THE SCHUR-HORN THEOREM FOR OPERATORS WITH FINITE SPECTRUM

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ABSTRACT. The carpenter problem in the context of II_1 factors, formulated by Kadison, asks: Let $\mathcal{A} \subset \mathcal{M}$ be a masa in a type II_1 factor and let E be the normal conditional expectation from $\mathcal{M}$ onto $\mathcal{A}$. Then, is it true that for every positive contraction A in $\mathcal{A}$, there is a projection P in $\mathcal{M}$ such that $E(P) = A$? In this note, we show that this is true if A has finite spectrum. We will then use this result to prove an exact Schur-Horn theorem for positive operators with finite spectrum in type II_1 factors and an approximate Schur-Horn theorem for general positive operators in type II_1 factors.

1. INTRODUCTION

Let $\mathcal{A}$ be a masa in a II_1 factor $\mathcal{M}$ and E the normal conditional expectation from $\mathcal{M}$ to $\mathcal{A}$. Kadison, in [7], asked the following question:

Question 1.1 (Kadison’s carpenter problem). Given any positive contraction B in $\mathcal{A}$, does there exist a projection P in $\mathcal{M}$ so that $E(P) = B$?

We refer the reader to the above cited paper for the discussion leading up to this problem. The best result to date is the result of [5] that says the following:

Theorem 1.2 (Dykema, Fang, Hadwin, Smith). *Kadison’s carpenter problem has an affirmative solution for generator masas in $L(F_2)$. Also, for any positive contraction B in a Cartan masa $\mathcal{A}$ in the hyperfinite II_1 factor $\mathcal{R}$, there is an automorphism θ of $\mathcal{A}$ and a projection P in $\mathcal{R}$ so that $E_{\mathcal{A}}(P) = \theta(B)$.*

There are several consequences of this result that the reader can work out independently. For general II_1 factors, far less is known. Indeed, everything that is known so far with the exception of the result mentioned above and some extensions proved in the same paper, is a straightforward interpretation of results for matrices. For instance, the matricial Schur-Horn theorem guarantees that λI can be lifted if λ is a rational number, but it is not known if irrational multiples of the identity can be lifted to projections. In this note we show that this is indeed the case. It will follow that elements with finite spectrum can be lifted to projections.

In this note, we will work in a slightly more general context. Kadison’s carpenter problem is a special case of a majorization problem for von Neumann algebras. The notion of majorization in von Neumann algebras goes back at least to Hiai’s work [6] in the 1980s. The following definition is an extension of classical majorization to the continuous case.

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Definition 1.3 (Majorization). Given two self-adjoint operators A, S in a finite factor $(\mathcal{M}, \tau)$, say that A is majorized by S and denoted by $A \prec S$ if

$$\tau(f(A)) \leq \tau(f(S))$$

for every continuous convex real valued function f defined on a closed interval $[c, d]$ containing the spectra of both A and S .

The condition implies that $\tau(A) = \tau(S)$. Majorization can be expressed in several ways, and these equivalences can be found in [6] and the references therein. A natural extension of Kadison's problem was formulated by Kadison and Arveson in [4].

Question 1.4 (Arveson and Kadison's Schur Horn problem). Let A be a positive element in $\mathcal{A}$ and S a positive element in $\mathcal{M}$ such that $A \prec S$. Then, is it true that there exists an element T in $\mathcal{O}(S) = \overline{\{USU^* \mid U \in \mathcal{U}(\mathcal{M})\}}^{\|\cdot\|}$ such that $E(T) = A$?

One does need to take the norm closure; see the example following Lemma 5.5 in the same paper.

This problem was solved in the affirmative for the generator and radial masas in the free group factors in [5], where it was also solved modulo an automorphism of the masa for Cartain masas in the hyperfinite II_1 factor. In this note, we will work with general masas inside general type II_1 factors. Our main result is the following theorem whose proof is an adaptation of the best known proof of the matricial Schur Horn theorem.

Theorem 1.5. *Let $\mathcal{A}$ be a masa in a II_1 factor $\mathcal{M}$ and let E be the normal conditional expectation from $\mathcal{M}$ to $\mathcal{A}$. Let $A \in \mathcal{A}$ and $S \in \mathcal{M}$ be positive operators with finite spectrum such that $A \prec S$. Then, there is a unitary U in $\mathcal{M}$ so that $E(USU^*) = A$.*

The theorem says that the Schur-Horn problem can be solved when both elements have finite spectrum. Routine calculations will then allow us to adapt the above theorem to deduce an approximate Schur-Horn theorem for general operators in a II_1 factor.

Theorem 1.6. *Let S be a positive operator in $\mathcal{M}$. Then, the norm closure of $E(\mathcal{U}(S))$ equals $\{A \in \mathcal{A}^+ \mid A \prec S\}$.*

In particular, letting $\mathcal{O}(S) = \overline{\{USU^* \mid U \in \mathcal{U}(\mathcal{M})\}}^{\|\cdot\|}$, we have that

$$\overline{E(\mathcal{O}(S))}^{\|\cdot\|} = \{A \in \mathcal{A} \mid A \prec S\}.$$

The conjectured Schur-Horn theorem of Arveson and Kadison says that we do not need to take the norm closure for equality, something that we are unable to prove in this paper. A weaker version of our theorem where the σ -SOT closure was taken in the place of the norm closure was proved by Argerami and Massey in [3]. Also, the above result was established for Cartan masas in the hyperfinite II_1 factors (and thus for general semi-regular masas; see [11]) in [5].

