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ON GENERALIZED HYPERGEOMETRIC EQUATIONS AND MIRROR MAPS

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ABSTRACT. This paper deals with generalized hypergeometric differential equations of order $n \geq 3$ having maximal unipotent monodromy at 0. We show that among these equations those leading to mirror maps with integral Taylor coefficients at 0 (up to simple rescaling) have special parameters, namely R-partitioned parameters. This result yields the classification of all generalized hypergeometric differential equations of order $n \geq 3$ having maximal unipotent monodromy at 0 such that the associated mirror map has the above integrality property.

1. INTRODUCTION

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be an element of $(\mathbb{Q} \cap]0,1[)^n$ for some integer $n \geq 3$. We consider the generalized hypergeometric differential operator given by

$$\mathcal{L}_{\boldsymbol{\alpha}} = \delta^n - z \prod_{k=1}^{n} (\delta + \alpha_k),$$

where $\delta = z\frac{d}{dz}$. It has maximal unipotent monodromy at 0. Frobenius' method yields a basis of solutions $y_{\boldsymbol{\alpha};1}(z), \ldots, y_{\boldsymbol{\alpha};n}(z)$ of $\mathcal{L}_{\boldsymbol{\alpha}} y(z) = 0$ such that

$$y_{\boldsymbol{\alpha};1}(z) \in \mathbb{C}(\{z\})^{\times}, \tag{1}$$

$$y_{\boldsymbol{\alpha};2}(z) \in \mathbb{C}(\{z\}) + \mathbb{C}(\{z\})^{\times} \log(z), \tag{2}$$

$$\vdots \quad \vdots \quad \vdots$$

$$y_{\boldsymbol{\alpha};n}(z) \in \sum_{k=0}^{n-2} \mathbb{C}(\{z\}) \log(z)^k + \mathbb{C}(\{z\})^{\times} \log(z)^{n-1}, \tag{3}$$

where $\mathbb{C}(\{z\})$ denotes the field of germs of meromorphic functions at $0 \in \mathbb{C}$. One can assume that $y_{\boldsymbol{\alpha};1}$ is the generalized hypergeometric series

$$y_{\boldsymbol{\alpha};1}(z) := F_{\boldsymbol{\alpha}}(z) := \sum_{k=0}^{+\infty} \frac{(\boldsymbol{\alpha})_k}{k!^n} z^k \in \mathbb{C}(\{z\}),$$

where the Pochhammer symbols $(\boldsymbol{\alpha})_k := (\alpha_1)_k \cdots (\alpha_n)_k$ are defined by $(\alpha_i)_0 = 1$ and, for $k \in \mathbb{N}^*$, $(\alpha_i)_k = \alpha_i(\alpha_i + 1) \cdots (\alpha_i + k - 1)$. One can also assume that

$$y_{\boldsymbol{\alpha};2}(z) = G_{\boldsymbol{\alpha}}(z) + \log(z) F_{\boldsymbol{\alpha}}(z),$$

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where

$$G_{\boldsymbol{\alpha}}(z)=\sum_{k=1}^{+\infty}\frac{(\boldsymbol{\alpha})_k}{k!^n}\left(nH_k(1)-\sum_{i=1}^{n}H_k(\alpha_i)\right)z^k\in\mathbb{C}(\{z\}),$$

with $H_k(x)=\sum_{k=0}^{n-1}\frac{1}{x+k}$.

For details on the generalized hypergeometric equations, we refer to Beukers and Heckman [2] and to the subsequent work of Katz [8].

Let us consider

$$\mathcal{Q}_{\boldsymbol{\alpha}}(z)=\exp\left(\frac{y_{\boldsymbol{\alpha};2}(z)}{y_{\boldsymbol{\alpha};1}(z)}\right)=z\exp\left(\frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)}\right).$$

This paper is concerned with the following problem: *describe the parameters $\boldsymbol{\alpha}$ such that, for some $\kappa\in\mathbb{N}^*$,*

$$\kappa^{-1}\mathcal{Q}_{\boldsymbol{\alpha}}(\kappa z)=z\exp\left(\frac{G_{\boldsymbol{\alpha}}(\kappa z)}{F_{\boldsymbol{\alpha}}(\kappa z)}\right)$$

has integral Taylor coefficients at 0. This kind of problem appears in mirror symmetry theory. In this context, the map $\mathcal{Q}_{\boldsymbol{\alpha}}(z)$ is usually called the canonical coordinate. In what follows, we will identify $\mathcal{Q}_{\boldsymbol{\alpha}}(z)$ with its Taylor expansion at 0 (which belongs to $z+z^2\mathbb{C}[[z]]$).

We shall first describe known results.

Definition 1. We say that $\boldsymbol{\alpha}$ is R-partitioned if, up to permutation,[1] it is the concatenation of uples of the form $\left(\frac{b}{m}\right)_{b\in[[1,m]],\gcd(b,m)=1}$ for $m\in\mathbb{N}^*$.

For instance, the 3-uple $\boldsymbol{\alpha}=(1/2,1/6,5/6)$ is R-partitioned, but not the 4-uple $\boldsymbol{\alpha}=(1/2,1/6,1/6,5/6)$.

We shall now make a short digression in order to recall the link between the fact that $\boldsymbol{\alpha}$ is R-partitioned and the fact that, up to rescaling, the Taylor coefficients of $F_{\boldsymbol{\alpha}}(z)$ are quotients of products of factorials of linear forms with integral coefficients. For details on what follows, see for instance [5, §7.1, Proposition 2].

The following properties are equivalent:

(i) There exist $\kappa\in\mathbb{N}^*$ and $e_1,\dots,e_r,f_1,\dots,f_s\in\mathbb{N}^*$ such that

$$F_{\boldsymbol{\alpha}}(\kappa z)=\sum_{k=0}^{+\infty}\frac{(e_1k)!\cdots(e_rk)!}{(f_1k)!\cdots(f_sk)!}z^k;$$

(ii) $\boldsymbol{\alpha}$ is R-partitioned.

Moreover, assume that $\boldsymbol{\alpha}$ is R-partitioned and let $\boldsymbol{N}=(N_1,\dots,N_\ell)\in(\mathbb{N}^*)^\ell$ be such that $\boldsymbol{\alpha}$ is, up to permutation, the concatenation of the uples $\left(\frac{b}{m}\right)_{b\in[[1,N_i]],\gcd(b,N_i)=1}$ for i varying in $[[1,\ell]]$. Consider

$$C_{\boldsymbol{N}}:=C_{N_1}\cdots C_{N_\ell}\in\mathbb{N}^*,$$

where

$$C_{N_i}=N_i^{\varphi(N_i)}\prod_{\substack{p\text{ prime}\\ p\mid N_i}}p^{\varphi(N_i)/(p-1)}$$

[1] We say that "up to permutation" $\boldsymbol{\alpha}=\boldsymbol{\beta}$ if there exists a permutation σ of $[[1,n]]$ such that, for all $i\in[[1,n]]$, $\alpha_i=\beta_{\sigma(i)}$.

