A NEW PROOF OF THE BILINEAR $T(1)$ THEOREM

JAROD HART

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ABSTRACT. A new simple proof of the bilinear $T(1)$ Theorem in the spirit of the proof of Coifman-Meyer of the celebrated result of David and Journé in the linear case is presented. This new proof is obtained independently of the linear $T(1)$ Theorem by combining recent bilinear square function bounds and a paraproduct construction.

1. INTRODUCTION

A central theme of harmonic analysis over the past few decades has been the study of singular integral operator bounds in terms of subtle cancellation conditions. The most famous result in this study is the $T(1)$ Theorem for linear Calderón-Zygmund operators, proved by David-Journé [7]:

$T(1)$ Theorem. *Let $T : \mathcal{S} \rightarrow \mathcal{S}'$ be a linear singular integral operator with a standard Calderón-Zygmund kernel. Then T can be extended to a bounded operator on $L^2 \rightarrow L^2$ if and only if T has the weak boundedness property and $T1, T^*1$ are in BMO .*

In [7] the authors considered smooth truncations of a singular integral operator and worked in operator norm topologies using the Cotlar-Knapp-Stein Lemma to prove the almost orthogonality of these truncations. In the bilinear setting, this approach quickly fails. The Cotlar-Knapp-Stein Lemma requires one to define truncated operators that map a Hilbert space into itself. In the linear setting this is not a problem, and in fact is natural to work in the Hilbert space L^2 . In the bilinear setting, the truncated operators map from a product of Lebesgue spaces into other Lebesgue spaces. This immediately disqualifies the use of the Cotlar-Knapp-Stein Lemma as it is used in [7].

There are other proofs of the $T(1)$ Theorem. For example, the simple proof of Coifman and Meyer in [5] avoided the almost orthogonality lemma by using a different representation of the operator in terms of smooth ones and directly estimated the operator norm using square functions.

Christ-Journé [6] and Grafakos-Torres [15] obtained versions of the $T(1)$ Theorem in a multilinear setting relying somehow on the linear result. We state an equivalent bilinear version that we will be considering in this work. See Section 2 for definitions.

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Theorem 1.1. *Let $T : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ be a bilinear singular integral operator with standard kernel. Then T can be extended to a bounded operator from $L^p \times L^q$ into L^r for all $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ if and only if T has the weak boundedness property and $T(1, 1), T^{*1}(1, 1), T^{*2}(1, 1)$ are in BMO .*

In [6], the authors used the conditions that T satisfies a certain weak boundedness property (different from the one we will use in this work) and $T(1, 1), T^{*1}(1, 1), T^{*2}(1, 1) \in BMO$ to prove the result using an approach similar to the proof of Coifman-Meyer. They rely on almost orthogonality in a different form and Carleson measure estimates. In [14] the authors replaced the assumptions that T is weakly bounded and $T^{*j}(1, 1) \in BMO$ for $j = 0, 1, 2$ with the single equivalent condition

$$\sum_{j=0}^2 \sup_{\xi, \eta \in \mathbb{R}^n} \left\| T^{*j} \left(e^{i\xi \cdot (\cdot)}, e^{i\eta \cdot (\cdot)} \right) \right\|_{BMO} < \infty.$$

This is also a bilinear version of an equivalent condition in the linear case proved in [7]. The proof in [15] followed an approach combining restricted boundedness properties as in the arguments in Stein's proof [21] of the T(1) Theorem and relying again on linear theory. We also point out that a version of a *reduced* bilinear T(1) Theorem in the context of Triebel-Lizorkin spaces was obtained by Bényi [1] (with yet another weak boundedness property) using smooth atomic and molecular decompositions as in the works of Meyer [20], Frazier-Han-Jawerth-Weiss [9], Frazier-Torres-Weiss [10], Torres [22] and others in the linear case. A different bilinear T(1) Theorem for modulation invariant operators generalizing the bilinear Hilbert transform (and hence outside the multilinear Calderón-Zygmund theory) was more recently obtained by Bényi-Demeter-Nahmod-Thiele-Torres-Villarroja [2].

As in [6], the proof we present here for the bilinear version of the T(1) Theorem follows the Coifman-Meyer approach. The main difference in our method, however, is that we are able to take advantage of recent bilinear square functions from [16] not available before, which simplify some of the arguments. The bilinear square function estimates reduce matters to some very natural computations avoiding the use of the linear T(1) Theorem. We also construct bilinear paraproducts that allow us to further simplify the proof to the case of a *reduced* theorem.

All of the results in this work can be extended to the m -linear setting with obvious modifications of the proofs, but for simplicity we restrict ourselves to working in the bilinear case.

2. FURTHER TECHNICAL DEFINITIONS AND BACKGROUND MATERIAL

As usual, define the Schwartz space $\mathcal{S}$ to be the collection of rapidly decreasing, C^∞ functions with the topology generated by the semi-norms

$$\rho_{\alpha, \beta}(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)|$$

for $\alpha, \beta \in \mathbb{N}_0^n$. We also define the Fourier transform of a function f by

$$\widehat{f}(\xi) = \int e^{-ix\xi} f(x) dx$$

whenever the integral converges. We denote by $\mathcal{D} \subset \mathcal{S}$ the subspace of functions $f \in \mathcal{S}$ with compact support, and $\mathcal{S}_0 \subset \mathcal{S}$ to be the subspace of functions $f \in \mathcal{S}$ with infinite vanishing moment, i.e. $\partial_\xi^\alpha \widehat{f}(0) = 0$ for all $\alpha \in \mathbb{N}_0^n$. Given a

function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, we use the typical notation for $t > 0$, $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$, $\varphi^y(x) = \varphi(x - y)$ and $\varphi^{y,t}(x) = \varphi(\frac{x-y}{t})$.