The paper has four sections apart from the introduction. In section 2, we show that scalars can be lifted to projections. In section 3, we push this through to show that the Schur-Horn problem can be solved for operators with finite spectrum. Section 4 contains the approximate Schur-Horn theorem. There is then a last section consisting of some remarks and observations.

Some words on notation: Given two operators A, B inside a von Neumann algebra $\mathcal{M}$ such that there is a projection P inside $\mathcal{M}$ with $A = PAP$ and $B = (I - P)B(I - P)$, in order to stress the fact that A and B live under the auspices of orthogonal projections, we will use the expression $A \oplus B$ to denote their sum. Next, given a self-adjoint operator A and a Borel measurable subset X of the real line, the expression $E_A(X)$ will denote the spectral projection of A corresponding to the subset X . This notation conflicts with the usage of $E_{\mathcal{A}}(A)$ or simply $E(A)$ where $\mathcal{A}$ is a subalgebra of $\mathcal{M}$, which denotes the image under a conditional expectation E ; hopefully this will not cause any confusion.

2. LIFTING SCALARS

The carpenter theorem for matrices mentioned in [7] ensures that rational multiples of the identity can be lifted to projections. In this section, we will show that irrational multiples of the identity can be lifted to as well. We begin with a simple observation.

Lemma 2.1. *Let P be a projection in a masa $\mathcal{A}$ inside a type II_1 factor $\mathcal{M}$ and let λ, a, b be positive scalars such that $\tau(S) = \lambda$, where $S = aP + b(I - P)$. Then, there is a unitary U in $\mathcal{M}$ and a projection Q in $\mathcal{A}$ such that letting $T = USU^*$, we have that*

- (1) $E(QTQ) = \lambda Q$.
- (2) $(I - Q)T(I - Q) = cR + d(I - Q - R)$ for some projection R in $\mathcal{A}$ with $R \leq I - Q$ and positive numbers c, d .
- (3) $\tau(Q) \geq \frac{1}{3}$.

Proof. The lemma is trivial if $a = b$, for then $a = b = \lambda$ and there is nothing to prove. We assume without loss of generality that $a > b$. Since $\tau(S) = \lambda$, we must then have that $a > \lambda > b$. We may also assume that $\tau(P) \leq \frac{1}{2}$. For, suppose we have proved the lemma in this case; the result when $\tau(P) > \frac{1}{2}$ can be derived by applying the lemma to $I - S$ and $(1 - \lambda)I$. We therefore assume that $\tau(P) \leq \frac{1}{2}$.

Let k be the largest integer such that $(k + 1)\tau(P) \leq 1$. Since $\tau(P) \leq \frac{1}{2}$, the positive integer k must be at least 1. Pick projections $Q_1, \dots, Q_k$, each of trace $\tau(P)$ in $\mathcal{A}$ that are mutually orthogonal and also orthogonal to P . Let $V_1, \dots, V_k$ be partial isometries in $\mathcal{M}$ such that

- (1) $V_1^*V_1 = Q_1$ and $V_1V_1^* = P$.
- (2) For $2 \leq i \leq k$, $V_iV_i^* = Q_{i-1}$ and $V_i^*V_i = Q_i$.

Pick θ_1 such that $a \cos^2(\theta_1) + b \sin^2(\theta_1) = \lambda$ and let U_1 be the operator

$$U_1 = \cos(\theta_1)P + \sin(\theta_1)V_1 - \sin(\theta_1)V_1^* + \cos(\theta_1)Q_1 + (I - P - Q_1).$$

We will identify the above operator with the operator matrix (using V_1 as the matrix unit E_{12}), an identification that is standard:

$$U_1 = \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 \\ -\sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Under this same identification, S is the operator

$$S = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

Let $S_1 = U_1 S U_1^*$. It is easy to check that U_1 is a unitary and that

$$S_1 = \begin{pmatrix} a \cos^2(\theta_1) + b \sin^2(\theta_1) & * & 0 \\ * & a \sin^2(\theta_1) + b \cos^2(\theta_1) & 0 \\ 0 & 0 & b \end{pmatrix} = \begin{pmatrix} \lambda & * & 0 \\ * & a_1 & 0 \\ 0 & 0 & b_1 \end{pmatrix},$$

where $a_1 = a \sin^2(\theta_1) + b \cos^2(\theta_1)$ and $b_1 = b$. Equating traces,

$$\lambda \tau(P) + a_1 \tau(P) + b_1(1 - 2\tau(P)) = \lambda.$$

Since $b_1 = b < \lambda$, we must have that $a_1 > \lambda$ and a fortiori $a_1 > b_1$.

Now, continue as above. Pick θ_2 such that $a_1 \cos^2(\theta_2) + b_1 \sin^2(\theta_2) = \lambda$ and let U_2 be the operator

$$U_2 = \cos(\theta_2)Q_1 + \sin(\theta_2)V_2 - \sin(\theta_2)V_2^* + \cos(\theta_2)Q_2 + (I - Q_1 - Q_2).$$

We may write the unitary U_2 as

$$U_2 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & \cos(\theta_2) & \sin(\theta_2) & 0 \\ 0 & -\sin(\theta_2) & \cos(\theta_2) & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and let $S_2 = U_2 S_1 U_2^*$. We have that

$$S_2 = \begin{pmatrix} \lambda & * & * & 0 \\ * & \lambda & * & 0 \\ * & * & a_2 & 0 \\ 0 & 0 & 0 & b_2 \end{pmatrix},$$

where $a_2 = a_1 \sin^2(\theta_2) + b_1 \cos^2(\theta_2)$ and $b_2 = b_1$. Equating traces,

$$2\lambda \tau(P) + a_2 \tau(P) + b_2(1 - 3\tau(P)) = \lambda.$$

Since $b_2 = b_1 = b < \lambda$, we must have that $a_2 > \lambda$ and a fortiori $a_2 > b_2$.