(φ denotes Euler's totient function). Then there exist $e_1,\dots,e_r,f_1,\dots,f_s\in\mathbb{N}^*$ such that

$$F_{\boldsymbol{\alpha}}(C_{\boldsymbol{N}}z)=\sum_{k=0}^{+\infty}\frac{(e_1k)!\cdots(e_rk)!}{(f_1k)!\cdots(f_sk)!}z^k.$$

This concludes the digression. We now state a result proved by Krattenthaler and Rivoal [10, §1.2, Theorem 1].

Theorem 2. *Assume that $\boldsymbol{\alpha}$ is R-partitioned and let $\boldsymbol{N}=(N_1,\dots,N_\ell)\in(\mathbb{N}^*)^\ell$ be such that $\boldsymbol{\alpha}$ is, up to permutation, the concatenation of the uples $\left(\frac{b}{m}\right)_{b\in[[1,N_i]],\gcd(b,N_i)=1}$ for i varying in $[[1,\ell]]$. Then*

$$C_{\boldsymbol{N}}^{-1}\mathcal{Q}_{\boldsymbol{\alpha}}(C_{\boldsymbol{N}}z)=z\exp\left(\frac{G_{\boldsymbol{\alpha}}(C_{\boldsymbol{N}}z)}{F_{\boldsymbol{\alpha}}(C_{\boldsymbol{N}}z)}\right)\in\mathbb{Z}[[z]].$$

Actually, special cases of this theorem were considered by Lian and Yau [11–13], and Zudilin formulated a general conjecture, which he proved in some cases in [16]. Zudilin's conjecture was proved by Krattenthaler and Rivoal in [9, 10], and these results were generalized by Delaygue [3–5]. The pioneering work is due to Dwork [7].

What about non-R-partitioned parameters $\boldsymbol{\alpha}$? The following theorem, which is our main result, answers this question.

Notation 3. Consider $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{Q}^n$. Let d be the least denominator in $\mathbb{N}^*$ of $\boldsymbol{\alpha}$ (i.e. d is the least common denominator in $\mathbb{N}^*$ of $\alpha_1,\dots,\alpha_n$). Let $k_1<\cdots<k_{\varphi(d)}$ be the integers in $[[1,d-1]]$ coprime to d. For any $j\in[[1,\varphi(d)]]$, we denote by $\mathcal{P}_j(\boldsymbol{\alpha})$ the set of primes congruent to $k_j \mod d$. Note that $\bigcup_{j\in[[1,\varphi(d)]]}\mathcal{P}_j(\boldsymbol{\alpha})$ coincides with the set of primes p coprime to d.

Theorem 4. *Consider $\boldsymbol{\alpha}\in(\mathbb{Q}\cap]0,1[)^n$ with $n\geq 3$. Let d be the least denominator in $\mathbb{N}^*$ of $\boldsymbol{\alpha}$. Assume that, for all $j\in[[1,\varphi(d)]]$, for infinitely many primes p in $\mathcal{P}_j(\boldsymbol{\alpha})$, we have*

$$\mathcal{Q}_{\boldsymbol{\alpha}}(z)=z\exp\left(\frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)}\right)\in\mathbb{Z}_p[[z]]$$

(where $\mathbb{Z}_p$ is the ring of p-adic integers). Then, $\boldsymbol{\alpha}$ is R-partitioned.

In particular, the following converse of Theorem 2 holds:

Corollary 5. *If $\boldsymbol{\alpha}\in(\mathbb{Q}\cap]0,1[)^n$ with $n\geq 3$ is such that there exists $\kappa\in\mathbb{N}^*$ with the property that*

$$\kappa^{-1}\mathcal{Q}_{\boldsymbol{\alpha}}(\kappa z)=z\exp\left(\frac{G_{\boldsymbol{\alpha}}(\kappa z)}{F_{\boldsymbol{\alpha}}(\kappa z)}\right)\in\mathbb{Z}[[z]],$$

then $\boldsymbol{\alpha}$ is R-partitioned.

This result is false for $n=2$; the detailed study of this case will appear elsewhere.

Remark 6. Let $\mathcal{Z}_{\boldsymbol{\alpha}}(q)\in q+q^2\mathbb{C}[[q]]$ be the compositional inverse of $\mathcal{Q}_{\boldsymbol{\alpha}}(z)\in z+z^2\mathbb{C}[[z]]$. This is a mirror map. For all $\kappa\in\mathbb{N}^*$, we have $(\kappa z)^{-1}\mathcal{Q}_{\boldsymbol{\alpha}}(\kappa z)\in\mathbb{Z}[[z]]$ if and only if $(\kappa q)^{-1}\mathcal{Z}_{\boldsymbol{\alpha}}(\kappa q)\in\mathbb{Z}[[q]]$. Therefore, we can reformulate our results in terms of mirror maps.

The main ingredients of the proof of Theorem 4 are:

A) Dieudonné-Dwork's Lemma (translates p-adic integrality properties of the Taylor coefficients of $\mathcal{Q}_{\boldsymbol{\alpha}}(z)$ in terms of p-adic congruences which do not involve the exponential function);
B) Dwork's congruences for generalized hypergeometric series (allow us to reduce the problem to solve the equation

$$\frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)} = \frac{G_{\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\beta}}(z)} \tag{4}$$

with respect to the unknown parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $(\mathbb{Q}\cap]0,1[)^n$);
C) differential Galois theory, and especially the detailed study of the generalized hypergeometric equations by Beukers and Heckman [2] and Katz [8] (basic tool for solving (4)).

We also give a result relating the auto-duality of the generalized hypergeometric equations to integrality properties of the Taylor coefficients of mirror maps.

Theorem 7. *Let us consider $\boldsymbol{\alpha} \in (\mathbb{Q}\cap]0,1[)^n$ with $n \geq 3$. Let d be the least denominator of $\boldsymbol{\alpha}$ in $\mathbb{N}^*$. The following assertions are equivalent:*

i) *for all prime p congruent to -1 modulo d, we have $\mathcal{Q}_{\boldsymbol{\alpha}}(z) \in \mathbb{Z}_p[[z]]$;*
ii) *for infinitely many primes p congruent to -1 modulo d, we have $\mathcal{Q}_{\boldsymbol{\alpha}}(z) \in \mathbb{Z}_p[[z]]$;*
iii) *$\mathcal{L}_{\boldsymbol{\alpha}}$ is self-dual.*

This paper is organized as follows. In section 2, we solve equation (4). In section 3, we give basic properties of an operator introduced by Dwork. Section 4 is devoted to the proof of Theorem 4. In section 5, we prove Theorem 7.

2. The equation $\frac{G_\alpha(z)}{F_\alpha(z)} = \frac{G_\beta(z)}{F_\beta(z)}$

Proposition 8. *Let us consider $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in $(\mathbb{Q}\cap]0,1[)^n$ with $n \geq 3$. The following assertions are equivalent:*

i) $\frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)} = \frac{G_{\boldsymbol{\beta}}(z)}{F_{\boldsymbol{\beta}}(z)}$;
ii) *up to permutation, $\boldsymbol{\alpha} = \boldsymbol{\beta}$.*

In other words, i) *holds if and only if $y_{\boldsymbol{\alpha};1}(z) = y_{\boldsymbol{\beta};1}(z)$.*

Before proceeding to the proof, we shall recall basic facts concerning differential Galois theory.