Definition 2.1. Let $T : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ be a bilinear operator. Then T is a bilinear singular integral operator with standard Calderón-Zygmund kernel if there exists a distribution $K \in (\mathcal{S} \times \mathcal{S} \times \mathcal{S})'$ so that

$$\langle T(f, g), h \rangle = \langle K, h \otimes f \otimes g \rangle$$

for all $f, g, h \in \mathcal{S}$, where K agrees with a locally integrable function $K(x, y, z)$ on $\mathbb{R}^{3n} \setminus \{(x, x, x) : x \in \mathbb{R}^n\}$, that is, for $\text{supp}(f) \cap \text{supp}(g) \cap \text{supp}(h) = \emptyset$,

$$\langle T(f, g), h \rangle = \iiint K(x, y, z) h(x) f(y) g(z) dx dy dz,$$

and K satisfies

$$(2.1) \quad |K(x, y, z)| \leq \frac{A}{(|x - y| + |x - z|)^{2n}} \text{ for } x \neq y, \text{ or } x \neq z,$$

$$(2.2) \quad |K(x, y, z) - K(x', y, z)| \leq \frac{A|x - x'|^\gamma}{(|x - y| + |x - z|)^{2n+\gamma}} \\ \text{for } |x - x'| \leq \max(|x - y|, |x - z|)/2,$$

$$(2.3) \quad |K(x, y, z) - K(x, y', z)| \leq \frac{A|y - y'|^\gamma}{(|x - y| + |x - z|)^{2n+\gamma}} \\ \text{for } |x - x'| \leq \max(|x - y|, |x - z|)/2,$$

$$(2.4) \quad |K(x, y, z) - K(x, y, z')| \leq \frac{A|z - z'|^\gamma}{(|x - y| + |x - z|)^{2n+\gamma}} \\ \text{for } |z - z'| \leq \max(|x - y|, |x - z|)/2$$

for some $0 < \gamma \leq 1$.

We wish to prove a boundedness criterion for bilinear singular integral operators using the weak boundedness property and conditions of the form $T(1, 1)$. Although $T(1, 1)$ is not defined a priori as $1 \notin \mathcal{S}$, we define $T(f, g)$ for $f, g \in C^\infty \cap L^\infty$ modulo a constant similar to the linear case.

Definition 2.2. Let $T : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ be a bilinear singular integral operator and $\eta \in \mathcal{D}$ with $\eta \geq 0$ and $\eta \equiv 1$ on a neighborhood of the origin. Define $\eta_R(x) = \eta(x/R)$ and

$$T(f, g) = \lim_{R \rightarrow \infty} T(f\eta_R, g\eta_R) - c_R$$

for $f, g \in C^\infty \cap L^\infty$, where this limit is taken in $\mathcal{D}'$ and

$$c_R = \iint_{|y|, |z| > 1} K(0, y, z) \eta_R(y) f(y) \eta_R(z) g(z) dy dz.$$

This limit exists in $\mathcal{D}'$ and is consistent with T modulo a constant just as in the linear case. So when computing $T(f, g)$ for $f, g \in C^\infty \cap L^\infty$, we must restrict ourselves to pairing $T(f\eta_R, g\eta_R)$ with functions $\phi \in \mathcal{D}$ where ϕ has integral zero to be consistent with the definition of T . For the details of these remarks, see for example [15].

We now define the weak boundedness property for linear and bilinear singular integral operators. We will not use the linear weak boundedness property in this

work, but it is worth stating to compare with the bilinear weak boundedness property, as there have been a number of different definitions used. See for example the definitions of the bilinear weak boundedness property in [6] and [1].

Definition 2.3. A function $\phi \in \mathcal{D}$ is a normalized bump of order M if $\text{supp}(\phi) \subset B(0, 1)$ and for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq M$,

$$\|\partial^\alpha \phi\|_{L^\infty} \leq 1.$$

Let $T : \mathcal{S} \rightarrow \mathcal{S}'$ be a linear singular integral operator. Then T has the weak boundedness property for linear singular integral operators if there exists an $M \in \mathbb{N}_0$ such that for all normalized bumps ϕ_1 and ϕ_2 of order M , $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|\langle T\phi_1^{x,t}, \phi_2^{y,t} \rangle| \lesssim t^n.$$

We remark that typically in the literature the above *WBP* is stated for the same translation $x = y$ of the smooth bumps. However, given the size estimate on the standard kernel, the apparently more general condition stated for any x, y is actually equivalent. This can be easily seen by considering separately the cases $|x - y| \lesssim t$ and using the size estimate (2.1) and the integral representation of T for $t \lesssim |x - y|$.

Likewise, we say that a bilinear singular integral operator $T : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ has the weak boundedness property for bilinear singular integral operators, written $T \in \text{WBP}$, if there exists an $M \in \mathbb{N}_0$ such that for all normalized bumps ϕ_1 , ϕ_2 , and ϕ_3 of order M , $x, y, z \in \mathbb{R}^n$ and $t > 0$,

$$(2.5) \quad |\langle T(\phi_1^{x,t}, \phi_2^{y,t}), \phi_3^{z,t} \rangle| \lesssim t^n.$$

Defining the formal transposes of T via

$$\langle T(f, g), h \rangle = \langle T^{*1}(h, g), f \rangle = \langle T^{*2}(f, h), g \rangle,$$

we see that the *WBP* is symmetric by transposition.