Proceeding this way $k - 2$ more times, we get an operator S_k of the form

$$S_k = \begin{pmatrix} \lambda & * & \dots & * & * & 0 \\ * & \lambda & \dots & * & * & 0 \\ \vdots & \vdots & \ddots & \vdots & * & 0 \\ * & * & * & \lambda & * & 0 \\ * & * & * & * & a_k & 0 \\ 0 & 0 & 0 & 0 & 0 & b_k \end{pmatrix}.$$

Let $Q = P + Q_1 + \dots + Q_{k-1}$ (if $k = 1$, let $Q = P$). We see that

(1) $E(QS_kQ) = \lambda Q$. This is because

$$\begin{aligned} E(QS_kQ) &= E(PS_kP) + E(Q_1S_kQ_1) + \dots + E(Q_{k-1}S_kQ_{k-1}) \\ &= \lambda P + \lambda Q_1 + \dots + \lambda Q_{k-1} \\ &= \lambda Q. \end{aligned}$$

(S_k is the operator T promised in the statement of the lemma).

- (2) $(I - Q)S_k(I - Q)$ has a two point spectrum in $(I - Q)M(I - Q)$.
 (3) $\tau(Q) = k\tau(P)$. Since $(k + 1)\tau(P) \leq 1 < (k + 2)\tau(P)$, we see that

$$\tau(Q) = k\tau(P) = \frac{k}{k+2}(k+2)\tau(P) > \frac{k}{k+2} \geq \frac{1}{3}.$$

The lemma follows. $\square$

We now use a simple limit argument to show that scalar multiples of the identity can be lifted to projections.

Theorem 2.2. *Let $\mathcal{A}$ be a masa in a II_1 factor $\mathcal{M}$ and let E be the normal conditional expectation from $\mathcal{M}$ to $\mathcal{A}$. Then for any $0 \leq \lambda \leq 1$, there is a projection P in $\mathcal{M}$ such that $E(P) = \lambda I$.*

Proof. Let P_0 be any projection of trace λ in $\mathcal{A}$. Using Lemma 2.1, construct a unitary U_1 and a projection Q_1 in $\mathcal{A}$ such that, letting $P_1 = U_1 P_0 U_1^*$,

- (1) $\tau(Q_1) \geq \frac{1}{3}$.
 (2) $E(Q_1 P_1 Q_1) = \lambda Q_1$.
 (3) $(I - Q_1)P_1(I - Q_1)$ has two point spectrum in $(I - Q_1)M(I - Q_1)$.

Let $R_1 = Q_1$. Next, for $k = 2, 3, \dots$, apply Lemma 2.1 to $\lambda(I - R_{k-1})$ and $(I - R_{k-1})Q_{k-1}(I - R_{k-1})$ inside the II_1 factor $(I - R_{k-1})M(I - R_{k-1})$ to construct a unitary U_k and a projection Q_k in $(I - R_{k-1})M(I - R_{k-1})$ and let

$$R_k = Q_1 \oplus Q_2 \oplus \dots \oplus Q_k \quad \text{and} \quad P_k = (R_{k-1} \oplus U_k)P_{k-1}(R_{k-1} \oplus U_k)^*.$$

Here we identify Q_k , which is a projection in $(I - R_{k-1})M(I - R_{k-1})$, with a projection in $\mathcal{M}$ dominated by $I - R_{k-1}$. Note that P_k is a projection. We have that

- (1) $E(Q_k P_k Q_k) = \lambda Q_k$, and thus

$$E(R_k P_k R_k) = \sum_{m=1}^k E(Q_m P_m Q_m) = \sum_{m=1}^k \lambda Q_m = \lambda R_k.$$

- (2) $\tau(I - R_k) \leq \frac{2}{3}\tau(I - R_{k-1}) \leq (\frac{2}{3})^k$, and hence R_k converges to I strongly.
 (3) $(I - R_k)P_k(I - R_k)$ has two point spectrum in $(I - R_k)\mathcal{M}(I - R_k)$.
 (4) We also have that $R_{k-1}P_{k-1}R_{k-1} = R_{k-1}P_k R_{k-1}$, and thus

$$(2.1) \quad R_l(P_m - P_n)R_l = 0 \quad \text{for any } n, m \geq l.$$

We now claim that P_k converges in the strong operator topology to a projection that we will call P and also that $E(P) = \lambda I$. For the first claim, since the projections R_k converge strongly to I by statement (2) above, for any $\epsilon > 0$ there is an N so that $\|(I - R_N)\|_2 < \epsilon$. For $n, m \geq N$,

$$\|(P_n - P_m)\|_2 \leq \|R_N(P_n - P_m)R_N\|_2 + 2\|(I - R_N)(P_n - P_m)\|_2.$$

