normal on H_s for $s > 0$ if and only if $\psi(z) = Az$ for some normal linear operator A on $\mathbb{C}^n$ with $\|A\| \leq 1$.*

For a linear operator V on $\mathbb{C}^n$ with $\|V\| \leq 1$, we denote by C_V the composition operator C_{ϕ_V} where $\phi_V(z) = Vz$ for $z \in \mathbb{B}_n$. Since ϕ_V is a linear fractional map of the form (1.1), the operator C_V is bounded on $\mathcal{H}$ and $C_V^* = C_{V^*}$. Define the conjugate-linear operator $J : \mathcal{H} \rightarrow \mathcal{H}$ by $(Jf)(z) = \overline{f(\bar{z})}$ for $f \in \mathcal{H}$ and $z \in \mathbb{B}_n$. That J is a conjugation on $\mathcal{H}$ follows from (1.3) and the fact that J is isometric on $H^2(\mathbb{B}_n)$. The following result is useful for constructing examples of complex symmetric composition operators.

Proposition 1.2. *If V is a complex symmetric linear operator on $\mathbb{C}^n$ with $\|V\| \leq 1$, then C_V is complex symmetric on $\mathcal{H}$.*

Proof. First suppose that V has a symmetric matrix representation with respect to the standard orthonormal basis of $\mathbb{C}^n$. Then V is complex symmetric with respect to the usual complex conjugation $\bar{z}$ for $z \in \mathbb{C}^n$; that is, $\overline{V^*z} = Vz$. Hence C_V is J -symmetric because for any $f \in \mathcal{H}$ and $z \in \mathbb{C}^n$, we have

$$(JC_V^*Jf)(z) = f(\overline{V^*z}) = f(Vz) = (C_Vf)(z).$$

In general, let U_V be a unitary operator on $\mathbb{C}^n$ such that $W = U_V^*VU_V$ has a symmetric matrix with respect to the standard basis of $\mathbb{C}^n$. Then C_V is unitarily equivalent to C_W ; that is, $C_V = C_{U_V^*}C_W C_{U_V}$ where C_{U_V} is unitary on $\mathcal{H}$. Hence if we define the conjugation $\mathcal{J}_V$ on $\mathcal{H}$ by $\mathcal{J}_V = C_{U_V^*}JC_{U_V}$, it follows that C_V is $\mathcal{J}_V$ -symmetric. $\square$

It is known that all 2×2 complex matrices are complex symmetric [4]. For $n = 2$ it follows that C_V is complex symmetric on $\mathcal{H}$ for any linear operator V on $\mathbb{C}^2$. As a consequence of Proposition 1.1 and Proposition 1.2, if V is complex symmetric on $\mathbb{C}^n$ but not normal, then C_V has the same properties on $\mathcal{H}$.

2. THE COMPLEX SYMMETRY OF C_{φ_a}

Not every complex symmetric composition operator is of the form C_V for some linear operator V on $\mathbb{C}^n$. The composition operator C_{φ_a} is bounded on $\mathcal{H}$ and complex symmetric for all $a \in \mathbb{B}_n$. This is because $C_{\varphi_a} \circ C_{\varphi_a} = I$ and Theorem 2 [7] states that operators algebraic of degree 2 are complex symmetric. Our main goal is to construct a conjugation $\mathcal{J}_a$ such that C_{φ_a} is $\mathcal{J}_a$ -symmetric on $\mathcal{H}$. The particular case $n = 1$ resolves a problem of Garcia and Hammond [3]: *If φ is an involutive disk automorphism, find an explicit conjugation $\mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}$ such that $C_\varphi = \mathcal{J}C_\varphi^*\mathcal{J}$.*

For any $a \in \mathbb{B}_n$, denote by $W_a : \mathcal{H} \rightarrow \mathcal{H}$ the unitary part in the polar decomposition of C_{φ_a} on $\mathcal{H}$. We need two lemmas for constructing the conjugation $\mathcal{J}_a$ from W_a . The first one is a general result about the unitary part in the polar decomposition of a linear involution on a Hilbert space.