The new tool available that we use to decompose bilinear singular integral operators is the following square function estimate, which is Theorem 1.1 and Remark 2.1 from [16], along with the minor correction in [17].

Theorem 2.4. Suppose Θ_k is defined for $k \in \mathbb{Z}$,

$$\Theta_k(f, g)(x) = \iint \theta_k(x, y, z) f(y) g(z) dy dz,$$

with kernels $\theta_k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ for $k \in \mathbb{Z}$ satisfying

$$(2.6) \quad |\theta_k(x, y, z)| \lesssim \frac{2^{2kn}}{(1 + 2^k|x - y|)^N (1 + 2^k|x - z|)^N},$$

$$(2.7) \quad |\theta_k(x, y, z) - \theta_k(x, y', z)| \lesssim 2^{2kn} (2^k|y - y'|)^\gamma,$$

$$(2.8) \quad |\theta_k(x, y, z) - \theta_k(x, y, z')| \lesssim 2^{2kn} (2^k|z - z'|)^\gamma,$$

$$(2.9) \quad \iint \theta_k(x, y, z) dy dz = 0$$

for some $N > n$ and $0 < \gamma \leq 1$. Then for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for with $1 < p, q, r < \infty$,

$$(2.10) \quad \left\| \left(\sum_{k \in \mathbb{Z}} |\Theta_k(f, g)|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}.$$

In fact, in [16] this theorem is proved for a more general accretive setting. Similar results to Theorem 2.4 were proved independently by Grafakos-Oliveira [13] and by Grafakos-Liu-Maldonado-Yang [12] for a certain range of exponents and under a variety of regularity and cancellation assumptions. Here we will need the specific version of Theorem 2.4 as stated above. For previous work on bilinear square functions see for example Maldonado-Naibo [19] and the references therein.

3. CONTINUOUS DECOMPOSITION OF T

In this section, we shall introduce approximation to the identity operators and Littlewood-Paley projection operators to decompose the operator T . We decompose T using operators depending on a continuous parameter t and then translate to the discrete setting of Theorem 2.4. Let $\varphi \in \mathcal{D}$ radial with $\text{supp}(\varphi) \subset B(0, 1/2)$ and $\widehat{\varphi}(0) = 1$, which will be fixed for Sections 3 and 4. Let P_t be defined by $P_t f = \varphi_t * f$ and let $P_t^2 f = \varphi_t * \varphi_t * f$. We now prove a proposition to decompose a bilinear singular T .

Proposition 3.1. *There exist operators $Q_t^{1,j}$, $Q_t^{2,j}$ and $t_{j,k} \in [2^{-(k+1)}, 2^{-k}]$ for $t > 0$, $j = 1, \dots, n$ and $k \in \mathbb{Z}$ such that for all $f, g, h \in \mathcal{S}_0$,*

$$\begin{aligned} \langle T(f, g), h \rangle &= \sum_{j=1}^n \sum_{k \in \mathbb{Z}} \left\langle \Theta_k^{0,j}(f, g), Q_{t_{j,k}}^{2,j} h \right\rangle + \left\langle \Theta_k^{1,j}(h, g), Q_{t_{j,k}}^{2,j} f \right\rangle \\ &\quad + \left\langle \Theta_k^{2,j}(f, h), Q_{t_{j,k}}^{2,j} g \right\rangle, \end{aligned}$$

where

$$(3.1) \quad \Theta_k^{i,j}(f, g) = \frac{2^{-(k+1)}}{t_{j,k}} Q_{t_{j,k}}^{1,j} * T^{*i}(P_{t_{j,k}}^2 f, P_{t_{j,k}}^2 g)$$

for $i = 0, 1, 2$ and $j = 1, 2, \dots, n$. Furthermore $Q_t^{1,j}$, $Q_t^{2,j}$ are convolution operators with kernels $\psi_t^{1,j}, \psi_t^{2,j} \in \mathcal{D}$ which are supported in $B(0, t)$ and have mean zero.

Proof. First define $Q_t^{1,j} f = -2(\partial_{x_j} \varphi)_t * f$ and $Q_t^{2,j} f = (x_j \varphi)_t * f$. Note that the kernels of these operators are smooth functions supported in $B(0, t)$ since φ_t is smooth with support in $B(0, t)$. Since P_t , $Q_t^{1,j}$ and $Q_t^{2,j}$ are smooth in the parameter $t \in (0, \infty)$, it follows that for $f \in \mathcal{S}$ and $t > 0$,

$$t \frac{d}{dt} P_t^2 f = \sum_{j=1}^n Q_t^{1,j} Q_t^{2,j} f$$

as a well defined operator that is continuous on $\mathcal{S}$. Recall that for $f \in \mathcal{S}$, $P_t^2 f \rightarrow f$ as $t \rightarrow 0$ and for $f \in \mathcal{S}_0$, $P_t^2 f \rightarrow 0$ as $t \rightarrow \infty$, where the convergence is in the topology of $\mathcal{S}$. It also follows that for $f, g, h \in \mathcal{S}$ that $P_t^2 f \otimes P_t^2 g \otimes P_t^2 h \rightarrow f \otimes g \otimes h$ as $t \rightarrow 0$ and for $f, g, h \in \mathcal{S}_0$ that $P_t^2 f \otimes P_t^2 g \otimes P_t^2 h \rightarrow 0$ as $t \rightarrow \infty$. This uses the fact that $\rho_{\alpha,\beta}(P_t^2 f)$ is uniformly bounded in t for any Schwartz semi-norm $\rho_{\alpha,\beta}$,

and similarly for g and h . By the continuity of T , for any $f, g, h \in \mathcal{S}_0$,