The first term is zero by (2.1). For the second term,

$$\|(I - R_N)(P_n - P_m)\|_2 \leq \|I - R_N\|_2 \|P_n - P_m\| \leq 2\epsilon.$$

Thus, $\|(P_n - P_m)\|_2 \leq 4\epsilon$ and the sequence $\{P_n\}$ is strongly convergent. Let P be the limit projection. For the second claim,

$$\begin{aligned} \|E(P) - \lambda I\|_2 &= \lim \|E(P_n) - \lambda I\|_2 \\ &= \lim \|\lambda R_n + E((I - R_n)P_n(I - R_n)) - \lambda I\|_2 \\ &= \lim \|-\lambda(I - R_n) + E((I - R_n)P_n(I - R_n))\|_2 \\ &\leq \lim \lambda\|(I - R_n)\|_2 + \|(I - R_n)P_n(I - R_n)\|_2 \\ &\leq \lim \lambda \left(\frac{2}{3}\right)^n + \|P_n\| \left(\frac{2}{3}\right)^n \\ &\leq \lim(\lambda + 1) \left(\frac{2}{3}\right)^n \\ &= 0. \end{aligned}$$

We conclude that $E(P) = \lambda I$. $\square$

We record a simple proposition.

Proposition 2.3. *Let A be a positive contraction in $\mathcal{A}$ that can be written as $A = \sum_n \lambda_n E_n$, where the E_n 's are orthogonal projections summing up to I . Then, there is a projection P in $\mathcal{M}$ such that $E(P) = A$.*

Proof. The element A may be written as $A = \sum_{n=1}^{\infty} \lambda_n E_n$, where the E_n 's are mutually orthogonal projections in $\mathcal{A}$ summing up to 1 and $0 \leq \lambda_n \leq 1$ for every n . The algebra $E_n \mathcal{M} E_n$ is a type II_1 factor, and we may find a projection P_n in $E_n \mathcal{M} E_n$ such that $E_{AE_n}(P_n) = \lambda_n E_n$ for every n . Let P be the projection $\sum_{n=1}^{\infty} P_n$. Here, we are identifying P_n , which is a projection in $E_n \mathcal{M} E_n$, with a projection in $\mathcal{M}$ that is dominated by E_n . Then,

$$E(P) = \sum_{n=1}^{\infty} E(P_n) = \sum_{n=1}^{\infty} E(E_n P_n E_n) = \sum_{n=1}^{\infty} \lambda_n E_n = A. \quad \square$$

3. THE SCHUR-HORN THEOREM FOR OPERATORS WITH FINITE SPECTRUM

We will now bootstrap the theorem in the previous section to get a Schur-Horn theorem for positive operators with finite spectrum. Recall the following reformulation of majorization in II_1 factors. Let A, S be positive contractions in a type II_1 factor $\mathcal{M}$ and let $f, g : [0, 1] \rightarrow [0, 1]$ be the (essentially unique, right-continuous, non-increasing) functions (the s number functions), see for instance [8], which satisfy

$$\tau(A^n) = \int_0^1 f^n(r) dm(r) \quad \text{and} \quad \tau(S^n) = \int_0^1 g^n(r) dm(r) \quad \text{for } n = 0, 1, \dots.$$

Here m stands for Lebesgue measure on $\mathbb{R}$. Then $A \prec S$ iff (see [6]),

$$\int_0^t f(r) dm(r) \leq \int_0^t g(r) dm(r), \quad 0 \leq t \leq 1, \quad \text{and} \quad \int_0^1 f(r) dm(r) = \int_0^1 g(r) dm(r).$$

Lemma 3.1. *Let $A = \lambda_1 E_1 \oplus \lambda_2 E_2$, where $E_1 + E_2 = I$ and $\lambda_1 \geq \lambda_2 \geq 0$, and $S = \mu_1 F_1 \oplus \mu_2 F_2$, where $F_1 + F_2 = I$ and $\mu_1 > \mu_2 \geq 0$, be two operators in a II_1 factor with $\tau(A) = \tau(S)$. If $\mu_1 \geq \lambda_1$ and $\mu_2 \leq \lambda_2$, then $A \prec S$.*

Proof. It is easy to see that if B is a positive contraction, then $B \prec P$ for any projection P with $\tau(P) = \tau(B)$. Let $c = \frac{1}{\mu_1 - \mu_2}$ and $d = -\frac{\mu_2}{\mu_1 - \mu_2}$. The operator $cS + dI$ may be checked to equal F_1 . It is hence a projection, and of course $\tau(cS + dI) = \tau(cA + dI)$.

$$cA + dI = (c\lambda_1 + d)E_1 + (c\lambda_2 + d)E_2 = \frac{\lambda_1 - \mu_2}{\mu_1 - \mu_2}E_1 + \frac{\lambda_2 - \mu_2}{\mu_1 - \mu_2}E_2.$$

Since $\lambda_2 \leq \lambda_1$, $\lambda_2 \geq \mu_2$ and $\lambda_1 \leq \mu_1$, we have that

$$0 \leq \frac{\lambda_2 - \mu_2}{\mu_1 - \mu_2} \leq \frac{\lambda_1 - \mu_2}{\mu_1 - \mu_2} \leq \frac{\mu_1 - \mu_2}{\mu_1 - \mu_2} = 1.$$

Thus, $cA + dI$ is a positive contraction. By the observation in the first line of the proof, $cA + dI \prec cS + dI$ and therefore $A \prec S$. $\square$

Lemma 3.2. *Let $A = \lambda_1 E_1 \oplus \lambda_2 E_2$ and $S = \mu_1 E_1 \oplus \mu_2 E_2$, where $E_1 + E_2 = I$, be positive operators in a type II_1 factor $\mathcal{M}$ with the same trace. If $\lambda_1 \leq \mu_1$, then $A \prec S$.*