2.1. **Differential Galois theory: A short introduction.**

For details on the content of this section, we refer to van der Put and Singer's book [15, §1.1-§1.4]. For an introduction to the subject, we also refer to the articles of Beukers [1, §2.1 and §2.2] and Singer [14, §1.1-§1.3].

The proof of Proposition 8 will use the formalism of differential modules. Nevertheless, for the convenience of the reader, we first introduce differential Galois groups in the framework of differential equations.

The following table summarizes some analogies between classical Galois theory and differential Galois theory (the concepts in the right hand column will be introduced in the next sections).

Galois theory	Differential Galois theory
Polynomial equations	Differential equations or differential modules
Fields	Differential fields
Splitting fields	Picard-Vessiot fields
Galois groups	Differential Galois groups
Finite groups	Linear algebraic groups

2.1.1. *Differential fields.* A differential field (k, d) is a field k endowed with a map $d : k \to k$ such that, for all $f, g \in k$, $d(f+g) = df + dg$ and $d(fg) = (df)g + f(dg)$.

The field of constants of the differential field (k, d) is the field defined by $\{f \in k \mid df = 0\}$.

Two differential fields (k, d) and $(\widetilde{k}, \widetilde{d})$ are isomorphic if there exists a field isomorphism $\varphi : k \to \widetilde{k}$ such that $\varphi \circ d = \widetilde{d} \circ \varphi$.

A differential field $(\widetilde{k}, \widetilde{d})$ is a differential field extension of a differential field (k, d) if $\widetilde{k}$ is a field extension of k and $\widetilde{d}_{|k} = d$; in this case, we denote $\widetilde{d}$ by d.

Let $(\widetilde{k}, d)$ be a differential field extension of a differential field (k, d) and consider $E \subset \widetilde{k}$. We say that $(\widetilde{k}, \widetilde{d})$ is the differential field generated by E over (k, d) if $\widetilde{k}$ is the field generated by $\{d^i f \mid f \in E, i \in \mathbb{N}\}$ over k.

Until the end of §2.1, *we let (k, d) be a differential field. We assume that its field of constants C is algebraically closed and that the characteristic of k is* 0.

2.1.2. *Picard-Vessiot fields and differential Galois groups for differential operators.* Consider a differential operator $L = \sum_{i=0}^{n} a_i d^i$ of order n with coefficients $a_0, \dots, a_n$ in k. There exists a differential field extension (K, d) of (k, d) such that

1) the field of constants of (K, d) is C;
2) the C-vector space of solutions of L in K given by
$$\mathrm{Sol}(L) = \{y \in K \mid Ly = 0\}$$
has dimension n;
3) (K, d) is the differential field generated by $\mathrm{Sol}(L)$ over (k, d).

Such a differential field (K, d) is called a Picard-Vessiot field for L over (k, d) and is unique up to isomorphism.

Remark 9. We can replace 2) by "$\mathrm{Sol}(L)$ has at least dimension n"; this a consequence of [15, Lemma 1.10].

We can replace 3) by "K is the field generated over k by $\{d^i y_j \mid j \in [[1, n]], i \in [[0, n-1]]\}$ for some (or, equivalently, for any) C-basis $y_1, \dots, y_n$ of $\mathrm{Sol}(L, K)$".

The corresponding differential Galois group G of L over (k, d) is the group made of the k-linear field automorphisms of K commuting with d:
$$G = \{\sigma \in \mathrm{Aut}(K/k) \mid d\sigma = \sigma d\}.$$
It follows from the definition[2] that any $\sigma \in G$ induces a C-linear automorphism of $\mathrm{Sol}(L)$, namely $\sigma_{|\,\mathrm{Sol}(L)}$. Thus one can identify G with an *algebraic* subgroup of $\mathrm{GL}(\mathrm{Sol}(L))$ via the faithful representation
$$\sigma \in G \mapsto \sigma_{|\,\mathrm{Sol}(L)} \in \mathrm{GL}(\mathrm{Sol}(L)). \tag{5}$$

[2]Indeed, for any $\sigma \in G$, for any $y \in \mathrm{Sol}(L)$, we have $0 = \sigma(Ly) = \sigma(\sum_{i=0}^{n} a_i d^i y) = \sum_{i=0}^{n} \sigma(a_i)\sigma(d^i y) = \sum_{i=0}^{n} a_i d^i \sigma(y) = L(\sigma(y))$, so $\sigma(\mathrm{Sol}(L)) \subset \mathrm{Sol}(L)$. It follows that the restriction $\sigma_{|\,\mathrm{Sol}(L)}$ of any element σ of G to $\mathrm{Sol}(L)$ is a C-linear automorphism of $\mathrm{Sol}(L)$.

If we choose a C-basis $y_1, \ldots, y_n$ of $\mathrm{Sol}(L)$, then (5) becomes

$$\sigma \in G \mapsto (m_{i,j}(\sigma))_{1\leq i,j\leq n} \in \mathrm{GL}_n(C), \tag{6}$$

where $(m_{i,j}(\sigma))_{1\leq i,j\leq n} \in \mathrm{GL}_n(C)$ is such that, for all $j \in [[1, n]]$,

$$\sigma(y_j) = \sum_{i=1}^{n} m_{i,j}(\sigma) y_i.$$

2.1.3. *Picard-Vessiot fields and differential Galois groups for differential modules.* A differential module (M, ∂) over (k, d) is a finite dimensional k-vector space M endowed with a map $\partial : M \to M$ such that, for all $f \in k$, for all $m, n \in M$, $\partial(m + n) = \partial m + \partial n$ and $\partial(fm) = (df)m + f(\partial m)$.

Let (M, ∂) be a differential module over (k, d) of dimension $\dim_k M = n$. We let $(e_i)_{1\leq i\leq n}$ be a k-basis of M. There exists a differential field extension (K, d) of (k, d) such that:

1) the field of constants of (K, d) is C;
2) the C-vector space of solutions of (M, ∂) in K given by[3]
$$\omega(M, \partial) = \mathrm{Ker}(d \otimes \partial : K \otimes_k M \to K \otimes_k M)$$
has dimension n;
3) K is the field generated over k by the entries of some (or any) matrix $(y_{i,j})_{1\leq i,j\leq n} \in \mathrm{M}_n(K)$ such that $(\sum_{i=1}^{n} y_{i,j} \otimes e_i)_{1\leq j\leq n}$ is a C-basis of $\omega(M, \partial)$.