$$\begin{aligned}
 \langle T(f, g), h \rangle &= \lim_{t \rightarrow 0} \langle T(P_t^2 f, P_t^2 g), P_t^2 h \rangle - \lim_{t \rightarrow \infty} \langle T(P_t^2 f, P_t^2 g), P_t^2 h \rangle \\
 &= \int_0^\infty t \frac{d}{dt} \langle P_t^2 T(P_t^2 f, P_t^2 g), h \rangle \frac{dt}{t} \\
 (3.2) \quad &= \sum_{j=1}^n \sum_{k \in \mathbb{Z}} \int_{2^{-(k+1)}}^{2^{-k}} \left(\langle Q_t^{1,j} Q_t^{2,j} T(P_t^2 f, P_t^2 g), h \rangle \right. \\
 &\quad + \langle P_t^2 T(Q_t^{1,j} Q_t^{2,j} f, P_t^2 g), h \rangle \\
 &\quad \left. + \langle P_t^2 T(P_t^2 f, Q_t^{1,j} Q_t^{2,j} g), h \rangle \right) \frac{dt}{t}.
 \end{aligned}$$

Since $P_t, Q_t^{i,j}$ for $i = 0, 1, 2$ and $j = 1, 2, \dots, n$ are continuous in the t parameter in the topology of $\mathcal{S}$ and T is continuous from $\mathcal{S} \times \mathcal{S}$ into $\mathcal{S}'$, it follows that $\langle T(P_t^2 f, P_t^2 g), Q_t^{1,j} Q_t^{2,j} h \rangle$ is a continuous function of t for $f, g, h \in \mathcal{S}$. Then by the mean value theorem for integrals for each $j = 1, 2, \dots, n$ and $k \in \mathbb{Z}$ there exists $t_{j,k} \in [2^{-(k+1)}, 2^{-k}]$ such that (3.2) is equal to

$$\begin{aligned}
 &\sum_{j=1}^n \sum_{k \in \mathbb{Z}} \left(\langle Q_{t_{j,k}}^{1,j} * T(P_{t_{j,k}}^2 f, P_{t_{j,k}}^2 g), Q_{t_{j,k}}^{2,j} h \rangle + \langle Q_{t_{j,k}}^{1,j} * T^{*1}(P_{t_{j,k}}^2 h, P_{t_{j,k}}^2 g), Q_{t_{j,k}}^{2,j} f \rangle \right. \\
 &\quad \left. + \langle Q_{t_{j,k}}^{1,j} * T^{*2}(P_{t_{j,k}}^2 f, P_{t_{j,k}}^2 h), Q_{t_{j,k}}^{2,j} g \rangle \right) \frac{2^{-(k+1)}}{t_{j,k}} \\
 &= \sum_{k \in \mathbb{Z}} \sum_{j=1}^n \left(\langle \Theta_k^{0,j}(f, g), Q_{t_{j,k}}^{2,j} h \rangle + \langle \Theta_k^{1,j}(h, g), Q_{t_{j,k}}^{2,j} f \rangle + \langle \Theta_k^{2,j}(f, h), Q_{t_{j,k}}^{2,j} g \rangle \right),
 \end{aligned}$$

where $\Theta_k^{i,j}(f, g)$ is defined for $i = 0, 1, 2$ and $j = 1, 2, \dots, n$ as in (3.1). Since $\varphi(x)$ is radial, it follows easily that $x_j \varphi(x)$ has mean zero. It is also trivial by the Fundamental Theorem of Calculus that $\partial_{x_j} \varphi(x)$ has mean zero. This completes the proof of Proposition 3.1. $\square$

4. REDUCED T(1) THEOREM

The goal of this section is to prove the reduced T(1) Theorem, which we state now.

Theorem 4.1. *Let $T : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ be a bilinear singular integral operator with standard kernel. If T has the weak boundedness property and $T(1, 1) = T^{*1}(1, 1) = T^{*2}(1, 1) = 0$, then T can be extended to a bounded operator from $L^p \times L^q$ into L^r for all $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

The strategy to prove Theorem 4.1 is to use Proposition 3.1 to decompose T , and then use Theorem 2.4 to bound the associated square functions. We will also need a lemma verifying (2.6)-(2.8) for the kernels of $\Theta_k^{i,j}$ from Proposition 3.1.

Lemma 4.2. *Let $T : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ be a bilinear singular integral operator. Let φ_t be as before and $\psi_t \in \mathcal{D}$ have mean zero with $\text{supp}(\psi) \subset B(0, 1)$. If $T \in WBP$, $T(1, 1) = 0$ and θ_k is defined by*

$$\theta_t(x, y, z) = \langle T((\varphi_t * \varphi_t)^y, (\varphi_t * \varphi_t)^z), \psi_t^x \rangle,$$

then for some $N > n$,

$$(4.1) \quad \sup_{t \in [2^{-(k+1)}, 2^{-k}]} |\theta_t(x, y, z)| \lesssim \frac{2^{2kn}}{(1 + 2^k|x - y|)^N (1 + 2^k|x - z|)^N},$$

$$(4.2) \quad \sup_{t \in [2^{-(k+1)}, 2^{-k}]} |\theta_t(x, y, z) - \theta_t(x, y', z)| \lesssim 2^{2kn} (2^k|y - y'|),$$

$$(4.3) \quad \sup_{t \in [2^{-(k+1)}, 2^{-k}]} |\theta_t(x, y, z) - \theta_t(x, y, z')| \lesssim 2^{2kn} (2^k|z - z'|),$$

$$(4.4) \quad \iint \theta_t(x, y, z) dy dz = 0 \text{ for all } t > 0.$$

Proof. To show (4.1), we first assume that $2^k(|x - y| + |x - z| + |y - z|) \leq 12$. The weak boundedness of T says that we may test T on normalized bumps of order M . Since up to a constant depending on M , φ and ψ are normalized bumps of order M , for $t \in [2^{-(k+1)}, 2^{-k}]$

$$\begin{aligned} |\theta_t(x, y, z)| &= t^{-3n} |\langle T((\varphi * \varphi)^{y,t}, (\varphi * \varphi)^{z,t}), \psi^{x,t} \rangle| \\ &\lesssim \frac{2^{2kn}}{(1 + 2^k|x - y|)^{n+\gamma/2} (1 + 2^k|x - z|)^{n+\gamma/2}}. \end{aligned}$$