Proof. It is easy to see that we must have $\mu_2 < \lambda_2$. The lemma now follows from Lemma 3.1. $\square$

Lemma 3.3. *Let A be a self-adjoint operator and S a positive contraction in a II_1 factor so that $A \prec S$. Then A is a positive contraction as well.*

Proof. Routine verification. $\square$

Proposition 3.4. *Let S be a positive operator in $\mathcal{M}$ with two point spectrum and let A be a positive contraction in $\mathcal{A}$ that has finite spectrum and so that $A \prec S$. Then, there is a unitary U in $\mathcal{M}$ such that $E(USU^*) = A$.*

Proof. Write $S = \mu_1 F_1 \oplus \mu_2 F_2$, where $\mu_1 \geq \mu_2$ and $F_1 \oplus F_2 = I$. Let $c = \frac{1}{\mu_1 - \mu_2}$ (note that $c > 0$) and $d = -\frac{\mu_2}{\mu_1 - \mu_2}$. The operator $cS + dI$ may be checked to equal F_1 and is hence a projection. We also have that $cA + dI \prec cS + dI = F_1$. By Lemma 3.3, $cA + dI$ must actually be a positive contraction. Also, of course, $\tau(cS + dI) = \tau(cA + dI)$. Now, by Proposition 2.3, there is a unitary U so that $E(U(cS + dI)U^*) = cA + dI$, and hence $E(USU^*) = A$. $\square$

When one or both operators have finite spectrum, majorization reduces to a simple condition.

Lemma 3.5. *Let A, S be positive operators in a II_1 factor with $\tau(A) = \tau(S)$ and let f, g be the s -number functions of A, S respectively, as above. Suppose A has finite spectrum, i.e., the s -number function f for A has the form*

$$f = \sum_{n=1}^N \lambda_n \chi_{[s_{n-1}, s_n)}$$

for some natural number N and some sequences $0 = s_0 < s_1 < \cdots < s_N = 1$ and $0 \leq \lambda_N < \lambda_{N-1} < \cdots < \lambda_1$. Then, $A \prec S$ iff for $n = 1, 2, \dots, N$,

$$\int_0^{s_n} f(r) dm(r) \leq \int_0^{s_n} g(r) dm(r) \text{ or, equivalently, } \tau(AE_A([0, s_n])) \leq \tau(SE_S([0, s_n])).$$

Proof. Routine verification. □

We now prove the promised special case of the Schur-Horn theorem.

Theorem 3.6 (The Schur-Horn theorem for operators with finite spectrum in a II_1 factor). *Let A and S be positive operators with finite spectrum in $\mathcal{A}$ and $\mathcal{M}$ respectively and so that $A \prec S$. Then, there is a unitary U in $\mathcal{M}$ so that $E(USU^*) = A$.*

Proof. We assume that A and S have spectrum consisting of N and M points respectively. Write $A = \sum_{n=1}^N \lambda_n E_n$ and $S = \sum_{n=1}^M \mu_n F_n$, where the $\{\lambda_n\}_1^N$ (respectively, the $\{\mu_n\}_1^M$) are distinct. Assume now that the theorem has been proved whenever the operators have finite spectra and the cumulative number of eigenvalues is less than $N + M$.

We may assume that none of the λ_i equal any of the μ_j . For suppose $\lambda_i = \mu_j$. Assume that $\tau(E_i) \leq \tau(F_j)$; the other case is handled similarly. We may, after conjugating by a unitary, write $A = \lambda_i E_i \oplus (A - \lambda_i E_i)$ and $S = \mu_j E_i \oplus (S - \mu_j E_i) = \lambda_i E_i \oplus (S - \lambda_i E_i)$. Clearly, $A - \lambda_i E_i \prec S - \lambda_i E_i$, and it is enough to prove the theorem for $A - \lambda_i E_i$, which has at most $N - 1$ point spectrum in $(I - E_i)\mathcal{A}$, and $S - \lambda_i E_i$, which has at most M point spectrum inside $(I - E_i)\mathcal{M}(I - E_i)$. We therefore assume that none of the λ_i equal any of the μ_j .

Since $\mathcal{A}$ is isomorphic to $L^\infty([0, 1], dm)$, we may find a maximal nest of projections $\{P_t : 0 \leq t \leq 1\}$ in $\mathcal{A}$ with $P_t \leq P_s$ for $0 \leq t \leq s \leq 1$ and $\tau(P_t) = t$ for $0 \leq t \leq 1$. Since A (respectively S) has N (respectively M) point spectrum, we may, after conjugating A and S by unitaries, assume that A and S have the form

$$A = \sum_{n=1}^N \lambda_n (P_{s_n} - P_{s_{n-1}}) \quad \text{and} \quad S = \sum_{n=1}^M \mu_n (P_{t_n} - P_{t_{n-1}})$$

for sequences $0 = s_0 < s_1 < s_2 < \cdots < s_N = 1$ and $0 = t_0 < t_1 < t_2 < \cdots < t_M = 1$ and positive scalars $\lambda_1 > \lambda_2 > \cdots > \lambda_N \geq 0$ and $\mu_1 > \mu_2 > \cdots > \mu_M \geq 0$.

