ON DOUBLY PERIODIC PHASES

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ABSTRACT. Every meromorphic function on $\mathbb{C}$ with doubly periodic phase is equal to an elliptic function multiplied by a meromorphic function determined by the periods.

The purpose of this note is to completely characterize the meromorphic functions on $\mathbb{C}$ that have doubly periodic phases. In particular, our result shows that the characterization suggested by G. Semmler and E. Wegert in [4] is too narrow. The phase of a function $f(z)$ defined on an open set $D \subset \mathbb{C}$ with values in $\widehat{\mathbb{C}}$ is the function $\frac{f(z)}{|f(z)|}$ defined on $\{z \in D : f(z) \in \widehat{\mathbb{C}} \setminus \{0, \infty\}\}$ with values in $\mathbb{T}$.

We recall the definition of the Weierstrass sigma-function ([5, 7]). If $p_1, p_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$, we denote by L the $\mathbb{Z}$ -module generated by p_1, p_2 and we put $\omega = \frac{p_2}{p_1}$. Since the series

$$\sum_{\lambda \in L \setminus \{0\}} \frac{1}{|\lambda|^3}$$

is convergent, the Weierstrass product ([5, 6])

$$z \prod_{\lambda \in L \setminus \{0\}} e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}} \left(1 - \frac{z}{\lambda}\right)$$

defines an entire function $\sigma(z)$ whose set of zeros is precisely L , and which is known as the Weierstrass sigma-function. The following transformation property is probably well-known, but we include a proof for completeness.

Lemma. *For every $\xi_0 \in \mathbb{C}$ and $j \in \{1, 2\}$ the Weierstrass sigma-function satisfies*

$$\frac{\sigma(z)}{\sigma(-\xi_0 + z)} \cdot \frac{\sigma(-\xi_0 + z + p_j)}{\sigma(z + p_j)} = e^{v_j},$$

where

$$v_j = -\frac{3\xi_0}{p_j} + \xi_0 p_j^2 \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{\lambda(\lambda + p_j)^2}.$$

Proof. The Weierstrass sigma-function satisfies ([5, 7])

$$(1) \quad \sigma(z + p_j) = -\sigma(z) e^{\left(z + \frac{p_j}{2}\right)\eta_j} \quad \forall z \in \mathbb{C} \quad \forall j \in \{1, 2\}$$

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for some constants η_j that can be expressed in terms of the Weierstrass zeta-function ([5, 7])

$$\zeta(z) = \frac{1}{z} + \sum_{\lambda \in L \setminus \{0\}} \left(\frac{1}{z - \lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right).$$

Indeed, the derivatives of the functions $z \mapsto \zeta(z + p_j) - \zeta(z)$, $j \in \{1, 2\}$, are identically equal to 0, and one has

$$\eta_j = \zeta(z + p_j) - \zeta(z), \quad j \in \{1, 2\}.$$

Moreover, if we let $z = -\frac{p_j}{2}$ in the previous equations, respectively, we see that

$$\eta_j = 2\zeta\left(\frac{p_j}{2}\right), \quad j \in \{1, 2\},$$

since ζ is an odd function of z . From the transformation property (1) of the Weierstrass sigma-function we have then

$$\begin{aligned} \frac{\sigma(z)}{\sigma(-\xi_0 + z)} \cdot \frac{\sigma(-\xi_0 + z + p_j)}{\sigma(z + p_j)} &= -e^{-2(z + \frac{p_j}{2})\zeta(\frac{p_j}{2})}(-1)e^{2(-\xi_0 + z + \frac{p_j}{2})\zeta(\frac{p_j}{2})} \\ &= e^{v_j} \end{aligned}$$

with

$$\begin{aligned} v_j &= -2\xi_0\zeta\left(\frac{p_j}{2}\right) \\ &= -\frac{4\xi_0}{p_j} - 2\xi_0 \sum_{\lambda \in L \setminus \{0\}} \left(\frac{2}{p_j - 2\lambda} + \frac{1}{\lambda} + \frac{p_j}{2\lambda^2} \right) \\ &= -\frac{4\xi_0}{p_j} - \xi_0 p_j^2 \sum_{\lambda \in L \setminus \{0\}} \frac{1}{\lambda^2(p_j - 2\lambda)} \\ &= -\frac{3\xi_0}{p_j} + \xi_0 p_j^2 \sum_{\lambda \in L \setminus \{0, p_j\}} \frac{1}{\lambda^2(2\lambda - p_j)} \\ &= -\frac{3\xi_0}{p_j} + \xi_0 p_j^2 \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{(\lambda + p_j)^2(2\lambda + p_j)} \\ &= -\frac{3\xi_0}{p_j} + \xi_0 p_j^2 \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{\lambda(\lambda + p_j)^2} + \xi_0 p_j^2 u(p_j) \end{aligned}$$

and

$$\begin{aligned} u(p_j) &= \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{(\lambda + p_j)^2(2\lambda + p_j)} - \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{\lambda(\lambda + p_j)^2} \\ &= - \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{\lambda(\lambda + p_j)(2\lambda + p_j)} \end{aligned}$$

for every $j \in \{1, 2\}$. But $u(p_j) = 0 \forall j \in \{1, 2\}$ since

$$\begin{aligned} u(-p_j) &= - \sum_{\lambda \in L \setminus \{0, p_j\}} \frac{1}{\lambda(\lambda - p_j)(2\lambda - p_j)} \\ &= - \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{(\lambda + p_j)\lambda(2\lambda + p_j)} \\ &= u(p_j) \end{aligned}$$

and

$$\begin{aligned} u(-p_j) &= - \sum_{\lambda \in L \setminus \{0, p_j\}} \frac{1}{\lambda(\lambda - p_j)(2\lambda - p_j)} \\ &= - \sum_{\lambda \in L \setminus \{0, -p_j\}} \frac{1}{(-\lambda)(-\lambda - p_j)(-2\lambda - p_j)} \\ &= -u(p_j). \end{aligned}$$