For $2^k(|x - y| + |x - z| + |y - z|) > 12$ and $t \in [2^{-(k+1)}, 2^{-k}]$, assume without loss of generality that $|x - y| \geq \max(|x - z|, |y - z|)$. We would like to use the integral representation of T , so we check that $(\varphi_t * \varphi_t)^y$ and ψ_t^x have disjoint support. For $v \in \text{supp}((\varphi_t * \varphi_t)^y)$ and $u \in \text{supp}(\psi_t^x)$, it follows that $|v - y| \leq t$ and $|u - x| \leq t$. Then for such u, v ,

$$|u - v| \geq |x - y| - |u - x| - |v - y| \geq \frac{1}{4}|x - y| + 3\frac{|x - y|}{4} - \geq \frac{1}{4}|x - y| + t.$$

Hence, using the integral representation of T and that ψ has mean zero,

$$\begin{aligned} |\theta_t(x, y, z)| &= \left| \iiint (K(u, v, w) - K(x, v, w)) \psi_t^x(u) (\varphi_t * \varphi_t)^y(v) (\varphi_t * \varphi_t)^z(w) du dv dw \right| \\ &\lesssim \int_{|u-x|<t} \int_{|v-y|<t} \int_{|w-z|<t} \frac{t^{-3n}|u-x|^\gamma dw dv du}{(|u-v| + |u-w| + |v-w|)^{2n+\gamma}} \\ &\leq \int_{|u-x|<t} \int_{|v-y|<t} \int_{|w-z|<t} \frac{t^{-(3n-\gamma)} dw dv du}{(|x-y|/4 + t)^{n+\gamma/2} (|x-y|/4 + t)^{n+\gamma/2}} \\ &\lesssim \frac{2^{2nk}}{(1 + 2^k|x - y|)^{n+\gamma/2} (1 + 2^k|x - z|)^{n+\gamma/2}}. \end{aligned}$$

It follows that θ_t satisfies (4.1). To show (4.2), it is sufficient to note that

$$|\nabla_y \theta_t(x, y, z)| = t^{-1} \sum_{j=1}^n |\langle T((\partial_{y_j} \varphi * \varphi)_t^y, (\varphi_t * \varphi_t)^z), \psi_t^x \rangle| \lesssim 2^{k(2n+1)}.$$

That same computation holds for (4.3) with $\nabla_z \theta_k(x, y, z)$. To prove (4.4), note that $\widehat{\varphi * \varphi}(0) = 1$ implies that for $R > 0$ sufficiently large, $\eta_R(x) = \varphi_t * \varphi_t * \chi_{B(0,R)}(x)$ satisfies $\eta_R \in \mathcal{D}$ and $\eta_R = 1$ on a neighborhood of the origin. So η_R is an appropriate

function to compute $T(1, 1)$ in the sense of Definition 2.2. Then it follows that $\Theta_t(1, 1) = 0$ for all $t > 0$, as verified in the following computation:

$$\begin{aligned} \iint \theta_t(x, y, z) dy \, dz &= \iint \langle T((\varphi_t * \varphi_t)^y, (\varphi_t * \varphi_t)^z), \psi_t^x \rangle dy \, dz \\ &= \lim_{R \rightarrow \infty} \langle T(\eta_R, \eta_R), \psi_t^x \rangle = 0. \end{aligned}$$

The last equality here is given by the assumption $T(1, 1) = 0$. $\square$

The proof of Theorem 4.1 is now easily achieved using Proposition 3.1, Lemma 4.2, and Theorem 2.4.

Proof. Recall the definition of P_t , $Q_t^{1,j}$, $Q_t^{2,j}$ and $\Theta_k^{i,j}$ for $i = 0, 1, 2$ and $j = 1, 2, \dots, n$ from Proposition 3.1. For $f, g, h \in \mathcal{S}_0$, we apply Proposition 3.1 to get

$$\begin{aligned} |\langle T(f, g), h \rangle| &\leq \sum_{j=1}^n \left\| \left(\sum_{k \in \mathbb{Z}} |\Theta_k^{0,j}(f, g)|^2 \right)^{\frac{1}{2}} \right\|_{L^r} \left\| \left(\sum_{k \in \mathbb{Z}} |Q_{t_{j,k}}^{2,j} h|^2 \right)^{\frac{1}{2}} \right\|_{L^{r'}} \\ &\quad + \sum_{j=1}^n \left\| \left(\sum_{k \in \mathbb{Z}} |\Theta_k^{1,j}(h, g)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}} \left\| \left(\sum_{k \in \mathbb{Z}} |Q_{t_{j,k}}^{2,j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \\ &\quad + \sum_{j=1}^n \left\| \left(\sum_{k \in \mathbb{Z}} |\Theta_k^{2,j}(f, h)|^2 \right)^{\frac{1}{2}} \right\|_{L^{q'}} \left\| \left(\sum_{k \in \mathbb{Z}} |Q_{t_{j,k}}^{2,j} g|^2 \right)^{\frac{1}{2}} \right\|_{L^q}. \end{aligned}$$