Reindex the set $\{s_1, \dots, s_{N-1}\} \cup \{t_1, \dots, t_{M-1}\}$ by $\{r_1, \dots, r_L\}$, where $r_1 < r_2 < \cdots < r_{L-1}$, and let $r_L = 1$. Then, we may write

$$A = \sum_{n=1}^L \gamma_n (P_{r_n} - P_{r_{n-1}}) \quad \text{and} \quad f = \sum_{n=1}^L \delta_n (P_{r_n} - P_{r_{n-1}}),$$

where $\gamma_n = \lambda_m$ for the unique value m so that $[r_{n-1}, r_n) \subset [s_{m-1}, s_m)$ and similarly for the numbers δ_n .

We will prove the theorem by induction on L . When $L = 1$, A and S are scalars, and thus $A = S = \tau(A)I$ and the theorem is trivial. Assume we have shown the following:

Statement 3.7. Let A and S be positive operators inside a masa, which we denote by $\mathcal{A}$, inside a type II_1 factor, which we denote by $\mathcal{M}$, so that $A = \sum_{n=1}^K \gamma_n (P_{r_n} - P_{r_{n-1}})$ and $S = \sum_{n=1}^K \delta_n (P_{r_n} - P_{r_{n-1}})$ for some sequences $0 = r_0 < r_1 < \cdots < r_{K-1} < r_K = 1$, $\gamma_1 \geq \cdots \geq \gamma_K$, $\delta_1 \geq \cdots \geq \delta_K$, where K is a natural number less than L . Then, there is a unitary U so that $E(USU^*) = A$.

We will now show that we can extend this to the case when the decompositions have length L as well. To avoid trivialities, we assume that $A \neq S$.

The majorization condition for the operators A and S that we are working with becomes the following: $\tau(A) = \tau(S)$, and for every $k = 1, \dots, L-1$ we have that

$$\int_0^{r_k} f(r) dm(r) = \sum_{n=1}^k \gamma_n(r_n - r_{n-1}) \leq \sum_{n=1}^k \delta_n(r_n - r_{n-1}) = \int_0^{r_k} g(r) dm(r).$$

In particular, $\gamma_1 \leq \delta_1$. If $\gamma_n \leq \delta_n$ for every $n = 1, \dots, L$ (and since $A \neq S$, at least one of these expressions must be a strict inequality), then

$$\tau(A) = \sum_{n=1}^L \gamma_n(r_n - r_{n-1}) < \sum_{n=1}^L \delta_n(r_n - r_{n-1}) = \tau(S),$$

which contradicts the fact that $A \prec S$ (which entails that $\tau(A) = \tau(S)$). Thus, there is a natural number $1 < l \leq L$ so that

$$\gamma_n < \delta_n \quad \text{for } n = 1, \dots, l \quad \text{and} \quad \gamma_{l+1} > \delta_{l+1}.$$

Suppose that $(\delta_l - \gamma_l)(r_l - r_{l-1}) \leq (\gamma_{l+1} - \delta_{l+1})(r_{l+1} - r_l)$ (the other case is handled similarly). Pick r so that $r_l < r \leq r_{l+1}$ and such that $(\delta_l - \gamma_l)(r_l - r_{l-1}) = (\gamma_{l+1} - \delta_{l+1})(r - r_l)$. Let

$$A_1 := \gamma_l(P_{r_l} - P_{r_{l-1}}) + \gamma_{l+1}(P_r - P_{r_l}) \quad \text{and} \quad S_1 := \delta_l(P_{r_l} - P_{r_{l-1}}) + \delta_{l+1}(P_r - P_{r_l}).$$

Then

$$\tau(S_1 - A_1) = (\delta_l - \gamma_l)(r_l - r_{l-1}) + (\delta_{l+1} - \gamma_{l+1})(r - r_l) = 0.$$

Combining this with the fact that $\gamma_l < \delta_l$ and using Lemma 3.2, we conclude that $A_1 \prec S_1$ inside the II_1 factor PMP where P is the projection $P = P_r - P_{r_{l-1}}$. Now, let

$$A_2 := A - A_1 = \gamma_{l+1}(P_{r_{l+1}} - P_r) + \sum_{n \neq l, l+1} \gamma_n(P_{r_n} - P_{r_{n-1}})$$

and, similarly,

$$S_2 := S - S_1 = \delta_{l+1}(P_{r_{l+1}} - P_r) + \sum_{n \neq l, l+1} \delta_n(P_{r_n} - P_{r_{n-1}}),$$

where the operators are considered in $(I - P)\mathcal{M}(I - P)$. We have

$$(1) \quad \sum_{n=1}^k \gamma_n(r_n - r_{n-1}) \leq \sum_{n=1}^k \delta_n(r_n - r_{n-1}) \quad \text{for } k = 1, \dots, l-1.$$

- (2) For $k \geq l+1$ (if $k = l+1$, the third term in the first expression below will not show up)

$$\begin{aligned}
 & \sum_{n=1}^{l-1} \gamma_n(r_n - r_{n-1}) + \gamma_{l+1}(r_{l+1} - r) + \sum_{n=l+2}^k \gamma_n(r_n - r_{n-1}) \\
 &= \sum_{n=1}^k \gamma_n(r_n - r_{n-1}) - \tau(A_1) \\
 &< \sum_{n=1}^k \delta_n(r_n - r_{n-1}) - \tau(A_1) \\
 &= \sum_{n=1}^k \delta_n(r_n - r_{n-1}) - \tau(S_1) \\
 &= \sum_{n=1}^{l-1} \delta_n(r_n - r_{n-1}) + \delta_{l+1}(r_{l+1} - r) + \sum_{n=l+2}^k \delta_n(r_n - r_{n-1}).
 \end{aligned}$$

- (3) $\tau(A_2) = \tau(A) - \tau(A_1) = \tau(S) - \tau(S_1) = \tau(S_2)$.