Remark 10. One can reformulate what precedes in terms of differential systems. Let $A = (a_{i,j})_{1\leq i\leq j\leq n} \in \mathrm{M}_n(k)$ be the opposite of the matrix representing the action of ∂ on M with respect to the basis $(e_i)_{1\leq i\leq n}$ i.e., for all $j \in [[1, n]]$, $\partial e_j = -\sum_{i=1}^{n} a_{i,j} e_i$. Then, an element $\sum_{k=1}^{m} f_k \otimes e_k$ of $K \otimes_k M$ belongs to $\omega(M, \partial)$ if and only if

$$\begin{aligned}(d \otimes \partial)\left(\sum_{k=1}^{m} f_k \otimes e_k\right) &= \sum_{k=1}^{m} ((df_k) \otimes e_k + f_k \otimes \partial e_k) \\ &= \sum_{k=1}^{m} \left((df_k) \otimes e_k + f_k \otimes \left(-\sum_{i=1}^{n} a_{i,k} e_i\right)\right) \\ &= \sum_{k=1}^{m} (df_k) \otimes e_k - \sum_{i=1}^{n} \left(\sum_{k=1}^{m} a_{i,k} f_k\right) \otimes e_i = 0,\end{aligned}$$

and this equality holds if only if, for all $k \in [[1, n]]$, $df_k = \sum_{j=1}^{m} a_{k,j} f_j$, i.e.

$$d\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

[3] The action of $d \otimes \partial$ on $K \otimes_k M$ is given by

$$(d \otimes \partial)(\sum_{k=1}^{m} f_k \otimes m_k) := \sum_{k=1}^{m} ((df_k) \otimes m_k + f_k \otimes \partial m_k).$$

It follows that 2) and 3) can be restated as "there exists $Y \in \mathrm{GL}_n(K)$ such that $dY = AY$ and K is the field generated over k by the entries of Y".

Such a differential field (K, d) is called a Picard-Vessiot field for (M, ∂) over (k, d) and is unique up to isomorphism. The differential Galois group G of (M, ∂) over (k, d) is then defined by

$$G = \{\sigma \in \mathrm{Aut}(K/k) \mid d\sigma = \sigma d\}.$$

It follows from the definition[4] that any $\sigma \in G$ induces a C-linear automorphism of $\omega(M, \partial)$, namely $(\sigma \otimes Id_M)_{|\omega(M,\partial)}$. One can identify G with an *algebraic* subgroup of $\mathrm{GL}(\omega(M, \partial))$ via the faithful representation

$$\sigma \in G \mapsto (\sigma \otimes Id_M)_{|\omega(M,\partial)}. \tag{7}$$

One can reformulate §2.1.2 in terms of differential modules. We consider L as in §2.1.2. We denote by (M_L, ∂_L) the differential module over (k, d) associated to L characterized by:

i) $M_L = k^n$.

ii) The opposite of the matrix representing the action of ∂_L on M with respect to the canonical k-basis $(e_i)_{1 \leq i \leq n}$ of $M_L = k^n$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix}.$$

Then a differential field (K, d) is a Picard-Vessiot field for L if and only if it is a Picard-Vessiot field for (M_L, ∂_L). Once such a Picard-Vessiot field (K, d) is fixed, the corresponding Galois groups of L and (M_L, ∂_L) are the same. Moreover, if $(y_j)_{1 \leq j \leq n}$ is a C-basis of $\mathrm{Sol}(L)$, then a C-basis of $\omega(M_L, \partial_L)$ is given by $\left(\sum_{i=0}^{n-1} d^i y_j \otimes e_i\right)_{1 \leq j \leq n}$; with respect to this basis, the representation (7) becomes the representation (6).

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[4]Indeed, for any $\sigma \in G$, for any $\sum_{k=1}^{m} f_k \otimes m_k \in \omega(M, \partial)$, we have

$$\begin{aligned} 0 &= (\sigma \otimes Id_M)\left((d \otimes \partial)\sum_{k=1}^{m} f_k \otimes m_k\right) \\ &= (\sigma \otimes Id_M)\left(\sum_{k=1}^{m}(df_k) \otimes m_k + f_k \otimes \partial m_k\right) \\ &= \sum_{k=1}^{m} \sigma(df_k) \otimes m_k + \sigma(f_k) \otimes \partial m_k \\ &= \sum_{k=1}^{m} d(\sigma(f_k)) \otimes m_k + \sigma(f_k) \otimes \partial m_k \\ &= (d \otimes \partial)\left((\sigma \otimes Id_M)\sum_{k=1}^{m} f_k \otimes m_k\right) \end{aligned}$$

so $\sigma \otimes Id_M$ leaves $\omega(M, \partial)$ globally invariant. It follows that the restriction $(\sigma \otimes Id_M)_{|\omega(M,\partial)}$ of any element σ of G to $\omega(M, \partial)$ is a C-linear automorphism of $\omega(M, \partial)$.

2.1.4. *Tannakian duality.* For what follows, we refer to [15, §2.4] (we refer the reader interested in tannakian categories to Deligne and Milne's work [6]). We let $\langle (M,d)\rangle$ be the smallest full subcategory of the category of differential modules over (k,d) containing (M,∂) and closed under all constructions of linear algebra (direct sums, tensor products, duals, subquotients; see [15, §2.2 and §2.4]). We let (K,d) be a Picard-Vessiot field for (M,∂) over (k,d) and we let G be the corresponding differential Galois group over (k,d). There is a C-linear equivalence of categories between $\langle (M,d)\rangle$ and the category of rational C-linear representations of the linear algebraic group G which is compatible with all constructions of linear algebra. Such an equivalence is given by a functor sending an object (N,∂_N) of $\langle (M,\partial)\rangle$ to the representation

$$\begin{array}{rcl}\rho_{(N,\partial_N)}: G & \rightarrow & \mathrm{GL}(\omega(N,\partial_N)) \\ \sigma & \mapsto & (\sigma\otimes Id_N)_{|\omega_{(N,\partial_N)}},\end{array}$$

where

$$\omega(N,\partial_N)=\mathrm{Ker}(d\otimes\partial_N : K\otimes_k N\rightarrow K\otimes_k N).$$

The differential Galois group of (N,∂_N) over (k,d) can be identified with the image of $\rho_{(N,\partial_N)}$.

In what follows, the base differential field (k,d) will be $(\mathbb{C}(z),d/dz)$. In order to simplify the notation, we will drop the derivatives ($(k,d)=k$, $(M,\partial)=M$, etc.).

2.2. **Proof of Proposition 8.** Of course the only nontrivial implication is i)$\Rightarrow$ii). We consider the differential modules $M_{\boldsymbol{\alpha}} := M_{\mathcal{L}_{\boldsymbol{\alpha}}}$ and $M_{\boldsymbol{\beta}} := M_{\mathcal{L}_{\boldsymbol{\beta}}}$ associated to $\mathcal{L}_{\boldsymbol{\alpha}}$ and $\mathcal{L}_{\boldsymbol{\beta}}$ respectively (see the end of §2.1.3). A Picard-Vessiot field over $\mathbb{C}(z)$ of the differential module $M=M_{\boldsymbol{\alpha}}\oplus M_{\boldsymbol{\beta}}$ is given by

$$K=\mathbb{C}(z)\left(y^{(i)}_{\boldsymbol{\alpha};j}(z), y^{(i)}_{\boldsymbol{\beta};j}(z)\mid (i,j)\in[[0,n-1]]\times[[1,n]]\right).$$