The kernels of $\Theta_k^{i,j}$ are given by

$$\theta_k^{i,j}(x, y, z) = \left\langle T^{*i}((\varphi_{t_{j,k}} * \varphi_{t_{j,k}})^y, (\varphi_{t_{j,k}} * \varphi_{t_{j,k}})^z), \psi_{t_{j,k}}^x \right\rangle,$$

where $\varphi_{t_{j,k}}(x - y)$ is the kernel of $P_{t_{j,k}}$ and $\psi_{t_{j,k}}(x - y)$ is the kernel of $Q_t^{1,j*}$. Then for each $i = 0, 1, 2$ and $j = 1, 2, \dots, n$ we have that $\varphi_t * \varphi_t$ and ψ_t satisfy the hypotheses of Lemma 4.2. Hence $\varphi_{t_{j,k}} * \varphi_{t_{j,k}}$, $\psi_{t_{j,k}}$ and $\Theta_k^{i,j}$ satisfy (2.6)-(2.9). Then by Theorem 2.4, it follows that $\Theta_k^{i,j}$ satisfies (2.10) for $i = 0, 1, 2$ and $j = 1, 2, \dots, n$. Then applying this bound with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ for $\Theta_k^{0,j}$, $\frac{1}{r'} + \frac{1}{q} = \frac{1}{p'}$ for $\Theta_k^{1,j}$ and $\frac{1}{p} + \frac{1}{r'} = \frac{1}{q'}$ for $\Theta_k^{2,j}$, it follows that

$$|\langle T(f, g), h \rangle| \lesssim \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^{r'}} + \|h\|_{L^{r'}} \|g\|_{L^q} \|f\|_{L^p} + \|f\|_{L^p} \|h\|_{L^{r'}} \|g\|_{L^q}.$$

Note that it is important here that $1 < p, q, r < \infty$ so that $1 < p', q', r' < \infty$, and we can apply Theorem 2.4 for $1 < p, q, r, p', q', r' < \infty$ and Littlewood-Paley estimates for $1 < p, q, r' < \infty$. Since $\mathcal{S}_0$ is dense in L^p for $1 < p < \infty$, this completes the proof of Lemma 4.2. $\square$

5. A PARAPRODUCT CONSTRUCTION AND THE FULL $T(1)$ THEOREM

In this section we construct paraproducts that allow the reduction of Theorem 1.1 to Theorem 4.1.

Lemma 5.1. *Given $\beta \in BMO$, there exists a bilinear Calderón-Zygmund operator L bounded on $L^p \times L^q \rightarrow L^r$ for all $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, satisfying*

$$L(1, 1) = \beta \text{ and } L^{*1}(1, 1) = L^{*2}(1, 1) = 0.$$

Proof. Let $P_t f = \varphi_t * f$ be a smooth approximation to the identity for some $\varphi \in \mathcal{D}$ that is radial with $\text{supp}(\varphi) \subset B(0, 1)$. Also $Q_t f = \psi_t * f$, where ψ is a radial compactly supported function with mean zero, and

$$\int_0^\infty \widehat{\psi}(te_1)^2 \frac{dt}{t} = 1,$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. Then it follows that

$$\int_0^\infty Q_t^2 f \frac{dt}{t} = f$$

in L^p for all $1 < p < \infty$ and in H^1 . This is a classical result in L^p for $1 < p < \infty$. For H^1 , this fact was originally due to Folland-Stein [8] in the discrete setting and to Wilson in [23] in the continuous setting as used here. Define L with kernel $\ell(x, y, z)$ by the following:

$$\begin{aligned} L(f, g) &= \int_0^\infty L_t(f, g) \frac{dt}{t} = \int_0^\infty Q_t((Q_t \beta)(P_t f P_t g)) \frac{dt}{t}, \\ \ell(x, y, z) &= \int_0^\infty \ell_t(x, y, z) \frac{dt}{t} = \int_0^\infty \int \psi_t(x - u) Q_t \beta(u) \varphi_t(u - y) \varphi_t(u - z) du \frac{dt}{t}. \end{aligned}$$

Define the non-negative measure $d\mu$ on $\mathbb{R}_+^{n+1}$ by

$$(5.1) \quad d\mu(x, t) = |Q_t \beta(x)|^2 dx \frac{dt}{t}.$$

We should first note that $Q_t \beta$ is well defined since $Q_t 1 = 0$. Further, Q_t are Littlewood-Paley-Stein projection operators, so the square function associated to Q_t is bounded on L^p for $1 < p < \infty$. It also follows that (5.1) is a Carleson measure. This type of result was originally proved by Carleson in [4] and further developed by Jones in [18]. A proof of this result can be found as Theorem 7.3.8(c) in [11]. Then for any $f, g, h \in \mathcal{S}$ with $\|h\|_{L^2} \leq 1$, we have for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$:

$$\begin{aligned} |\langle L(f, g), h \rangle| &\leq \int_0^\infty \left| \int (Q_t \beta(x) P_t f(x) P_t g(x)) Q_t h(x) dx \right| \frac{dt}{t} \\ &\leq \left(\int_0^\infty \int |P_t f(x)|^2 |P_t g(x)|^2 |Q_t \beta(x)|^2 dx \frac{dt}{t} \right)^{\frac{1}{2}} \left\| \left(\int_0^\infty |Q_t h|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^2} \\ &\lesssim \left(\int_0^\infty \int |P_t f(x)|^p |Q_t \beta(x)|^2 dx \frac{dt}{t} \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty \int |P_t g(x)|^q |Q_t \beta(x)|^2 dx \frac{dt}{t} \right)^{\frac{1}{q}} \|h\|_{L^2} \\ &\lesssim \|f\|_{L^p} \|g\|_{L^q}. \end{aligned}$$

