A LOWER BOUND IN THE TAIL LAW OF THE ITERATED LOGARITHM FOR LACUNARY TRIGONOMETRIC SERIES

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ABSTRACT. Salem and Zygmund obtained an upper bound for a tail law of the iterated logarithm for sums of the form $\sum_{k=N}^{\infty} a_k \cos(n_k x) + b_k \sin(n_k x)$, where n_k satisfies a Hadamard gap condition and $\sum_{k=1}^{\infty} a_k^2 + b_k^2 < \infty$. Here we obtain a lower bound in their result under the same hypotheses.

1. INTRODUCTION AND MAIN RESULT

The law of the iterated logarithm (LIL) first arose in the work of Khintchine [5] who sought to obtain the exact rate of convergence in Borel's theorem on normal numbers. This result was generalized by Kolmogorov [6] to sums of independent random variables.

Recall that an increasing sequence of positive numbers $\{n_k\}$ is said to satisfy the Hadamard gap condition if there exists a $q > 1$ such that $\frac{n_{k+1}}{n_k} > q$ for all k . A trigonometric series which has the form $S(\theta) = \sum_{k=1}^{\infty} a_k \cos(n_k \theta) + b_k \sin(n_k \theta)$, where n_k satisfies a Hadamard gap condition and a_k, b_k are real, is often called a q -lacunary series. Let $S_m(\theta)$ denote the m th partial sum of the series. Salem and Zygmund [7] obtained an LIL in the context of a lacunary trigonometric series:

Theorem 1 (Salem and Zygmund). *Suppose that $S(\theta)$ is a q -lacunary series and n_k are positive integers. Set $B_m^2 = \frac{1}{2} \sum_{k=1}^m (a_k^2 + b_k^2)$ and $M_m = \max_{1 \leq k \leq m} (a_k^2 + b_k^2)^{\frac{1}{2}}$. Suppose also that $B_m \rightarrow \infty$ as $m \rightarrow \infty$ and M_m satisfies the Kolmogorov-type condition: $M_m^2 \leq K_m \frac{B_m^2}{\log \log(e^e + B_m^2)}$ for some sequence of numbers $K_m \downarrow 0$. Then*

$$\limsup_{m \rightarrow \infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log \log B_m}} \leq 1$$

for almost every θ in the unit circle T .

Toward a lower bound, progress was first made by Erdős and Gál [3]. Later, M. Weiss [8] gave a complete analogue of Kolmogorov's LIL in this setting.

Theorem 2 (M. Weiss). *Suppose $S_m(\theta) = \sum_{k=1}^m (a_k \cos n_k \theta + b_k \sin n_k \theta)$ is a q -lacunary series. Set $B_m = \left(\frac{1}{2} \sum_{k=1}^m (a_k^2 + b_k^2)\right)^{\frac{1}{2}}$ and $M_m = \max_{1 \leq k \leq m} (a_k^2 + b_k^2)^{\frac{1}{2}}$. Suppose also that $B_m \rightarrow \infty$ as $m \rightarrow \infty$ and M_m satisfies the Kolmogorov-type*

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condition: $M_m^2 \leq K_m \frac{B_m^2}{\log \log (e^e + B_m^2)}$ for some sequence of numbers $K_m \downarrow 0$. Then

$$\limsup_{m \rightarrow \infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log \log B_m}} = 1$$

for almost every θ in the unit circle.

In their paper [7], Salem and Zygmund also introduced a “tail” law of the iterated logarithm for lacunary series. In this LIL, they considered the tail sums of convergent series instead of partial sums of divergent series. We state their result:

Theorem 3 (Salem and Zygmund). *Let $\tilde{S}_N(\theta) = \sum_{k=N}^{\infty} (a_k \cos n_k \theta + b_k \sin n_k \theta)$ where $\frac{n_{k+1}}{n_k} > q > 1$ and $c_k^2 = a_k^2 + b_k^2$ satisfies $\sum_{k=1}^{\infty} c_k^2 < \infty$. Define $\tilde{B}_N = (\frac{1}{2} \sum_{k=N}^{\infty} c_k^2)^{\frac{1}{2}}$ and $\tilde{M}_N = \max_{k \geq N} |c_k|$. Suppose that $\tilde{B}_1 < \infty$ and that $\tilde{M}_N^2 \leq K_N \left(\frac{\tilde{B}_N^2}{\log \log \frac{1}{\tilde{B}_N}} \right)$ for some sequence of numbers $K_N \downarrow 0$ as $N \rightarrow \infty$. Then*

$$\limsup_{N \rightarrow \infty} \frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2 \log \log \frac{1}{\tilde{B}_N}}} \leq 1$$

for a.e. $\theta \in [0, 2\pi]$.