We thus conclude that we also have that $A_2 \prec S_2$.

By Proposition 3.4, there is a unitary U_1 inside PMP so that $E(U_1 S_1 U_1^*) = A_1$. Also, the induction hypothesis holds for the operators A_2 and M_2 inside $(I - P)\mathcal{M}(I - P)$ since the partition decomposition for A_2 and S_2 has length $L - 1$. We may therefore find a unitary U_2 inside so that $E(U_2 S_2 U_2^*) = A_2$. Thus, letting $U = U_1 \oplus U_2$, we have that $E(USU^*) = A$. $\square$

4. AN APPROXIMATE SCHUR-HORN THEOREM

Theorem 3.6 allows us to prove an approximate version of the Schur-Horn theorem for general positive operators.

Theorem 4.1. *Let S be a positive operator in a II_1 factor $\mathcal{M}$ and let $\mathcal{A}$ be a masa in $\mathcal{M}$. Then, the norm closure of $E(\mathcal{U}(S))$ equals $\{A \in \mathcal{A}^+ \mid A \prec S\}$.*

Proof. Choose A in $\mathcal{A}^+$ so that $A \prec S$. By scaling, if needed, we assume that A and S are strict contractions. Fix $n > 0$ and define the mutually orthogonal projections

$$P_k = E_A\left(\left[\frac{k-1}{n}, \frac{k}{n}\right)\right) \quad \text{for } 1 \leq k \leq n.$$

Next, define $\alpha_k = \tau(AP_k)$ for $1 \leq k \leq n$ and consider the operator $B = \sum_{k=1}^n \alpha_k P_k$. Since $\tau(C)I \prec C$ for any positive operator C , we have that $B \prec A$, and hence $B \prec S$. We also have that

$$\|A - B\| = \left\| \sum_{k=1}^n (A - \alpha_k) P_k \right\| \leq \left\| \sum_{k=1}^n \frac{1}{n} P_k \right\| = \frac{1}{n}.$$

Choose numbers $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = 1$ and orthogonal projections $Q_1, \dots, Q_n$ in $\{S' \cap \mathcal{M}\}$ such that $Q_k \leq E_S([t_{k-1}, t_k])$ and $\tau(Q_k) = \tau(P_k)$ for $1 \leq k \leq n$. To see why this is possible, proceed as follows: Let $t_0 = 0$ and pick t_1 such that $\tau(E_S([0, t_1])) \leq \tau(P_1) \leq \tau(E_S([0, t_1]))$. If S has no atom at t_1 , then let $Q_1 = E_S([0, t_1])$. If S has an atom at t_1 , pick a subprojection R of $E_S(\{t_1\})$ such that $\tau(E_S([0, t_1])) + \tau(R) = \tau(P_1)$ and let $Q_1 = E_S([0, t_1]) + R$. Continue this process for $n = 2, \dots$.

Next, pick positive operators $T_1, \dots, T_n$ all with finite spectrum such that for $1 \leq k \leq n$,

$$T_k \prec SQ_k, \quad \tau(T_k) = \tau(SQ_k) \quad \text{and} \quad \|SQ_k - T_k\| \leq \frac{1}{n}.$$

This is done in exactly the same way as the choice of the operator B given the operator A , in the first part of this proof. Let T be the operator $T = T_1 + \dots + T_n$. Then, the above conditions imply

$$T \prec S \quad \text{and} \quad \|S - T\| \leq \frac{1}{n}.$$

Also, for $1 \leq k \leq m$,

$$\tau(T(Q_1 + \dots + Q_k)) = \tau(S(Q_1 + \dots + Q_k)) \geq \tau(A(P_1 + \dots + P_k)) = \tau(B(P_1 + \dots + P_k)).$$

The middle inequality is because $A \prec S$. Hence, by Lemma 3.5, $B \prec T$. Since B and T have finite spectrum, there is a unitary U so that $B = E(UTU^*)$. We calculate that

$$\|A - E(USU^*)\| \leq \|A - B\| + \|B - E(UTU^*)\| + \|E(UTU^* - USU^*)\| \leq \frac{1}{n} + 0 + \frac{1}{n}$$

and see that A can be arbitrarily well approximated by elements in $E(\mathcal{U}(S))$. Since A was arbitrary, we have that the norm closure of $E(\mathcal{U}(S))$ equals $\{A \in \mathcal{A}^+ \mid A \prec S\}$. $\square$

5. DISCUSSION

The proofs given above can be easily adapted to masas in type *III* factors that admit a faithful normal conditional expectation. Cartan masas by definition satisfy this property, but not all masas do; by a result of Takesaki [12], if every masa in a von Neumann algebra admits a normal conditional expectation, then it is finite. Suppose $\mathcal{A}$ is a masa in a type *III* factor $\mathcal{M}$ admitting a normal conditional expectation $E : \mathcal{M} \rightarrow \mathcal{A}$. Let $A \in \mathcal{A}$ and S be positive operators. For any self-adjoint operator T , let $\alpha(T) = \min(\{x \in \sigma(T)\})$. For any unitary U in $\mathcal{M}$, we have that $\|E(USU^*)\| \leq \|S\|$ and that $\alpha(E(USU^*)) \geq \alpha(S)$. It is now easy to see that a necessary condition for the existence of an element $T \in \mathcal{O}(S)$ such that $E(T) = A$ is that $\|A\| \leq \|S\|$ and $\alpha(A) \geq \alpha(S)$.

