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COMPLETE CMC HYPERSURFACES IN THE HYPERBOLIC SPACE WITH PRESCRIBED GAUSS MAPPING

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ABSTRACT. Our aim in this paper is to show that a complete hypersurface $x : M^n \to \mathbb{H}^{n+1}$ immersed with constant mean curvature into the hyperbolic space $\mathbb{H}^{n+1}$ is totally umbilical provided that its Gauss mapping ν has some suitable behavior. In this setting, our first result requires that the image $\nu(M)$ lies in a totally umbilical spacelike hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$, while in our second one we suppose that M^n has scalar curvature bounded from below and that $\nu(M)$ is contained in the closure of a domain enclosed by a totally umbilical spacelike hypersurface of $\mathbb{S}_1^{n+1}$ determined by some vector a of the Minkowski space $\mathbb{L}^{n+2}$, with the tangential component of a with respect to M^n having Lebesgue integrable norm.

1. Introduction and statements of the results

Let $x : M^n \to \mathbb{Q}^{n+1}$ be an immersion of an orientable Riemannian manifold M^n in a space form $\mathbb{Q}^{n+1}$ and $\nu : M^n \to \mathbb{Q}^{n+1}$ its associated Gauss mapping. When $\mathbb{Q}^{n+1}$ is a Euclidean space and x is a complete graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, its Gauss mapping is contained in an open hemisphere of $\mathbb{S}^n$. The behavior of the Gauss mapping gives a deeper consequence for the immersion. For instance, one of the most celebrated theorems of the theory of minimal surfaces in $\mathbb{R}^3$ is Bernstein's theorem [4], which establishes that the only complete minimal graphs in $\mathbb{R}^3$ are the planes. This result was extended under the weaker hypothesis that the image of the Gauss mapping of M^2 lies in an open hemisphere of $\mathbb{S}^2$, as we can see in the work of Barbosa and do Carmo [3]. Independently, de Giorgi [5] and Simons [14] showed that a compact minimal hypersurface M^n of the Euclidean sphere $\mathbb{S}^{n+1}$ must be a totally geodesic sphere provided that the image of its Gauss mapping lies in an open hemisphere of $\mathbb{S}^{n+1}$. Such a result was extended by Nomizu and Smyth in [9] to constant mean curvature hypersurfaces of $\mathbb{S}^n$, proving that a compact connected orientable manifold M^n of dimension $n \geq 2$ immersed in the sphere $\mathbb{S}^{n+1}$ with constant mean curvature is a hypersphere if the Gauss image of M^n lies in a closed hemisphere of $\mathbb{S}^{n+1}$.

More recently, using the Lorentz model of the hyperbolic space $\mathbb{H}^{n+1}$ (for details, see Section 2), the second and third authors [2] showed that a constant mean curvature complete hypersurface M^n which is contained in a geodesic ball of $\mathbb{H}^{n+1}$ and such that the image of the Gauss mapping lies in a totally umbilical spacelike

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hypersurface of the de Sitter space $\mathbb{S}_1^{n+1}$ must be a totally umbilical geodesic sphere of $\mathbb{H}^{n+1}$. Moreover, in the case that M^n is contained between two horospheres (hyperspheres) of $\mathbb{H}^{n+1}$ determined by some nonzero null (spacelike) vector a of the Minkowski space $\mathbb{L}^{n+2}$ and with the image of its Gauss mapping contained in a totally umbilical spacelike hypersurface of $\mathbb{S}_1^{n+1}$ determined by a, they proved that M^n must be a horosphere (hypersphere) of $\mathbb{H}^{n+1}$.

Here, motivated by the works previously described, we apply a suitable Simons-type formula due to Nomizu and Smyth [10] jointly with the well known generalized maximum principle of Omori-Yau [11, 15] in order to obtain the following extension of the results of [2] mentioned above.

Theorem 1.1. *The only complete constant mean curvature hypersurfaces immersed in $\mathbb{H}^{n+1}$ such that the image of the Gauss mapping lies in a totally umbilical spacelike hypersurface of $\mathbb{S}_1^{n+1}$ are the totally umbilical ones.*

In [8], Montiel proved that if a complete spacelike hypersurface Σ^n in the de Sitter space $\mathbb{S}_1^{n+1}$ with constant mean curvature $H \geq 1$ is such that the image of its Gauss mapping is contained in the closure of the interior domain enclosed by a horosphere of $\mathbb{H}^{n+1}$, then its mean curvature is, in fact, equal to 1. When $n = 2$, this implies that Σ^2 is also a totally umbilical surface and, hence, the image of its Gauss mapping is exactly a horosphere of $\mathbb{H}^3$. In our second rigidity theorem, we establish a sort of dual for the result of Montiel mentioned above. For this, we use as our main analytical tool an extension of the classical Hopf theorem on a complete noncompact Riemannian manifold due to Yau [16] (cf. Lemma 3.2).