In this paper we will obtain a lower bound in Salem and Zygmund’s tail law of the iterated logarithm. Our main result is:

Theorem 4. *With the same notation and hypotheses as in the previous theorem, we have*

$$\limsup_{N \rightarrow \infty} \frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2 \log \log \frac{1}{\tilde{B}_N}}} \geq 1$$

for a.e. $\theta \in [0, 2\pi]$.

The proof will make use of a mix of techniques used by Salem and Zygmund combined with ideas from the study of dyadic martingales. We remark that the proof we give can also be adapted to give a proof of the lower bound in the M. Weiss LIL which is different from the original. We first fix some notation and state some lemmas which will be used in the course of the proof.

A dyadic subinterval of the unit interval $[0, 1]$ is an interval of the form $[\frac{j}{2^n}, \frac{j+1}{2^n})$, where $n = 0, 1, 2, \dots$ and $j = 0, 1, \dots, 2^n - 1$. As is to be expected, we will need a Borel-Cantelli-type lemma for independent, or at least weakly dependent, random variables. This is provided by the following, whose proof can be found in Bañuelos and Moore [1, p. 79]:

Lemma 5. *For $k = 1, 2, \dots$, suppose F_k is a collection of dyadic cubes whose union is $[0, 1]$ such that F_{k+1} is a refinement of F_k . Suppose that the maximum length of the elements of F_k tends to zero. Suppose $\mathcal{E}_k \subset F_k$ has the property*

$$\forall Q \in F_k, \quad \left| Q \cap \bigcup_{J \in \mathcal{E}_{k+1}} J \right| > |Q| \frac{C}{k}.$$

Set $E_k = \bigcup_{J \in \mathcal{E}_k} J$. Then for a.e. x , $x \in E_k$ i.o.

It is well known that lacunary series exhibit central limit theorem-type behavior. The following, due to Gaposhkin [4], will give us this type of behavior for trigonometric polynomials with a favorable error.

Theorem 6 (Gaposhkin). *Suppose that n_k satisfies $\frac{n_{k+1}}{n_k} \geq q > 1$ and λ_{Nk} satisfies the following conditions:*

$$\sum_{k=1}^{\infty} \lambda_{Nk}^2 = 1, \quad \Lambda_N = \max_k |\lambda_{Nk}| = o(1) \quad \text{as } N \rightarrow \infty.$$

Set $F_N(y) = P(\sum_{k=1}^{\infty} \lambda_{Nk} \omega_{n_k} < y)$, where $\omega_{n_k} = \sqrt{2} \cos(2\pi n_k x)$. Then the distributions $F_N(y)$ converge towards the normal law; moreover,

$$\sup_{-\infty < y < \infty} |F_N(y) - \Phi(y)| \leq c(q) \Lambda_N^{1/4},$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} dt$ and the constant $c(q)$ depends only on q .

Finally, we need a classical result on the exponential square integrability of lacunary series (see Zygmund [9], vol. I, p. 215).

Theorem 7. *Consider the series $g(x) = \sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x)$ with $\alpha^2 = \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$, where $\frac{n_{k+1}}{n_k} \geq q > 1$. If $\alpha \leq 1$, then $\int_0^{2\pi} \exp(Cg^2) dx \leq A$, provided $C \leq C_0(q)$, with A an absolute constant.*

2. THE PROOF OF THE THEOREM

Proof. We note that we may write

$$a_k \cos n_k x + b_k \sin n_k x = \sqrt{a_k^2 + b_k^2} \cos(n_k x - \phi_k).$$

We consider cosine series of the form $\sum_{k=1}^{\infty} a_k \cos(n_k x)$; the proof is identical in the general case. To facilitate working with dyadic cubes, we rescale and consider $\sum_{k=1}^{\infty} a_k \cos(2\pi n_k x)$ on $[0, 1]$.