In the last inequality, we use a result also due to Carleson. A proof of this estimate can be found in [11], stated as Theorem 7.3.7. So we have proved that L is bounded from $L^p \times L^q$ into L^2 for any $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. We now show that

$\ell(x, y, z)$ satisfies (2.1)-(2.4) as well. To see that ℓ satisfies (2.1), first consider ℓ_t :

$$\begin{aligned} |\ell_t(x, y, z)| &\leq \|Q_t\beta\|_{L^\infty} \int |\psi_t(x-u)\varphi_t(u-y)\varphi_t(u-z)|du \\ &\lesssim \int \frac{t^{-n}}{(1+t^{-1}|x-u|)^{4(n+1)}} \\ &\quad \times \frac{t^{-2n}}{(1+t^{-1}|u-y|)^{4(n+1)}(1+t^{-1}|u-z|)^{4(n+1)}} du \\ &\lesssim \frac{t^{-2n}}{(1+t^{-1}\max(|x-y|, |x-z|))^{4(n+1)}} \\ &\leq \frac{t^{-2n}}{(1+t^{-1}|x-y|)^{2(n+1)}(1+t^{-1}|x-z|)^{2(n+1)}}. \end{aligned}$$

Note that here we use $\|Q_t\beta\|_{L^\infty} \lesssim \|\beta\|_{BMO}$. Similarly for (2.2) we have

$$\begin{aligned} &|\ell_t(x, y, z) - \ell_t(x', y, z)| \\ &\leq \int |\psi_t(x-u) - \psi_t(x'-u)| |Q_t\beta(u)\varphi(u-y)\varphi(u-z)|du \\ &\lesssim \int \frac{t^{-n}(t^{-1}|x-x'|)}{(1+t^{-1}|x-u|)^{4(n+1)}} \frac{t^{-2n}}{(1+t^{-1}|u-y|)^{4(n+1)}(1+t^{-1}|u-z|)^{4(n+1)}} du \\ &\lesssim \frac{t^{-2n}(t^{-1}|x-x'|)}{(1+t^{-1}|x-y|)^{2(n+1)}(1+t^{-1}|x-z|)^{2(n+1)}}. \end{aligned}$$

It then follows by taking $d = |x-y| + |x-z|$ that

$$|\ell(x, y, z)| \leq d^{-2(n+1)} \int_0^d \frac{t^{-2n}}{t^{-2(n+1)}} \frac{dt}{t} + \int_d^\infty t^{-2n} \frac{dt}{t} \lesssim d^{-2n}.$$

Similarly we have

$$\begin{aligned} |\ell(x, y, z) - \ell(x', y, z)| &\lesssim |x-x'| d^{-2(n+1)} \int_0^d \frac{t^{-(2n+1)}}{t^{-2(n+1)}} \frac{dt}{t} + |x-x'| \int_d^\infty t^{-(2n+1)} \frac{dt}{t} \\ &\lesssim |x-x'| d^{-(2n+1)}. \end{aligned}$$

With symmetric arguments for (2.3) and (2.4), the kernel ℓ satisfies (2.1)-(2.4). So L is a Calderón-Zygmund operator and is bounded on $L^p \times L^q \rightarrow L^r$ for all $1 < p, q < \infty$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $\eta \in \mathcal{D}$ with $\eta \equiv 1$ on $B(0, 1)$ and $\text{supp}(\eta) \subset B(0, 2)$. Let $\phi \in \mathcal{D}$ with mean zero and N such that $\text{supp}(\phi) \subset B(0, N)$. Then to compute $L(1, 1)$ in the sense of Definition 2.2,

$$\begin{aligned} \langle L(1, 1), \phi \rangle &= \lim_{R \rightarrow \infty} \int_0^{R/4} \int Q_t \phi(x) [P_t \eta_R(x)]^2 Q_t \beta(x) dx \frac{dt}{t} \\ (5.2) \quad &+ \lim_{R \rightarrow \infty} \int_{R/4}^\infty \int Q_t \phi(x) [P_t \eta_R(x)]^2 Q_t \beta(x) dx \frac{dt}{t}. \end{aligned}$$

We may write this only if the two limits on the right hand side of the equation exist. As we are taking $R \rightarrow \infty$ and N is a fixed quantity determined by ϕ , without loss of generality assume that $R > 2N$. Note that for $t \leq R/4$ and $|x| < N + t$,

$$\text{supp}(\varphi_t(x - \cdot)) \subset B(x, t) \subset B(0, N + 2t) \subset B(0, R).$$

Since $\eta_R \equiv 1$ on $B(0, R)$, it follows that $P_t \eta_R(x) = 1$ for all $|x| < N + t$ when $t \leq R/4$. Therefore

$$\lim_{R \rightarrow \infty} \int_0^{R/4} \int Q_t \phi(x) [P_t \eta_R(x)]^2 Q_t \beta(x) dx \frac{dt}{t} = \int_0^\infty \int Q_t^2 \phi(x) \beta(x) dx \frac{dt}{t} = \langle \beta, \phi \rangle,$$

where we have used that Calderón's reproducing formula holds in H^1 . For any $t > 0$,

$$(5.3) \quad \|P_t \eta_R\|_{L^1} \lesssim \|\varphi_t\|_{L^1} \|\eta_R\|_{L^1} \lesssim R^n,$$

$$(5.4) \quad \|P_t \eta_R\|_{L^\infty} \leq \|\varphi_t\|_{L^1} \|\eta_R\|_{L^\infty} = 1,$$

and for any $x \in \mathbb{R}^n$,

$$(5.5) \quad |Q_t \phi(x)| = \left| \int (\psi_t(x-y) - \psi_t(x)) \phi(y) dy \right| \lesssim t^{-(n+1)}.$$