We let G be the corresponding differential Galois group and we use the notation ($\omega(N)$, ρ_N, etc.) of §2.1.4. If we choose the basis of $\omega(M)$ which is the concatenation of the bases of $\omega(M_{\boldsymbol{\alpha}})$ and $\omega(M_{\boldsymbol{\beta}})$ described at the end of §2.1.3, then the representation $\rho_M=\rho_{M_{\boldsymbol{\alpha}}}\oplus\rho_{M_{\boldsymbol{\beta}}}$ of G is identified with

$$\sigma\in G\mapsto\begin{pmatrix}(m_{\boldsymbol{\alpha};i,j}(\sigma))_{1\leq i,j\leq n} & 0\\ 0 & (m_{\boldsymbol{\beta};i,j}(\sigma))_{1\leq i,j\leq n}\end{pmatrix}\in\mathrm{GL}_{2n}(\mathbb{C}),$$

where, for all $\sigma\in G$,

$$\begin{cases}\sigma(y_{\boldsymbol{\alpha};j}(z))=\sum_{i=1}^n m_{\boldsymbol{\alpha};i,j}(\sigma)y_{\boldsymbol{\alpha};i}(z);\\ \sigma(y_{\boldsymbol{\beta};j}(z))=\sum_{i=1}^n m_{\boldsymbol{\beta};i,j}(\sigma)y_{\boldsymbol{\beta};i}(z).\end{cases}$$

Strategy of the proof. We are going to prove that there exists a character χ of G such that either the representation $\rho_{M_{\boldsymbol{\alpha}}}$ or its dual $\rho^*_{M_{\boldsymbol{\alpha}}}$ is conjugate to $\chi\otimes\rho_{M_{\boldsymbol{\beta}}}$ (see Lemma 13 below). Then a detailed study of both cases will lead to the fact that, up to permutation, $\boldsymbol{\alpha}=\boldsymbol{\beta}$, which is the desired result.

In order to achieve these goals, we first establish a bound for $\rho_M(G)$ (Lemma 11) and we describe $\rho_{M_{\boldsymbol{\alpha}}}(G)$ and $\rho_{M_{\boldsymbol{\beta}}}(G)$ (Lemma 12).

Lemma 11. *We have*

$$\rho_M(G)\subset\left\{\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\mid A,B\in\mathrm{GL}_n(\mathbb{C}), B_{n,1}A_{n,2}=A_{n,1}B_{n,2}\right\}. \tag{8}$$

Proof. Hypothesis i) implies that

$$\frac{y_{\boldsymbol{\alpha};1}(z)}{y_{\boldsymbol{\alpha};2}(z)} = \frac{y_{\boldsymbol{\beta};1}(z)}{y_{\boldsymbol{\beta};2}(z)},$$

so, for any $\sigma \in G$,

$$\sigma(y_{\boldsymbol{\beta};1}(z))\sigma(y_{\boldsymbol{\alpha};2}(z)) = \sigma(y_{\boldsymbol{\alpha};1}(z))\sigma(y_{\boldsymbol{\beta};2}(z)),$$

i.e.

$$\left(\sum_{i=1}^{n} m_{\boldsymbol{\beta};i,1}(\sigma) y_{\boldsymbol{\beta};i}(z)\right)\left(\sum_{i=1}^{n} m_{\boldsymbol{\alpha};i,2}(\sigma) y_{\boldsymbol{\alpha};i}(z)\right)$$
$$= \left(\sum_{i=1}^{n} m_{\boldsymbol{\alpha};i,1}(\sigma) y_{\boldsymbol{\alpha};i}(z)\right)\left(\sum_{i=1}^{n} m_{\boldsymbol{\beta};i,2}(\sigma) y_{\boldsymbol{\beta};i}(z)\right).$$

Therefore, using (1)–(3) from section 1, we get

$$(m_{\boldsymbol{\beta};n,1}(\sigma)m_{\boldsymbol{\alpha};n,2}(\sigma) - m_{\boldsymbol{\alpha};n,1}(\sigma)m_{\boldsymbol{\beta};n,2}(\sigma))\log(z)^{2n-2} \in \sum_{k=0}^{2n-3} \mathbb{C}(\{z\})\log(z)^k,$$

and hence the expected equality holds:

$$m_{\boldsymbol{\beta};n,1}(\sigma)m_{\boldsymbol{\alpha};n,2}(\sigma) = m_{\boldsymbol{\alpha};n,1}(\sigma)m_{\boldsymbol{\beta};n,2}(\sigma). \qquad \square$$

Lemma 12. *The Galois groups $\rho_{M_{\boldsymbol{\alpha}}}(G)$ and $\rho_{M_{\boldsymbol{\beta}}}(G)$ of $M_{\boldsymbol{\alpha}}$ and $M_{\boldsymbol{\beta}}$ respectively satisfy the following property:*[5] *$\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}}$ and $\rho_{M_{\boldsymbol{\beta}}}(G)^{0,\mathrm{der}}$ are conjugate to either $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$.*

Proof. This is proved in [2] and also in [8, Chapter 3, Theorem 3.5.8]. $\square$

Lemma 13. *There exists a character χ of G such that either $\rho_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$ or $\rho^*_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$.*

Proof. This lemma would follow from Goursat-Kolchin-Ribet [8, Proposition 1.8.2] (applied to $\rho_1 := \rho_{M_{\boldsymbol{\alpha}}}$ and $\rho_2 := \rho_{M_{\boldsymbol{\beta}}}$) if we knew that:

(a) $\rho_M(G)^{0,\mathrm{der}} \neq \begin{pmatrix} \rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}} & 0 \\ 0 & \rho_{M_{\boldsymbol{\beta}}}(G)^{0,\mathrm{der}} \end{pmatrix}$;

(b) if $n = 8$, then $\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}}$ is not conjugate to $\mathrm{SO}_8(\mathbb{C})$.

Indeed, (a) means that the conclusion of [8, Proposition 1.8.2] does not hold. But Lemma 12 implies that the irreducibility and the simplicity hypothesis [8, Proposition 1.8.2, Hypothesis (1)] is satisfied, and Lemma 12 together with [8, Example 1.8.1] implies that the Goursat adaptedness hypothesis [8, Proposition 1.8.2, Hypothesis (2)] is also satisfied, except if $n = 8$ and if $\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}}$ and $\rho_{M_{\boldsymbol{\beta}}}(G)^{0,\mathrm{der}}$ are conjugate to $\mathrm{SO}_8(\mathbb{C})$, but this is excluded by (b). Therefore, at least one of the remaining hypotheses [8, Proposition 1.8.2, Hypothesis (3) or (4)] is not satisfied; i.e. there exists a character χ of G such that $\rho_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$ or $\rho^*_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$.

It remains to prove our claims (a) and (b).

[5] Let G be a linear algebraic group. We will denote by G^0 the neutral component of G (= connected component of G which contains the neutral element of G) and G^{der} its derived subgroup (= commutator subgroup); $G^{0,\mathrm{der}}$ stands for the derived subgroup of the neutral component G^0 of G.

Lemma 11 implies that if G_1 and G_2 are some conjugates of either $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$, then

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \not\subset \rho_M(G). \tag{9}$$

In particular, (a) is true.