Lemma 2.1. *If T is an invertible operator on a Hilbert space with $T^2 = I$ and with polar decomposition $T = U|T|$, then $U^2 = I$. In particular, U is self-adjoint.*

Proof. We first prove that $|T|^{-1} = |T^*|$. It is known that if two positive operators A and B commute, then $\sqrt{A}$ and $\sqrt{B}$ also commute. Hence

$$(T^*T)(TT^*) = (TT^*)(T^*T)$$

implies that $|T||T^*| = |T^*||T|$. So $(|T^*||T|)^2 = |T^*|^2|T|^2 = (TT^*)(T^*T) = I$. Since the product of two commuting positive operators is again positive, the uniqueness of the square root of a positive operator implies $|T||T^*| = |T^*||T| = I$. We also note that the observation $T^*(T^*T) = (TT^*)T^*$ implies $T^*|T| = |T^*|T^*$. Now since $U = T^*|T|$ it follows that

$$U^2 = (|T^*|T^*)(T^*|T|) = |T^*||T| = I.$$

Since U is unitary, $U^2 = I$ implies $U^* = U$. $\square$

By Lemma 2.1, the unitary operator W_a is self-adjoint and satisfies $W_a^2 = I$ on $\mathcal{H}$ for any $a \in \mathbb{B}_n$.

Lemma 2.2. *If $a \in \mathbb{B}_n \cap \mathbb{R}^n$, then $JW_a = W_aJ$.*

Proof. The polar decomposition $C_{\varphi_a} = W_a |C_{\varphi_a}|$ implies that $W_a = C_{\varphi_a}^* |C_{\varphi_a}|$. We first show that $J C_{\varphi_a} = C_{\varphi_a} J$ for $a \in \mathbb{B}_n \cap \mathbb{R}^n$. For $a = (a_1, \dots, a_n) \in \mathbb{B}_n \cap \mathbb{R}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{B}_n$, we have $J \langle z, a \rangle = \overline{\langle z, a \rangle} = \langle z, a \rangle$. If $j = 1, \dots, n$, then by (1.2), $P_a z = \frac{\langle z, a \rangle}{\langle a, a \rangle} a$ and $Q_a z = z - \frac{\langle z, a \rangle}{\langle a, a \rangle} a$, the j -th component function of φ_a denoted by $[\varphi_a]_j$ is

$$\begin{aligned} [\varphi_a]_j(z) &= \left[\frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle} \right]_j = \left[\frac{(1 - \frac{\langle z, a \rangle}{\langle a, a \rangle}) + s_a \frac{\langle z, a \rangle}{\langle a, a \rangle}) a - s_a z}{1 - \langle z, a \rangle} \right]_j \\ &= \frac{(1 - \frac{\langle z, a \rangle}{\langle a, a \rangle}) + s_a \frac{\langle z, a \rangle}{\langle a, a \rangle}) a_j - s_a z_j}{1 - \langle z, a \rangle}. \end{aligned}$$

By $J \langle z, a \rangle = \langle z, a \rangle$ it follows that $J[\varphi_a]_j = [\varphi_a]_j$, and hence for any $j = 1, \dots, n$ we have

$$J C_{\varphi_a} J z_j = J C_{\varphi_a} z_j = J[\varphi_a]_j(z) = [\varphi_a]_j(z) = C_{\varphi_a} z_j.$$