In what follows, we denote by $a^\top$ the tangential component of a vector $a \in \mathbb{L}^{n+2}$ with respect to an immersion $x : M^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ and, according to the terminology established in [8], we say that the image of its Gauss mapping *ν is contained in the closure of a domain enclosed by a totally umbilical spacelike hypersurface of $\mathbb{S}_1^{n+1}$ determined by some vector $a \in \mathbb{L}^{n+2}$* when the angle function $\langle \nu, a\rangle$ does not change sign on M^n.

Theorem 1.2. *The only complete constant mean curvature hypersurfaces immersed in $\mathbb{H}^{n+1}$ with scalar curvature bounded from below and whose image of the Gauss mapping is contained in the closure of a domain enclosed by a totally umbilical spacelike hypersurface of $\mathbb{S}_1^{n+1}$ determined by some vector $a \in \mathbb{L}^{n+2}$, with $a^\top$ having Lebesgue integrable norm along them, are the totally umbilical ones.*

The proofs of Theorems 1.1 and 1.2 are given in Section 3.

2. Preliminaries

Throughout this paper we consider the Lorentz model of the hyperbolic space $\mathbb{H}^{n+1}$ obtained by furnishing the hyperquadric $\{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = -1,\ p_{n+2} > 0\}$ with the Riemannian metric induced by the Lorentz metric of the Minkowski space $\mathbb{L}^{n+2}$. In this setting, let $x : M^n \to \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$ be a connected orientable hypersurface immersed into $\mathbb{H}^{n+1}$. We recall that a unit normal globally defined vector field ν of M^n can be regarded as a mapping $\nu : M^n \to \mathbb{S}_1^{n+1}$, where $\mathbb{S}_1^{n+1}$ denotes the $(n+1)$-dimensional unitary de Sitter space; that is,

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2}; \langle p, p\rangle = 1\}.$$

In order to set up the notation, we will denote by ∇^0,$\overline{\nabla}$ and ∇ the Levi-Civita connections of $\mathbb{L}^{n+2}$, $\mathbb{H}^{n+1}$ and M^n, respectively. Then the Gauss and Weingarten formulas for M^n in $\mathbb{H}^{n+1}$ are given, respectively, by

$$\nabla^0{}_X Y = \nabla_X Y + \langle AX, Y\rangle\nu + \langle X, Y\rangle x$$

and

$$AX = -\overline{\nabla}_X\nu = -\nabla^0{}_X\nu,$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M)$, where A stands for the shape operator of M^n with respect to ν.

In what follows, for a fixed arbitrary vector $a \in \mathbb{L}^{n+2}$, let us consider the *height* and the *angle* functions, defined respectively by $l_a = \langle x, a\rangle$ and $f_a = \langle \nu, a\rangle$. A direct computation allows us to conclude that the gradient of such functions are given by $\nabla l_a = a^\top$ and $\nabla f_a = -A(a^\top)$, where $a^\top$ is the orthogonal projection of a onto the tangent bundle TM, which is given by

$$a^\top = a - f_a\nu + l_a x.$$

Taking into account that $\nabla^0 a = 0$ and using the Gauss and Weingarten formulas, we have

$$\nabla_X a^\top = f_a AX + l_a X, \tag{2.1}$$

for all $X \in \mathfrak{X}(M)$. We now use (2.1) and the Codazzi equation to deduce

$$\nabla_X A(a^\top) = f_a A^2 X + l_a AX + (\nabla_{a^\top} A)(X)$$

for all $X \in \mathfrak{X}(M)$. Thus, according to [12] (see also [1]), it follows from the last two identities that

$$\Delta l_a = nHf_a + nl_a \tag{2.2}$$

and

$$\Delta f_a = -|A|^2 f_a - nHl_a - n\langle\nabla H, a^\top\rangle, \tag{2.3}$$

where $H = (1/n)\mathrm{trace}(A)$ is the mean curvature of M^n.

For what follows, it is also convenient to consider the *traceless operator* associated to the second fundamental form, $\Phi : \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is given by

$$\Phi(X) = AX - HX,$$

for all $X \in \mathfrak{X}(M)$. It is easily checked that the Hilbert-Schmidt norm of Φ (that is, $|\Phi|^2 = \mathrm{tr}(\Phi^*\Phi)$, where Φ^* stands for the adjoint of Φ) satisfies

$$|\Phi|^2 = |A|^2 - nH^2.$$

Consequently, we have that $|\Phi|^2 = 0$ if and only if M^n is totally umbilical.