Therefore

$$\begin{aligned} (5.6) \quad & \int_{R/4}^\infty \int |Q_t \phi(x) [P_t \eta_R(x)]^2 Q_t \beta(x)| dx \frac{dt}{t} \\ & \leq \int_{R/4}^\infty \|P_t \eta_R\|_{L^1} \|P_t \eta_R\|_{L^\infty} \|Q_t \beta\|_{L^\infty} \|Q_t \phi\|_{L^\infty} \frac{dt}{t} \\ & \lesssim \int_{R/4}^\infty R^n t^{-(n+1)} \frac{dt}{t} \lesssim R^{-1}. \end{aligned}$$

Hence the second limit in (5.2) exists and tends to 0 as $R \rightarrow \infty$. Then $\langle L(1, 1), \phi \rangle = \langle \beta, \phi \rangle$ for all $\phi \in \mathcal{D}$ with mean zero and hence $L(1, 1) = \beta$ as an element of BMO . Again for any $\phi \in \mathcal{D}$ with mean zero and $\text{supp}(\phi) \subset B(0, N)$, we have

$$\begin{aligned} (5.7) \quad \langle L^{*1}(1, 1), \phi \rangle &= \lim_{R \rightarrow \infty} \int_0^{R/4} \int_{|x| < N+t} Q_t \beta(x) P_t \phi(x) P_t \eta_R(x) Q_t \eta_R(x) dx \frac{dt}{t} \\ &+ \lim_{R \rightarrow \infty} \int_{R/4}^\infty \int_{|x| < N+t} Q_t \beta(x) P_t \phi(x) P_t \eta_R(x) Q_t \eta_R(x) dx \frac{dt}{t}. \end{aligned}$$

Once more, without loss of generality take $R > 2N$. When $|x| < N + t$ and $t \leq R/4$

$$\text{supp}(\psi_t(x - \cdot)) \subset B(x, t) \subset B(0, N + 2t) \subset B(0, R),$$

and hence $Q_t \eta_R(x) = Q_t 1(x) = 0$. With this it is apparent that the first limit in (5.7) is 0. Similar to (5.3)-(5.5), for the terms of (5.7) we have $\|P_t \eta_R\|_{L^1} \lesssim R^n$, $\|Q_t \eta_R\|_{L^\infty} \lesssim 1$, and $\|P_t \phi\|_{L^\infty} \lesssim t^{-(n+1)}$. So the second term of (5.7) tends to 0 as $R \rightarrow \infty$, just like the second term in computing $L(1, 1)$ from (5.6). Then $L^{*1}(1, 1) = 0$, and a similar argument proves that $L^{*2}(1, 1) = 0$, which concludes the proof of Lemma 5.1. $\square$

Finally we prove the full bilinear T(1) Theorem, that is, Theorem 1.1.

Proof. By Lemma 5.1, there exist bilinear Calderón-Zygmund operators L_0, L_1, L_2 such that $L_i^{*i}(1, 1) = T^{*i}(1, 1)$ and $L_j^{*i}(1, 1) = 0$ for $i \neq j$. Define $S : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ by $S(f, g) = T(f, g) - L_0(f, g) - L_1(f, g) - L_2(f, g)$, which by Theorem 4.1 is bounded from $L^p \times L^q$ into L^r for all $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Therefore T is bounded on the same spaces. It follows from a bilinear Calderón-Zygmund weak type endpoint estimate and interpolation that T is bounded from $L^p \times L^q$ into L^r for all $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ (see for example [14] or [15]).

The necessity of the conditions in the theorem are by now also well known; see for example [15]. $\square$

We conclude by observing that our version of the $T(1)$ Theorem still easily applies to one of the main applications of the result. Namely it applies to bilinear pseudodifferential operators

$$T(f, g)(x) = \int p(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix(\xi+\eta)} d\xi d\eta$$

with symbols in the classes $BS_{1,\delta}^0$, $0 \leq \delta < 1$. These operators have standard kernels, and the classes $BS_{1,\delta}^0$ are closed under transposition for $\delta < 1$. Such an operator trivially satisfies the *WBP* with normalized bumps of order $M = 2[n/2+1]$ where $[x]$ is the integer part of x ,

$$\begin{aligned} |\langle T(\phi_1^{x,t}, \phi_2^{y,t}), \phi_3^{z,t} \rangle| &\leq \|p\|_{L^\infty} \|\widehat{\phi_1^{x,t}}\|_{L^1} \|\widehat{\phi_2^{y,t}}\|_{L^1} \|\phi_3^{z,t}\|_{L^1} \\ &\lesssim t^n \|p\|_{L^\infty} \sum_{j=0}^{M/2} \sum_{k=0}^{M/2} \|\Delta^j \phi_1\|_{L^\infty} \|\Delta^k \phi_2\|_{L^\infty} \|\phi_3\|_{L^\infty}. \end{aligned}$$

Formally we have that $T(1, 1) = p(\cdot, 0, 0) \in L^\infty \subset BMO$, and similarly for the transposes of T . Then boundedness of T follows from Theorem 1.1. See [3] for further definitions, details and other related works.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045
 E-mail address: jhart@math.ku.edu