Let M be a fixed large positive number. Define $N_1 \leq N_2 \leq \dots$ by

$$N_l = \min \left\{ N : \frac{1}{2} \sum_{k=N+1}^{\infty} a_k^2 < \frac{1}{M^l} \right\}.$$

Let $\varepsilon > 0$ and assume $\varepsilon \ll 1$. Choose $\delta > 0$ so that $(1 + \delta)(1 - \varepsilon^2) > 1$. Finally choose $0 < \mu < 1$ so that

$$\frac{\mu \log(8\sqrt{\pi})}{1 + \varepsilon} - \frac{1}{2} \frac{\mu \log \mu}{1 + \varepsilon} + \frac{1}{1 + \varepsilon} < 1.$$

Using the definition of N_l and the fact that $|a_N| < \varepsilon \sqrt{\sum_{k=N}^{\infty} a_k^2}$, for N sufficiently large, we can assume that l is sufficiently large so that

$$(2.1) \quad (1 - \varepsilon^2) \frac{1}{M^l} < \frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 < \frac{1}{M^l}.$$

Consequently,

$$(2.2) \quad (1 - \varepsilon^2)M < \frac{\sum_{k=N_l+1}^{\infty} a_k^2}{\sum_{k=N_l+1+1}^{\infty} a_k^2} < \frac{M}{1 - \varepsilon^2}.$$

By Theorem 7 (rescaled to $[0, 1]$) and Chebyshev's inequality we obtain

$$\left| \left\{ x \in [0, 1] : \left| \sum_{k=N_{l+1}+1}^{\infty} a_k \cos(2\pi n_k x) \right| \geq \lambda \sqrt{\sum_{k=N_{l+1}+1}^{\infty} a_k^2} \right\} \right| \leq A e^{-C\lambda^2},$$

for every $\lambda > 0$.

Put

$$\lambda = \left(\sum_{k=N_{l+1}+1}^{\infty} a_k^2 \right)^{-\frac{1}{2}} \sqrt{\frac{(1+\delta)}{CM} \sum_{k=N_{l+1}+1}^{\infty} a_k^2 \log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_{l+1}+1}^{\infty} a_k^2}}}}$$

and use (2.2) to obtain the estimate

$$\begin{aligned} & \left| \left\{ x \in [0, 1] : \left| \sum_{k=N_{l+1}+1}^{\infty} a_k \cos(2\pi n_k x) \right| \right. \right. \\ & \quad \left. \left. > \sqrt{\frac{(1+\delta)}{CM} \sum_{k=N_{l+1}+1}^{\infty} a_k^2 \log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_{l+1}+1}^{\infty} a_k^2}}} \right\} \right| \\ & \leq A \exp(-(1+\delta)(1-\varepsilon^2) \log \log M^{\frac{1}{2}}) = A \left(l \frac{\log M}{2} \right)^{-(1+\delta)(1-\varepsilon^2)}. \end{aligned}$$

Since $(1+\delta)(1-\varepsilon^2) > 1$, by the Borel-Cantelli lemma for a.e. x we have

$$(2.3) \quad \frac{\left| \sum_{k=N_{l+1}+1}^{\infty} a_k \cos(2\pi n_k x) \right|}{\sqrt{\sum_{k=N_{l+1}+1}^{\infty} a_k^2 \log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_{l+1}+1}^{\infty} a_k^2}}}}} \leq \sqrt{\frac{1+\delta}{CM}}$$

for all sufficiently large l (depending on x). The definition of N_l and (2.1) yields

$$(2.4) \quad \frac{1}{2} \sum_{k=N_{l+1}}^{N_{l+1}} a_k^2 > \frac{1}{M^l} \left(1 - \varepsilon^2 - \frac{1}{M} \right) > \left(\frac{1}{2} \sum_{k=N_{l+1}}^{\infty} a_k^2 \right) \left(1 - \varepsilon^2 - \frac{1}{M} \right).$$

By hypotheses, for all sufficiently large N ,

$$\max_{k \geq N} a_k^2 \leq K_N \left(\frac{\frac{1}{2} \sum_{k=N}^{\infty} a_k^2}{\log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N}^{\infty} a_k^2}}} \right) \leq \frac{\varepsilon}{2} \left(\frac{\frac{1}{2} \sum_{k=N}^{\infty} a_k^2}{\log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N}^{\infty} a_k^2}}} \right).$$