In order to prove (b), we argue by contradiction: we assume that $n=8$ and that $\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}}$ is conjugate to $\mathrm{SO}_8(\mathbb{C})$. Then [8, Proposition 3.5.8.1 and Theorem 3.4] implies that there exists a permutation $\nu \in \mathfrak{S}_8$ of $[[1,8]]$ such that, for all $i \in [[1,8]]$, $\alpha_i + \alpha_{\nu(i)} \in \mathbb{Z}$ and that $\sum_{i=1}^{8} \alpha_i \in 1/2 + \mathbb{Z}$. Let $\mathcal{O}$ be the set of orbits of $[[1,8]]$ under the action of the subgroup of $\mathfrak{S}_8$ generated by ν. Consider $i_0 \in \Omega \in \mathcal{O}$ and set $\omega := \sharp\Omega$. If ω is even, then

$$\sum_{i\in\Omega} \alpha_i = \sum_{k=0}^{\omega/2-1} \left(\alpha_{\nu^{2k}(i_0)} + \alpha_{\nu^{2k+1}(i_0)}\right) \in \mathbb{Z}.$$

Assume that $\omega = 2\omega'+1$ is odd. We have

$$\begin{array}{rcl} \alpha_i + \alpha_{\nu(i)} & \in & \mathbb{Z}, \\ \alpha_{\nu(i)} + \alpha_{\nu^2(i)} & \in & \mathbb{Z}, \\ \vdots & \vdots & \vdots \\ \alpha_{\nu^{2\omega'-1}(i)} + \alpha_{\nu^{2\omega'}(i)} & \in & \mathbb{Z}, \\ \alpha_{\nu^{2\omega'}(i)} + \alpha_{\nu^{2\omega'+1}(i)} = \alpha_{\nu^{2\omega'}(i)} + \alpha_i & \in & \mathbb{Z}. \end{array}$$

This implies that, for all $k \in \mathbb{Z}$, $\alpha_{\nu^k(i_0)} = 1/2$, so

$$\sum_{i\in\Omega} \alpha_i = \sum_{k=0}^{2\omega'} \alpha_{\nu^k(i_0)} \in 1/2 + \mathbb{Z}.$$

But the number of orbits with odd cardinality is even (because $\sum_{\Omega\in\mathcal{O}} \sharp\Omega = 8$ is even). It follows clearly that

$$\sum_{i=1}^{8} \alpha_i = \sum_{\Omega\in\mathcal{O}} \sum_{i\in\Omega} \alpha_i \in \mathbb{Z}.$$

This yields a contradiction. □

In order to conclude the proof of Proposition 8, it remains to study both cases $\rho_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$ and $\rho^*_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$ and to prove that in both cases, up to permutation, $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

(1) Assume that $\rho_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$. By tannakian duality, there exists a rank one object L of $\langle M \rangle$ such that $M_{\boldsymbol{\alpha}} \cong L \otimes M_{\boldsymbol{\beta}}$. Since $\mathcal{L}_{\boldsymbol{\alpha}}$ is regular singular with singularities in $\{0,1,\infty\}$, we get that L is regular singular and that its nontrivial monodromies are at most at $0,1,\infty$. Since the monodromies at 1 of both $M_{\boldsymbol{\alpha}}$ and $M_{\boldsymbol{\beta}}$ are pseudo-reflections ([2, Proposition 2.10]), we get that the monodromy of L at 1 is trivial. Moreover, the monodromies at 0 of both $M_{\boldsymbol{\alpha}}$ and $M_{\boldsymbol{\beta}}$ are unipotent, so the monodromy of L at 1 is also trivial. Therefore, the monodromy representation of L is trivial, and hence L is trivial. So $M_{\boldsymbol{\alpha}} \cong M_{\boldsymbol{\beta}}$, and hence, up to permutation, $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

(2) Assume that $\rho^*_{M_{\boldsymbol{\alpha}}} \cong \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$. By tannakian duality, there exists a rank one object L of $\langle M \rangle$ such that $M^*_{\boldsymbol{\alpha}} \cong L \otimes M_{\boldsymbol{\beta}}$. We now distinguish several cases depending on the Galois groups of $M_{\boldsymbol{\alpha}}$ and $M_{\boldsymbol{\beta}}$ (we recall that $\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}}$ and $\rho_{M_{\boldsymbol{\beta}}}(G)^{0,\mathrm{der}}$ are conjugate to either $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$ by virtue of Lemma 12).

 (a) Assume first that $\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}}$ is, up to conjugation, either $\mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$. Then $\rho_{M_{\boldsymbol{\alpha}}}(G)$ is conjugated to some subgroup of either $\mathbb{C}^*\mathrm{O}_n(\mathbb{C})$ or $\mathbb{C}^*\mathrm{Sp}_n(\mathbb{C})$ (because the normalizers of $\mathrm{SO}_n(\mathbb{C})$ and $\mathrm{Sp}_n(\mathbb{C})$ in $\mathrm{GL}_n(\mathbb{C})$ are $\mathbb{C}^*\,\mathrm{O}_n(\mathbb{C})$ and $\mathbb{C}^*\,\mathrm{Sp}_n(\mathbb{C})$ respectively). It follows that $\rho^*_{M_{\boldsymbol{\alpha}}} \cong \eta \otimes \rho_{M_{\boldsymbol{\alpha}}}$ for some character η of G (for instance, if $\rho_{M_{\boldsymbol{\alpha}}}(G) \subset P\,\mathrm{O}_n(\mathbb{C})P^{-1}$ for some $P \in \mathrm{GL}_n(\mathbb{C})$, then $\eta = \eta' \circ \rho_{M_{\boldsymbol{\alpha}}}$ where η' is the character of $\mathbb{C}^* P\,\mathrm{O}_n(\mathbb{C})P^{-1}$ defined, for $c \in \mathbb{C}^*$ and $A \in P\,\mathrm{O}_n(\mathbb{C})P^{-1}$, by $\eta'(cA) = c^{-2}$). So $\rho_{M_{\boldsymbol{\alpha}}} \cong \eta^{-1} \otimes \chi \otimes \rho_{M_{\boldsymbol{\beta}}}$, and we are reduced to the previous case. So, up to permutation, $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

 (b) The case that $\rho_{M_{\boldsymbol{\beta}}}(G)^{0,\mathrm{der}}$ is, up to conjugation, either $\mathrm{SO}_n(\mathbb{C})$ or $\mathrm{Sp}_n(\mathbb{C})$ is similar.