The Schur-Horn problem in type *III* factors is more tractable than in the type *II*₁ case. Standard arguments allow us to prove the following lemma.

Lemma 5.1. *Let $S = \sum_{n=1}^N \mu_n F_n$ be a positive contraction with finite spectrum in a type *III* factor $\mathcal{M}$ with $\|S\| = 1$ and $\alpha(S) = 0$. Then, $\mathcal{O}(S)$ contains a non-trivial projection (and thus every projection).*

With this in hand, it is easy to see that if $A \in \mathcal{A}$ and $S \in \mathcal{M}$ are positive elements with finite spectrum so that $\|A\| \leq \|S\|$ and $\alpha(A) \geq \alpha(S)$, then we can solve the Schur-Horn problem for A and S . There is further a simple condition that allows us to determine when we can find a unitary so that $E(USU^*) = A$.

Suppose 0 is the point spectrum of A so that there is a projection P in $\mathcal{A}$ so that $PAP = 0$. Suppose we write $A = E(T)$ for some positive operator T ; then $E(PTP) = 0$ and hence $PTP = 0$. Thus, 0 must be in the point spectrum of T . If $A = E(USU^*)$, we get that 0 must be in the point spectrum of USU^* and hence in the point spectrum of S . Similarly, if 1 is in the point spectrum of a positive contraction A and $A = E(USU^*)$ for some positive contraction S and a unitary U ,

then 1 must be in the point spectrum of S as well. These necessary conditions are also sufficient.

Theorem 5.2. *Let $\mathcal{A}$ be a masa inside a type III factor $\mathcal{M}$ admitting a faithful normal conditional expectation E , let A and S be positive operators with finite spectrum in $\mathcal{A}$ and $\mathcal{M}$ respectively and let E be the normal conditional expectation onto $\mathcal{A}$. Assume further that $\alpha(A) \geq \alpha(S)$ and $\|A\| \leq \|S\|$. Then:*

- (1) *There is an element $T \in \mathcal{O}(S)$ such that $E(T) = A$.*
- (2) *Assume additionally that if either 0 and $\|S\|$ are in the point spectrum of A , then they are in the point spectrum of S as well. Then, there is a unitary U such that $E(USU^*) = A$.*

We omit the details, as they are a straightforward adaptation of the proof of Theorem 3.6.

In general, we could ask:

Question 5.3. Let A and S be positive operators in $\mathcal{A}$ and $\mathcal{M}$ respectively, where $\mathcal{A}$ is a masa inside a type III factor admitting a normal conditional expectation and so that $\|A\| \leq \|S\|$ and $\alpha(A) \geq \alpha(S)$. Then, is there an element T in $\mathcal{O}(S)$ so that $E(T) = A$?

Lyapunov's theorem [10], which states that the range of any non-atomic vector valued measure taking values in $\mathbb{C}^n$ is compact and convex, was reformulated in operator algebraic language by Lindenstrauss [9] to say the following: Let Φ be a weak* continuous linear map from a non-atomic abelian von Neumann algebra into $\mathbb{C}^n$. Then, for any positive contraction A , there is a projection P such that $\Phi(A) = \Phi(P)$. Akemann and Anderson, in their superb monograph [2], called any theorem concerning linear maps $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are subsets of linear spaces. This assures us that $\text{Ran}(\Phi) = \text{Ran}(\Phi|_{\partial(\mathcal{X})})$ a Lyapunov type theorem. Clearly, Kadison's carpenter problem is a Lyapunov type problem. Anderson and Akemann proved a variety of Lyapunov theorems and showed, quite surprisingly, that Lyapunov theorems are substantially more tractable when the maps considered are singular. The one of most interest to us is

Theorem 5.4 (Akemann and Anderson). *Let $\mathcal{A}$ be a masa in an type II_1 factor $\mathcal{M}$. Let F be a singular conditional expectation from $\mathcal{M}$ to $\mathcal{A}$. Then, for any positive contraction A in $\mathcal{A}$, there is a projection P in $\mathcal{M}$ so that $F(P) = A$.*

There are plenty of singular conditional expectations onto masas in II_1 factors [1], though none of them are trace preserving. The corresponding Schur-Horn problem cannot be any other than

Question 5.5. Let $\mathcal{A}$ be a masa in an type II_1 factor $\mathcal{M}$. Let F be a singular conditional expectation from $\mathcal{M}$ to $\mathcal{A}$. Suppose $A \in \mathcal{A}$ and $S \in \mathcal{M}$ are positive contractions that are not multiples of the identity such that $\|A\| \leq \|S\|$ and $\alpha(A) \geq \alpha(S)$. Then, is there an element $T \in \mathcal{O}(S)$ such that $F(T) = A$?

Finally, an answer to the following related question, which we are unable to solve, should help in solving the Schur-Horn and carpenter problems in type II_1 factors.

Question 5.6. Let A be a positive operator in a masa $\mathcal{A}$ inside a II_1 factor $\mathcal{M}$. Then, does the norm closure of $\mathfrak{L}(A) = \{S \in \mathcal{M} \mid \exists T \in \mathcal{O}(S) \text{ so that } E(T) = A\}$ equal $\{S \in \mathcal{M} \mid A \prec S\}$?

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