Then for l for large enough,

$$\max_{k \geq N_{l+1}} a_k^2 < \frac{K_{N_{l+1}}}{(1-\varepsilon^2-\frac{1}{M})} \frac{1}{\log l} \frac{1}{2} \sum_{k=N_{l+1}}^{N_{l+1}} a_k^2 < \frac{\varepsilon/2}{(1-\varepsilon^2-\frac{1}{M})} \frac{1}{\log l} \frac{1}{2} \sum_{k=N_{l+1}}^{N_{l+1}} a_k^2.$$

We may assume that ε is small enough and M large enough so that $1 - \varepsilon^2 - \frac{1}{M} > \frac{1}{2}$. Thus,

$$(2.5) \quad \max_{k \geq N_{l+1}} \frac{|a_k|}{\sqrt{\frac{1}{2} \sum_{k=N_{l+1}}^{N_{l+1}} a_k^2}} \leq \sqrt{\frac{2K_{N_{l+1}}}{\log l}} \leq \sqrt{\frac{\varepsilon}{\log l}}.$$

Suppose l is large so that $\mu \log l \gg 1$. We define a sequence of positive integers $l_1, l_2, \dots, l_{\square}$, where for simplicity we write $\square = \left\lceil \frac{\mu \log l}{1 + \varepsilon} \right\rceil$ ($\square$ represents the greatest integer function) as follows:

Let l_1 be the first time such that

$$\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2 \geq \frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2$$

so that

$$\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 < \frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2.$$

Likewise, let l_2 be the first time such that

$$\frac{1}{2} \sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2 \geq \frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2$$

so that

$$\frac{1}{2} \sum_{k=N_l+l_1+1}^{N_l+l_2-1} a_k^2 < \frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2.$$

Similarly we define $l_3, \dots, l_{\square}$. By (2.5) we have $\forall j = 1, 2, \dots, \square$ with $l_0 = 0$,

$$\frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2 \leq \frac{1}{2} \sum_{k=N_l+l_{j-1}+1}^{N_l+l_j} a_k^2 < (1 + \varepsilon) \frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2.$$

Adding these from $j = 1$ to $j = \square$ gives

$$\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_{\square}} a_k^2 \leq \left[\frac{\mu \log l}{1 + \varepsilon} \right] \frac{1 + \varepsilon}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2 \leq \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2,$$

which shows that $N_l + l_{\square}$ does not exceed N_{l+1} . Consider a dyadic cube Q such that $|Q| = 2^{-L}$, where L is chosen so that $2^L \leq n_{N_l} < 2^{L+1}$. By Theorem 6 and (2.5) we have

$$\begin{aligned} & \frac{1}{|Q|} \left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} \frac{a_k}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2}} \cos(2\pi n_k x) > \eta \right\} \right| \\ & \geq \frac{1}{\sqrt{2\pi}} \int_{\eta}^{\infty} e^{-\frac{t^2}{2}} dt - c(q) (2\mu K_{N_l+1})^{\frac{1}{8}}. \end{aligned}$$

Set $\eta = \sqrt{\frac{2}{\mu}}$. We can assume that l is large enough (depending only on q, ε , and μ) so that

$$\frac{1}{|Q|} \left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} \frac{a_k}{\sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}} \sqrt{2} \cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}} \right\} \right| \geq \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2}{\mu}}}^{\infty} e^{-\frac{t^2}{2}} dt.$$

Rearrange and use the estimate $\int_{\xi}^{\infty} e^{-\frac{u^2}{2}} du \geq \frac{\xi}{1+\xi^2} e^{-\frac{\xi^2}{2}}$ (valid for $\xi > 0$; see e.g.

Chung [2, p. 231]) and the fact that $\mu < 1$ to obtain

$$(2.6) \quad \left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k \cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2} \right\} \right| \geq \frac{1}{8\sqrt{\pi}} \sqrt{\mu} e^{-\frac{1}{\mu}} |Q|.$$

Let

$$G(Q) = \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k \cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2} \right\}.$$

Put $h(x) = \sum_{k=N_l+1}^{N_l+l_1} a_k \cos(2\pi n_k x)$. Then $|h'(x)| \leq 2\pi \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} n_k^2}$.