Since C_{φ_a} and J are both multiplicative operators on $\mathcal{H}$, we have $J C_{\varphi_a} J z^\alpha = C_{\varphi_a} z^\alpha$ for any multi-index $\alpha \in \mathbb{N}^n$. Therefore $J C_{\varphi_a} J = C_{\varphi_a}$ for $a \in \mathbb{B}_n \cap \mathbb{R}^n$. From this it follows that $J C_{\varphi_a}^* = C_{\varphi_a}^* J$, and hence $J |C_{\varphi_a}| = |C_{\varphi_a}| J$. Since $W_a = C_{\varphi_a}^* |C_{\varphi_a}|$, we get $J W_a = W_a J$ for $a \in \mathbb{B}_n \cap \mathbb{R}^n$. $\square$

So for $a \in \mathbb{B}_n \cap \mathbb{R}^n$, Lemmas 2.1 and 2.2 imply that the conjugate-linear isometry defined by $\mathcal{J}_a = J W_a$ is an involution on $\mathcal{H}$; that is, $\mathcal{J}_a^2 = (W_a J)(J W_a) = W_a^2 = I$. So $\mathcal{J}_a$ is a conjugation on $\mathcal{H}$, and furthermore:

Theorem 2.3. *If $a \in \mathbb{B}_n \cap \mathbb{R}^n$, then C_{φ_a} is $\mathcal{J}_a$ -symmetric on $\mathcal{H}$.*

Proof. From Lemma 2.2 and the self-adjointness of W_a we get that

$$\begin{aligned} C_{\varphi_a}^* \mathcal{J}_a &= C_{\varphi_a}^* J W_a = C_{\varphi_a}^* W_a J = C_{\varphi_a}^* W_a^* J = C_{\varphi_a}^* |C_{\varphi_a}| C_{\varphi_a} J \\ &= W_a C_{\varphi_a} J = W_a J C_{\varphi_a} = \mathcal{J}_a C_{\varphi_a} \end{aligned}$$

and hence that C_{φ_a} is $\mathcal{J}_a$ -symmetric when $a \in \mathbb{B}_n \cap \mathbb{R}^n$. $\square$

For the general case $a \in \mathbb{B}_n$, we will show that C_{φ_a} is unitarily equivalent to $C_{\varphi_{\tilde{a}}}$ for some $\tilde{a} \in \mathbb{B}_n \cap \mathbb{R}^n$. For $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, denote by $U_\Theta : \mathcal{H} \rightarrow \mathcal{H}$ the bounded composition operator defined by $(U_\Theta f)(z) = f(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)$ for $f \in \mathcal{H}$. Then U_Θ is unitary on $H^2(\mathbb{B}_n)$ and hence also on $\mathcal{H}$ by (1.3). If the vector $-\Theta = (-\theta_1, \dots, -\theta_n) \in \mathbb{R}^n$, then $U_\Theta^* = U_{-\Theta}$. Now fix $a \in \mathbb{B}_n$ and choose $\Theta \in \mathbb{R}^n$ such that $\tilde{a} = (e^{i\theta_1} a_1, \dots, e^{i\theta_n} a_n) \in \mathbb{B}_n$. Since $\langle U_\Theta^* z, a \rangle = \sum_{j=1}^n e^{-i\theta_j} z_j \bar{a}_j = \langle z, \tilde{a} \rangle$ and $s_a = s_{\tilde{a}}$, for $j = 1, \dots, n$ we have

$$\begin{aligned} U_\Theta^* C_{\varphi_a} U_\Theta z_j &= U_\Theta^* C_{\varphi_a} e^{i\theta_j} z_j = U_\Theta^* e^{i\theta_j} [\varphi_a]_j = U_\Theta^* e^{i\theta_j} \frac{(1 - \frac{\langle z, a \rangle}{\langle a, a \rangle}) + s_a \frac{\langle z, a \rangle}{\langle a, a \rangle}) a_j - s_a z_j}{1 - \langle z, a \rangle} \\ &= \frac{(1 - \frac{\langle U_\Theta^* z, a \rangle}{\langle a, a \rangle}) + s_a \frac{\langle U_\Theta^* z, a \rangle}{\langle a, a \rangle}) e^{i\theta_j} a_j - s_a e^{i\theta_j} U_\Theta^* z_j}{1 - \langle U_\Theta^* z, a \rangle} \\ &= \frac{(1 - \frac{\langle z, \tilde{a} \rangle}{\langle \tilde{a}, \tilde{a} \rangle}) + s_{\tilde{a}} \frac{\langle z, \tilde{a} \rangle}{\langle \tilde{a}, \tilde{a} \rangle}) \tilde{a}_j - s_{\tilde{a}} z_j}{1 - \langle z, \tilde{a} \rangle} = C_{\varphi_{\tilde{a}}} z_j, \end{aligned}$$