□

Theorem. Let $f(z)$ be a nonconstant function meromorphic on $\mathbb{C}$. If the phase of $f(z)$ is doubly periodic with primitive periods p_1, p_2 and $\omega = \frac{p_2}{p_1} \in \mathbb{C} \setminus \mathbb{R}$, then there exists an elliptic function $g(z)$ with periods p_1, p_2 such that

$$f(z) = e^{az} g(z) \frac{\sigma(z)}{\sigma(-\xi_0 + z)},$$

where $\xi_0 \in \mathbb{C}$, and $a \in \mathbb{C}$ satisfies $\Im(ap_j) = \Im(v_j) + 2m_j\pi$, $j \in \{1, 2\}$, for some $m_1, m_2 \in \mathbb{Z}$.

Proof. We shall use the observation ([4]) that if the phase of a meromorphic function $f(z)$ on $\mathbb{C}$ has a period p , then $f(z+p) = e^\alpha f(z)$ for some constant $\alpha \in \mathbb{R}$. Let α_1, α_2 be two real numbers such that $f(z+p_1) = e^{\alpha_1} f(z)$ and $f(z+p_2) = e^{\alpha_2} f(z)$. Let F denote the parallelogram with the two vectors p_1, p_2 as adjacent sides. We have $f'(z+p_j) = e^{\alpha_j} f'(z)$ and $\frac{f'(z+p_j)}{f(z+p_j)} = \frac{f'(z)}{f(z)} \forall j \in \{1, 2\}$, so the function $\frac{f'(z)}{f(z)}$ is obviously elliptic with periods p_1, p_2 and its integral over F is therefore 0. In particular, $f(z)$ has the same number of zeros and poles inside F . Let Ξ be the multiset (we take into consideration the multiplicities that occur) consisting of all zeros of $f(z)$ inside F and let Γ be the multiset consisting of all poles of $f(z)$ inside F . We then have

$$\begin{aligned} \sum_{\xi \in \Xi} \xi - \sum_{\gamma \in \Gamma} \gamma &= \frac{1}{2\pi i} \int_F \frac{zf'(z)}{f(z)} dz \\ &= \frac{1}{2\pi i} \int_0^{p_1} (z - (z+p_2)) \frac{f'(z)}{f(z)} dz \\ &\quad - \frac{1}{2\pi i} \int_0^{p_2} (z - (z+p_1)) \frac{f'(z)}{f(z)} dz \\ &= -\frac{p_2}{2\pi i} \int_0^{p_1} d \log f(z) + \frac{p_1}{2\pi i} \int_0^{p_2} d \log f(z) \\ &= -\frac{p_2}{2\pi i} (\alpha_1 + 2n_1\pi i) + \frac{p_1}{2\pi i} (\alpha_2 + 2n_2\pi i) \\ &\equiv -\xi_0 \pmod{L} \end{aligned}$$

for some $n_1, n_2 \in \mathbb{Z}$, where ξ_0 is the unique number inside F that is congruent to $\frac{\alpha_1 p_2 - \alpha_2 p_1}{2\pi i}$ modulo L . The congruence above is equivalent to

$$\sum_{\xi \in \{\xi_0\} \cup \Xi} \xi - \sum_{\gamma \in \{0\} \cup \Gamma} \gamma \equiv 0 \pmod{L}.$$

Hence, by Abel's theorem ([1–3, 5]), there exists an elliptic function $g(z)$ with periods p_1, p_2 such that its multisets of zeros and poles inside F are $\{\xi_0\} \cup \Xi$ and $\{0\} \cup \Gamma$, respectively. The sets of zeros and poles of the meromorphic function $\frac{g(z)}{f(z)}$ are then $\xi_0 + L$ and L , respectively. Since the meromorphic function $\frac{\sigma(-\xi_0 + z)}{\sigma(z)}$ has exactly the same zeros and poles as $\frac{g(z)}{f(z)}$, there exists an entire function $h(z)$ such that

$$f(z) = e^{h(z)} g(z) \frac{\sigma(z)}{\sigma(-\xi_0 + z)}.$$

The conditions $f(z + p_j) = e^{\alpha_j} f(z)$, $j \in \{1, 2\}$, now imply

$$e^{h(z+p_j)} \frac{\sigma(z+p_j)}{\sigma(-\xi_0 + z + p_j)} = e^{\alpha_j + h(z)} \frac{\sigma(z)}{\sigma(-\xi_0 + z)};$$

hence, using the Lemma,

$$\begin{aligned} e^{h(z+p_j) - h(z) - \alpha_j} &= \frac{\sigma(z)}{\sigma(-\xi_0 + z)} \cdot \frac{\sigma(-\xi_0 + z + p_j)}{\sigma(z + p_j)} \\ &= e^{v_j}. \end{aligned}$$

The function $h'(z)$ is therefore doubly periodic and has no poles; hence $h'(z)$ is constant. So $h(z) = az + b$ for some $a, b \in \mathbb{C}$. Moreover, $ap_j - \alpha_j = v_j + 2m_j\pi i$ for $j \in \{1, 2\}$ and some $m_1, m_2 \in \mathbb{Z}$. We must have $ap_j - v_j - 2m_j\pi i = \alpha_j \in \mathbb{R}$, $j \in \{1, 2\}$, that is,

$$\Im(ap_j) = \Im(v_j) + 2m_j\pi, \quad j \in \{1, 2\}.$$

□

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