Choose L_1 so that $2^{L_1} \leq n_{N_l+l_1} < 2^{L_1+1}$. Then $\sqrt{\sum_{k=N_l+1}^{N_l+l_1} n_k^2} \leq 2^{L_1+1} \frac{q}{\sqrt{q^2-1}}$.

Thus,

$$|h'(x)| \leq 2\pi \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} n_k^2} \leq 2\pi 2^{L_1+1} \frac{q}{\sqrt{q^2-1}} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}.$$

Let $x \in G(Q)$ and consider y with $|x - y| < 2^{-L_1}$. Then

$$(2.7) \quad |h(y)| \geq |h(x)| - B \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}$$

(where $B = B(q)$) and hence

$$|h(y)| \geq \sqrt{\frac{2}{\mu}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2} - B \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} = \left(\frac{\sqrt{2} - \sqrt{2\mu}B}{\sqrt{\mu}} \right) \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2}.$$

From (2.6), this last estimate, and the definition of l_1 , we conclude that there exists a collection of dyadic subcubes $\{Q'\}$ of Q with each $|Q'| = 2^{-L_1}$ such that $\forall x \in Q'$,

$$\sum_{k=N_l+1}^{N_l+l_1} a_k \cos(2\pi n_k x) \geq \left(\frac{\sqrt{2} - \sqrt{2\mu}B}{\sqrt{\mu}} \right) \sqrt{\frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_l+l_1} a_k^2}$$

and with $\left| \bigcup_{Q' \subset Q} Q' \right| > \frac{1}{8\sqrt{\pi}} \sqrt{\mu} e^{-\frac{1}{\mu}} |Q|$.

Again on each Q' we use Theorem 6 to obtain

$$\left| \left\{ x \in Q' : \sum_{k=N_l+l_1+1}^{N_l+l_2} a_k \cos(2\pi n_k x) > \sqrt{\frac{2}{\mu}} \sqrt{\frac{1}{2} \sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2} \right\} \right| \geq \frac{\sqrt{\mu} e^{-\frac{1}{\mu}}}{8\sqrt{\pi}} |Q'|.$$

Then as above there exists a collection of dyadic subcubes $\{Q''\}$ of Q' with $|Q''| = 2^{-L_2}$, where L_2 satisfies $2^{L_2} \leq n_{N_l+l_2} < 2^{L_2+1}$, such that $\forall x \in Q''$,

$$\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k \cos(2\pi n_k x) \geq \left(\frac{\sqrt{2} - \sqrt{2\mu}B}{\sqrt{\mu}} \right) \sqrt{\frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

and with $\left| \bigcup_{Q'' \subset Q'} Q'' \right| > \frac{1}{8\sqrt{\pi}} \sqrt{\mu} e^{-\frac{1}{\mu}} |Q'|$. We continue this process. Eventually we come to a subcollection of cubes $\{I\}$ with $|I| = 2^{-L_\square}$, where $\square$ is the greatest integer contained in $\frac{\mu \log l}{1+\varepsilon}$ and $L_\square$ is the number satisfying $2^{L_\square} \leq n_{N_l+l_\square} < 2^{L_\square+1}$, such that $\forall x \in I$,

$$\sum_{k=N_l+l_\square+1}^{N_l+l_\square} a_k \cos(2\pi n_k x) \geq \left(\frac{\sqrt{2} - \sqrt{2\mu}B}{\sqrt{\mu}} \right) \sqrt{\frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

Moreover, $\left| \bigcup_{I \subset \tilde{Q}} I \right| > \frac{1}{8\sqrt{\pi}} \sqrt{\mu} e^{-\frac{1}{\mu}} |\tilde{Q}|$, where $\tilde{Q}$ is the previous generation cube. On each such I , we use Theorem 8.25, Chapter 5, of [9] (rescaled to I) to conclude that

$$\frac{1}{|I|} \left| \left\{ x \in I : \sum_{k=N_l+l_\square+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \lambda_q \sqrt{\frac{1}{2} \sum_{k=N_l+l_\square+1}^{N_{l+1}} a_k^2} \right\} \right| \geq \mu_q,$$

where λ_q and μ_q are positive constants depending only on q . Let $G(I)$ denote the above set. Choose $\tilde{L}$ so that $2^{\tilde{L}} \leq n_{N_{l+1}} < 2^{\tilde{L}+1}$. Let

$$h(x) = \sum_{k=N_l+l_\square+1}^{N_{l+1}} a_k \cos(2\pi n_k x).$$

Using (2.5) we can then estimate

$$|h'(x)| \leq 2\pi \sum_{k=N_l+l_\square+1}^{N_{l+1}} |a_k| n_k \leq 2\pi \sqrt{\frac{2K_{N_{l+1}}}{\log l}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2} \cdot \frac{2^{\tilde{L}+1} q}{q-1}.$$