which implies that $U_{\Theta}^* C_{\varphi_a} U_{\Theta} = C_{\varphi_{\tilde{a}}}$ on $\mathcal{H}$. To define the conjugation $\mathcal{J}_a$ when $a \in \mathbb{B}_n$, we let $\mathcal{J}_a = U_{\Theta} \mathcal{J}_{\tilde{a}} U_{\Theta}^*$. Hence we have proved

Theorem 2.4. *If $a \in \mathbb{B}_n$, then C_{φ_a} is $\mathcal{J}_a$ -symmetric on $\mathcal{H}$, where $\mathcal{J}_a = U_{\Theta} \mathcal{J}_{\tilde{a}} U_{\Theta}^*$ and $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ is chosen such that $\tilde{a} = (e^{i\theta_1} a_1, \dots, e^{i\theta_n} a_n) \in \mathbb{B}_n$.*

REFERENCES

- [1] William Arveson, *Subalgebras of C^* -algebras. III. Multivariable operator theory*, Acta Math. **181** (1998), no. 2, 159–228, DOI 10.1007/BF02392585. MR1668582 (2000e:47013)
- [2] Carl C. Cowen and Barbara D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397026 (97i:47056)
- [3] S. R. Garcia and C. Hammond, *Which weighted composition operators are complex symmetric?* Operator Theory: Advances and Applications **236** (2014), 171–179.
- [4] Stephan Ramon Garcia and Mihai Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), no. 3, 1285–1315 (electronic), DOI 10.1090/S0002-9947-05-03742-6. MR2187654 (2006j:47036)
- [5] Stephan Ramon Garcia and Mihai Putinar, *Complex symmetric operators and applications. II*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3913–3931 (electronic), DOI 10.1090/S0002-9947-07-04213-4. MR2302518 (2008b:47005)
- [6] Stephan Ramon Garcia and Warren R. Wogen, *Complex symmetric partial isometries*, J. Funct. Anal. **257** (2009), no. 4, 1251–1260, DOI 10.1016/j.jfa.2009.04.005. MR2535469 (2011g:47005)
- [7] Stephan Ramon Garcia and Warren R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362** (2010), no. 11, 6065–6077, DOI 10.1090/S0002-9947-2010-05068-8. MR2661508 (2011g:47086)
- [8] Trieu Le, *Self-adjoint, unitary, and normal weighted composition operators in several variables*, J. Math. Anal. Appl. **395** (2012), no. 2, 596–607, DOI 10.1016/j.jmaa.2012.05.065. MR2948251
- [9] Walter Rudin, *Function theory in the unit ball of $\mathbb{C}^n$* , Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 241, Springer-Verlag, New York, 1980. MR601594 (82i:32002)
- [10] Ruhan Zhao and Kehe Zhu, *Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$* (English, with English and French summaries), Mém. Soc. Math. Fr. (N.S.) **115** (2008), vi+103 pp. (2009). MR2537698 (2010g:32010)

ABDUS SALAM SCHOOL OF MATHEMATICAL SCIENCES, NEW MUSLIM TOWN, LAHORE, PAKISTAN
Current address: Departamento de Matemática, ICMC-USP, São Carlos-SP, Brazil
E-mail address: waleed_math@hotmail.com