 (c) In order to conclude the proof it is sufficient to prove that the case $\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}} = \rho_{M_{\boldsymbol{\beta}}}(G)^{0,\mathrm{der}} = \mathrm{SL}_n(\mathbb{C})$ does not hold. We argue by contradiction: we assume that $\rho_{M_{\boldsymbol{\alpha}}}(G)^{0,\mathrm{der}} = \rho_{M_{\boldsymbol{\beta}}}(G)^{0,\mathrm{der}} = \mathrm{SL}_n(\mathbb{C})$. Since $M^*_{\boldsymbol{\alpha}} \cong M_{1-\boldsymbol{\alpha}}$ ([8, Theorem 3.4]), we have $M_{1-\boldsymbol{\alpha}} \cong L \otimes M_{\boldsymbol{\beta}}$. Arguing as above, we see that L is trivial (and $\boldsymbol{\beta} = 1 - \boldsymbol{\alpha}$). So the character χ is trivial and $\rho^*_{M_{\boldsymbol{\alpha}}} \cong \rho_{M_{\boldsymbol{\beta}}}$. This, together with inclusion (8), implies that there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that, for all $A \in \mathrm{SL}_n(\mathbb{C})$,

$$(PA^{-t}P^{-1})_{n,1}A_{n,2} = A_{n,1}(PA^{-t}P^{-1})_{n,2}.$$

 It follows that, for all $A \in \mathrm{GL}_n(\mathbb{C})$,

$$A_{n,1} = 0 \text{ and } A_{n,2} \neq 0 \Rightarrow (PA^{-t}P^{-1})_{n,1} = 0.$$

 Using a simple density argument, we get that, for all $A \in \mathrm{GL}_n(\mathbb{C})$,

$$A_{n,1} = 0 \Rightarrow (PA^{-t}P^{-1})_{n,1} = 0.$$

 This yields a contradiction by virtue of the following lemma.

Lemma 14. *For any $P \in \mathrm{GL}_n(\mathbb{C})$, there exists $A \in \mathcal{E} := \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A_{n,1} = 0\}$ such that $(PA^{-t}P^{-1})_{n,1} \neq 0$.*

Proof. We argue by contradiction: we assume that, for all $A \in \mathcal{E}$, we have $(PA^{-t}P^{-1})_{n,1} = 0$.

Setting $X = (x_1, \ldots, x_n) := P^{-1}(1, 0, \ldots, 0)^t \neq 0$, we see that the hyperplane $H := P^{-1}(\mathbb{C}^{n-1} \times \{0\})^t$ of $\mathrm{M}_{n,1}(\mathbb{C})$ is such that $\mathcal{E}^{-t}X \subset H$.

Using the fact that

$$\begin{pmatrix} \mathrm{GL}_{n-1}(\mathbb{C}) & 0 \\ 0 & \mathbb{C}^* \end{pmatrix} \subset \mathcal{E}, \tag{10}$$

we get that either $(x_1, \ldots, x_{n-1}) = (0, \ldots, 0)$ or $x_n = 0$ (because otherwise we would have $((\mathbb{C}^*)^n)^t \subset \mathcal{E}^{-t}X \subset H$; this would contradict the fact that H is a hyperplane of $\mathrm{M}_{n,1}(\mathbb{C})$). We are thus led to distinguish two cases:

(1) Assume that $(x_1, \ldots, x_{n-1}) = (0, \ldots, 0)$ and hence $x_n \neq 0$. We denote by $(E(i,j), (i,j) \in [[1,n]]^2)$ the canonical basis of $\mathrm{M}_n(\mathbb{C})$. For all $i \in [[2, n-1]]$,

$I_n+E(n,i) \in \mathcal{E}$ and $(I_n+E(n,i))^{-t} = I_n - E(i,n)$, so $(I_n+E(n,i))^{-t}X = (0,\dots,0,-x_n,0,\dots,0,x_n)^t \in H$, where x_n is at the ith and nth positions. Moreover $(I_n)^{-t}X = (0,\dots,0,x_n)^t$ belongs to H. So $H = (\{0\}\times\mathbb{C}^{n-1})^t$. But $I_n+E(n-1,1)+E(n,2) \in \mathcal{E}$ and $(I_n+E(n-1,1)+E(n,2))^{-t} = I_n - E(1,n-1) - E(2,n) + E(1,n)$, so $(I_n+E(n-1,1)+E(n,2))^{-t}X = (x_n,\dots)^t \in H$; this contradicts the equality $H = (\{0\}\times\mathbb{C}^{n-1})^t$.

(2) Assume that $x_n = 0$ and hence $(x_1,\dots,x_{n-1}) \neq (0,\dots,0)$. Using the inclusion (10), we see that $((\mathbb{C}^*)^{n-1}\times\{0\})^t \subset H$ and hence $H = (\mathbb{C}^{n-1}\times\{0\})^t$. Let $i_0 \in [[1,n-1]]$ be such that $x_{i_0} \neq 0$. We have $I_n+E(i_0,n) \in \mathcal{E}$ and $(I_n+E(i_0,n))^{-t} = I_n - E(n,i_0)$, so $(I_n+E(i_0,n))^{-t}X = (x_1,\dots,x_{n-1},-x_{i_0}) \in H$; this contradicts the equality $H = (\mathbb{C}^{n-1}\times\{0\})^t$. ☐

3. Dwork's map $\alpha \mapsto \alpha' =: \mathfrak{D}_p(a)$

For any prime number p, for any p-adic integer α in $\mathbb{Q}$, we denote by $\mathfrak{D}_p(\alpha)$ the unique p-adic integer in $\mathbb{Q}$ such that

$$p\mathfrak{D}_p(\alpha) - \alpha \in [[0,p-1]].$$

In other words,

$$\mathfrak{D}_p(\alpha) = \frac{\alpha+j}{p},$$

where j is the unique integer in $[[0,p-1]]$ such that $\alpha \equiv -j \mod p\mathbb{Z}_p$. The map $\alpha \mapsto \mathfrak{D}_p(\alpha)$ was used by Dwork in [7] (and denoted by $\alpha \mapsto \alpha'$).

Proposition 15. *Assume that $\alpha \in \mathbb{Q}\cap]0,1[$. Let $m,a \in \mathbb{N}^*$ be such that $\alpha = a/m$ and $\gcd(a,m)=1$ (so $\gcd(m,p)=1$). Then*

$$\mathfrak{D}_p(\alpha) = \frac{x}{m} \in \mathbb{Q}\cap]0,1[,$$

where x is the unique integer in $[[1,m-1]]$ such that $px \equiv a \mod m$.

In particular, $\mathfrak{D}_p(\alpha)$ does not depend on the prime p coprime to m in a fixed arithmetic progression $k+\mathbb{N}m$.

Proof. Since $\mathfrak{D}_p(\alpha) = \frac{\alpha+j}{p} = \frac{a+jm}{pm}$, we have to prove that $x := \frac{a+jm}{p}$ belongs to $[[1,m-1]]$ and that $px \equiv a \mod m$. We first note that $x \in \mathbb{Z}$ because $\alpha \equiv -j \mod p\mathbb{Z}_p$ so $a \equiv -jm \mod p\mathbb{Z}$. The inequality $a+jm>0$ is obvious. Moreover, $\alpha+j \leq \alpha+p-1 < 1+p-1 = p$, so $\frac{a+jm}{p} = \frac{m}{p}(\alpha+j) < m$. Lastly, $px = a+jm \equiv a \mod m$. ☐

We will need the following result:

Proposition 16. *For all $j \in [[1,\varphi(m)]]$, consider $p_j \in \mathcal{P}_j(\alpha)$. Then, up to permutation, we have*

$$(\mathfrak{D}_{p_1}(\alpha),\dots,\mathfrak{D}_{p_{\varphi(m)}}(\alpha)) = \left(\frac{b}{m}\right)_{b\in[[1,m-1]],\gcd(b,m)=1}.$$