Let $x \in G(I)$ and consider y such that $|x-y| < 2^{-\tilde{L}}$. Then

$$|h(y)| \geq |h(x)| - c(q) \sqrt{\frac{K_{N_{l+1}}}{\log l}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

Thus, there exists a collection of dyadic subcubes $\{J\}$ of I with $|J| = 2^{-\tilde{L}}$ such that $\forall x \in J$,

$$\begin{aligned} \sum_{k=N_l+l_\square+1}^{N_{l+1}} a_k \cos(2\pi n_k x) &> \lambda_q - c(q) \sqrt{\frac{K_{N_{l+1}}}{\log l}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\ &> -c(q) \sqrt{\frac{K_{N_{l+1}}}{\log l}} \sqrt{\frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}. \end{aligned}$$

Moreover, $|I \cap \bigcup_{J \subset I} J| > \mu_q |I|$.

Finally, adding the estimates from all of the above generations, we have

$$\begin{aligned} & \sum_{k=N_l+1}^{N_l+l_1} a_k \cos(2\pi n_k x) + \cdots + \sum_{k=N_l+l_{\square}-1}^{N_l+l_{\square}} a_k \cos(2\pi n_k x) \\ & + \sum_{k=N_l+l_{\square}+1}^{N_{l+1}} a_k \cos(2\pi n_k x) \\ & > \left(\left\lceil \frac{\mu \log l}{1+\varepsilon} \right\rceil \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}} \right) - c(q)\sqrt{K_{N_l+1}\mu} \right) \sqrt{\frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2} \end{aligned}$$

on a subcollection $\{J\}$ of dyadic subcubes of Q with $|J| = 2^{-\tilde{L}}$ and with

$$\begin{aligned} |Q \cap \bigcup J| & > |Q| \left(\frac{1}{8\sqrt{\pi}} \sqrt{\mu} e^{-\frac{1}{\mu}} \right)^{\left\lceil \frac{\mu \log l}{1+\varepsilon} \right\rceil} \mu_q \geq |Q| \left(\frac{1}{8\sqrt{\pi}} \sqrt{\mu} e^{-\frac{1}{\mu}} \right)^{\frac{\mu \log l}{1+\varepsilon}} \mu_q \\ & = |Q| l^{-\left(\frac{\mu \log 8\sqrt{\pi}}{1+\varepsilon} - \frac{\mu \log \mu}{2(1+\varepsilon)} + \frac{1}{1+\varepsilon} \right)} \mu_q \geq C \frac{|Q|}{l}. \end{aligned}$$

Consequently, $|Q \cap \bigcup J| > C \frac{|Q|}{l}$. We may assume that l is large enough so that

$$\left\lceil \frac{\mu \log l}{1+\varepsilon} \right\rceil / \left(\frac{\mu \log l}{1+\varepsilon} \right) > \frac{1}{(1+\varepsilon)^{1/2}}$$

and

$$\left(\frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^{3/2}} \right) - c(q) \frac{\sqrt{K_{N_l+1}}}{\log l} \geq \frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^2}$$

so that then

$$\begin{aligned} & \left(\left\lceil \frac{\mu \log l}{1+\varepsilon} \right\rceil \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}} \right) - c(q)\sqrt{K_{N_l+1}\mu} \right) \sqrt{\frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\ & \geq \left(\left(\frac{\mu \log l}{(1+\varepsilon)^{3/2}} \right) \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{\sqrt{\mu}} \right) - c(q)\sqrt{K_{N_l+1}\mu} \right) \sqrt{\frac{1}{\mu \log l} \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\ & = \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^{3/2}} - c(q) \frac{\sqrt{K_{N_l+1}}}{\log l} \right) \sqrt{\log l \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\ & \geq \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^2} \right) \sqrt{\log l \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}. \end{aligned}$$

Then

$$\left\{ x \in Q : \sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x) > \left(\frac{\sqrt{2}-\sqrt{2\mu}B}{(1+\varepsilon)^2} \right) \sqrt{\log l \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2} \right\} \supseteq \bigcup J.$$

Thus, if we let F_l denote the family of dyadic cubes Q in $[0, 1]$ of sidelength 2^{-L} (recall $2^L \leq n_{N_l} < 2^{L+1}$) and let $\mathcal{E}_{l+1}$ denote the union of those cubes J of sidelength $2^{-\tilde{L}}$ (recall $2^{\tilde{L}} \leq n_{N_{l+1}} < 2^{\tilde{L}+1}$) found in all of Q using the above argument, then, for

large enough l (depending only on ε and M), the hypotheses of Lemma 5 are satisfied, so that for a.e. x there exists an infinite sequence of numbers $N_1 < N_2 < \dots$ such that

$$(2.8) \quad \frac{\sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x)}{\sqrt{\log l \frac{1}{2} \sum_{k=N_l+1}^{N_{l+1}} a_k^2}} > \left(\frac{\sqrt{2} - \sqrt{2\mu}B}{(1+\varepsilon)^2} \right)$$

for all l sufficiently large. By (2.1), for sufficiently large l ,

$$\log l > \frac{\log(l \log \sqrt{M} - \log \sqrt{1-\varepsilon^2})}{1+\varepsilon} > \frac{\log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2}}}{1+\varepsilon}.$$

Then using this and (2.4) in (2.8) we have

$$\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k \cos(2\pi n_k x)}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 \log \log \left(\frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2}} \right)}} \geq \frac{\sqrt{2} - \sqrt{2\mu}B}{(1+\varepsilon)^{\frac{5}{2}}} \sqrt{1-\varepsilon^2 - \frac{1}{M}}.$$

Consequently, for a.e. x and large enough l (depending on x),

$$\frac{\sum_{k=N_l+1}^{\infty} a_k \cos(2\pi n_k x) - \sum_{k=N_{l+1}+1}^{\infty} a_k \cos(2\pi n_k x)}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 \log \log \left(\frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2}} \right)}} \geq \frac{\sqrt{2} - \sqrt{2\mu}B}{(1+\varepsilon)^{5/2}} \sqrt{1-\varepsilon^2 - \frac{1}{M}}.$$

But from (2.3) for a.e. x we have

$$\frac{|\sum_{k=N_{l+1}+1}^{\infty} a_k \cos(\pi n_k x)|}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 \log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2}}}} \leq \sqrt{\frac{2(1+\delta)}{CM}}$$

for sufficiently large l (depending on x).

Hence for a.e. x and large enough l we have

$$\frac{\sum_{k=N_l+1}^{\infty} a_k \cos(2\pi n_k x)}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2 \log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=N_l+1}^{\infty} a_k^2}}}} \geq \frac{\sqrt{2} - \sqrt{2\mu}B}{(1+\varepsilon)^{5/2}} \sqrt{1-\varepsilon^2 - \frac{1}{M}} - \sqrt{\frac{2(1+\delta)}{CM}}.$$

Thus, for a.e. x ,

$$\limsup_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} a_k \cos(2\pi n_k x)}{\sqrt{\frac{1}{2} \sum_{k=n}^{\infty} a_k^2 \log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=n}^{\infty} a_k^2}}}} \geq \frac{\sqrt{2} - \sqrt{2\mu}B}{(1+\varepsilon)^{5/2}} \sqrt{1-\varepsilon^2 - \frac{1}{M}} - \sqrt{\frac{2(1+\delta)}{CM}}.$$

By the choice of δ , we can take $\delta \searrow 0$ as $\varepsilon \searrow 0$. Also, by the choice of μ , we have $\mu \searrow 0$ as $\varepsilon \searrow 0$. Then letting $M \nearrow \infty$, $\varepsilon \searrow 0$, and $\delta \searrow 0$ we obtain

$$\limsup_{n \rightarrow \infty} \frac{\sum_{k=n}^{\infty} a_k \cos(2\pi n_k x)}{\sqrt{2 \frac{1}{2} \sum_{k=n}^{\infty} a_k^2 \log \log \frac{1}{\sqrt{\frac{1}{2} \sum_{k=n}^{\infty} a_k^2}}}} \geq 1.$$

□

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