Proof. Proposition 15 ensures that $\mathfrak{D}_{p_i}(\alpha) = \frac{x_i}{m}$, where x_i is the unique integer in $[[1,m-1]]$ such that $p_ix_i \equiv a \mod m$. The result follows from the fact that, up to permutation, $(x_1,\dots,x_{\varphi(m)}) = (b)_{b\in[[1,m-1]],\gcd(b,m)=1}$. ☐

Proposition 17. *Let us consider* $\boldsymbol{\alpha} \in (\mathbb{Q} \cap]0,1[)^n$. *Let* d *be the least denominator of* $\boldsymbol{\alpha}$ *in* $\mathbb{N}^*$. *The following properties are equivalent:*

i) *for all* $j \in [[1, \varphi(d)]]$, *there exists* $p_j \in \mathcal{P}_j(\boldsymbol{\alpha})$ *such that, up to permutation,* $\mathfrak{D}_{p_j}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$;
ii) $\boldsymbol{\alpha}$ *is R-partitioned.*

Proof. For any $k \in [[1, n]]$, we let $m_k, a_k \in \mathbb{N}^*$ be such that $\alpha_k = a_k/m_k$ and $\gcd(a_k, m_k) = 1$. Note that, for any $k \in [[1, n]]$, the set $\{\alpha_j \mid m_j = m_k\}$ is stable by $\mathfrak{D}_p(\cdot)$ for any prime p coprime to m_k; this follows from Proposition 15. Therefore, we can assume without loss of generality that $m_1 = \cdots = m_n$. In this case, using Proposition 16, it is easily seen that, up to permutation, $\boldsymbol{\alpha}$ coincides with $(\frac{b}{m_1})_{b \in [[1, m_1]], \gcd(b, m_1) = 1}$ concatenated with itself a certain number of times. □

4. Proof of Theorem 4

Let us recall the hypotheses. We consider $\boldsymbol{\alpha} \in (\mathbb{Q} \cap]0,1[)^n$ with $n \geq 3$. We let d be the least denominator in $\mathbb{N}^*$ of $\boldsymbol{\alpha}$. We assume that, for all $j \in [[1, \varphi(d)]]$, for infinitely many primes p in $\mathcal{P}_j(\boldsymbol{\alpha})$, we have

$$\mathcal{Q}_{\boldsymbol{\alpha}}(z) = z \exp\left(\frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)}\right) \in \mathbb{Z}_p[[z]].$$

We will need the following Dieudonné-Dwork Lemma (for a proof, see [16, Lemma 5] for instance).

Lemma 18 (Dieudonné-Dwork's Lemma)**.** *Let us consider* $f(z) \in z\mathbb{Q}[[z]]$ *and let* p *be a prime number. The following assertions are equivalent:*

1) $e^{f(z)} \in \mathbb{Z}_p[[z]]$;
2) $f(z^p) = pf(z) \mod p\mathbb{Z}_p[[z]]$.

Implication 1) ⇒ 2) of Dieudonné-Dwork's Lemma ensures that, for all $j \in [[1, \varphi(d)]]$, for infinitely many primes p in $\mathcal{P}_j(\boldsymbol{\alpha})$,

$$\frac{G_{\boldsymbol{\alpha}}(z^p)}{F_{\boldsymbol{\alpha}}(z^p)} = p\frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)} \mod p\mathbb{Z}_p[[z]].$$

On the other hand, Dwork [7, Theorem 4.1] ensures that, for all prime p coprime to d,

$$\frac{G_{\mathfrak{D}_p(\boldsymbol{\alpha})}(z^p)}{F_{\mathfrak{D}_p(\boldsymbol{\alpha})}(z^p)} = p\frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)} \mod p\mathbb{Z}_p[[z]].$$

Consequently, for all $j \in [[1, \varphi(d)]]$, for infinitely many primes p in $\mathcal{P}_j(\boldsymbol{\alpha})$,

$$\frac{G_{\mathfrak{D}_p(\boldsymbol{\alpha})}(z)}{F_{\mathfrak{D}_p(\boldsymbol{\alpha})}(z)} = \frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)} \mod p\mathbb{Z}_p[[z]]. \tag{11}$$

But $\mathfrak{D}_p(\boldsymbol{\alpha})$ does not depend on $p \in \mathcal{P}_j(\boldsymbol{\alpha})$. So, for all $j \in [[1, \varphi(d)]]$, for all prime $p \in \mathcal{P}_j(\boldsymbol{\alpha})$,

$$\frac{G_{\mathfrak{D}_p(\boldsymbol{\alpha})}(z)}{F_{\mathfrak{D}_p(\boldsymbol{\alpha})}(z)} = \frac{G_{\boldsymbol{\alpha}}(z)}{F_{\boldsymbol{\alpha}}(z)}$$

(apply to the Taylor coefficients of both sides of (11) the elementary fact that if a and b are elements of $\mathbb{Q}$ such that $a \equiv b \mod p\mathbb{Z}_p$ for infinitely many primes p, then $a = b$). Using Proposition 8, we get that, up to permutation, $\mathfrak{D}_p(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ for all prime p coprime to d. Proposition 17 yields the desired result: $\boldsymbol{\alpha}$ is R-partitioned.

5. Auto-duality and integrality

Proposition 19. *Let us consider $\boldsymbol{\alpha} \in (\mathbb{Q} \cap]0,1[)^n$. Let d be the least denominator of $\boldsymbol{\alpha}$ in $\mathbb{N}^*$. Then, for all primes p congruent to -1 modulo d, we have $\mathfrak{D}_p(\boldsymbol{\alpha}) = 1 - \boldsymbol{\alpha}$.*

Proof. For any $k \in [[1,n]]$, we let $m_k, a_k \in \mathbb{N}^*$ be such that $\alpha_k = a_k/m_k$ and $\gcd(a_k, m_k) = 1$. Using Proposition 15, we get, for any $k \in [[1,n]]$,

$$\mathfrak{D}_p(\alpha_k) = \frac{m_k - a_k}{m_k} = 1 - \alpha_k.$$

□

The following result follows from [8, Theorem 3.4].

Proposition 20. *Let us consider $\boldsymbol{\alpha} \in (\mathbb{Q} \cap]0,1[)^n$. The operator $\mathcal{L}_{\boldsymbol{\alpha}}$ is auto-dual (i.e. isomorphic to its dual) if and only if, up to permutation, $\boldsymbol{\alpha} = 1 - \boldsymbol{\alpha}$.*

Arguing as in section 4, one can prove the following result:

Theorem 21. *Let us consider $\boldsymbol{\alpha} \in (\mathbb{Q} \cap]0,1[)^n$ with $n \geq 3$. Let d be the least denominator of $\boldsymbol{\alpha}$ in $\mathbb{N}^*$. The following assertions are equivalent:*

i) *for all primes p congruent to -1 modulo d, we have $\mathcal{Q}_{\boldsymbol{\alpha}}(z) \in \mathbb{Z}_p[[z]]$;*
ii) *for infinitely many primes p congruent to -1 modulo d, we have $\mathcal{Q}_{\boldsymbol{\alpha}}(z) \in \mathbb{Z}_p[[z]]$;*
iii) *$\mathcal{L}_{\boldsymbol{\alpha}}$ is auto-dual.*

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