Now, we will recall the description of the totally umbilical spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ due to Montiel in [7]. Let $\mathcal{L}_\rho$ be the spacelike hypersurface immersed into $\mathbb{S}_1^{n+1}$ given by

$$\mathcal{L}_\rho = \{p \in \mathbb{S}_1^{n+1}; \langle p, a\rangle = \rho\},$$

where $a \in \mathbb{L}^{n+2}$, $\langle a, a\rangle = 1, 0, -1$ and $\rho^2 > \langle a, a\rangle$. Then, for $p \in \mathcal{L}_\rho$,

$$\nu(p) = \frac{1}{\sqrt{\rho^2 - \langle a, a\rangle}}(a - \rho p) \in \mathbb{H}^{n+1}$$

is a unit normal vector field for $\mathcal{L}_\rho$. Consequently, the shape operator A of $\mathcal{L}_\rho$ is given by

$$AX = \frac{\rho}{\sqrt{\rho^2 - \langle a, a\rangle}} X,$$

for all smooth vector fields X tangent to $\mathcal{L}_\rho$ (cf. [7], Example 1). Hence, $\mathcal{L}_\rho$ is totally umbilical with constant mean curvature $H = \dfrac{\rho}{\sqrt{\rho^2 - \langle a, a\rangle}}$. In fact, it can be verified that:

(1) if a is a unit spacelike vector, then $\mathcal{L}_\rho$ is isometric to an n-dimensional hyperbolic space of constant sectional curvature $-\dfrac{1}{\rho^2 - 1}$ and $H^2 > 1$;
(2) if a is a nonzero null vector, then $\mathcal{L}_\rho$ is isometric to a Euclidean space $\mathbb{R}^n$ and $H^2 = 1$;
(3) if a is a unit timelike vector, then $\mathcal{L}_\rho$ is isometric to an n-dimensional sphere of constant sectional curvature $\dfrac{1}{\rho^2 + 1}$ and $0 \leq H^2 < 1$.

To close this section, we quote a suitable characterization of totally umbilical hypersurfaces in a semi-Riemannian space form due to Kim et al. [6], which corresponds to a converse for a theorem due to Sharma and Duggal in [13].

Lemma 2.1. *Let M^n be a connected semi-Riemannian hypersurface of a semi-Riemannian space form $\overline{M}^{n+1}(c)$. Suppose that $\overline{M}^{n+1}(c)$ carries a conformal vector field V whose tangential component $V^\top$ on M^n becomes a conformal vector field. Then, one of the following holds:*

(i) *M^n is a totally umbilical hypersurface;*
(ii) *the restriction of V to M^n reduces to a tangent vector field on M^n.*

3. Proofs of Theorems 1.1 and 1.2

In order to prove our first result, we quote the well known generalized maximum principle due to Omori-Yau [11] and Yau [15].

Lemma 3.1. *Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u : M^n \to \mathbb{R}$ be a smooth function which is bounded from above on M^n. Then there exists a sequence $(p_k)_{k\geq 1}$ in M^n such that*

$$\lim_k u(p_k) = \sup_M u, \quad \lim_k |\nabla u|(p_k) = 0 \quad \textit{and} \quad \limsup_k \Delta u(p_k) \leq 0.$$

Proof of Theorem 1.1. Let $x : M^n \to \mathbb{H}^{n+1}$ be a complete immersed hypersurface with constant mean curvature H and let us denote by A its Weingarten operator with respect to a globally defined normal vector field ν. Using the characterization of totally umbilical spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ described in Section 2, we have from our hypothesis under the image of the Gauss mapping ν of M^n that there exists a vector $a \in \mathbb{L}^{n+2}$ and a real number τ such that the angle function f_a of M^n satisfies $f_a = \langle \nu, a\rangle = \tau$ on M^n, with $\tau^2 > \langle a, a\rangle$.

If $\tau = 0$, then it immediately follows from (2.1) that the Hessian of the height function $l_a = \langle x, a\rangle$ satisfies $\nabla^2 l_a = l_a g$, where g stands for the Riemannian metric of M^n. Consequently, we conclude that $\nabla l_a = a^\top$ is a conformal vector field on M^n. From Lemma 2.1, since $H = 0$ in this case, we obtain that M^n is a totally geodesic hypersurface of $\mathbb{H}^{n+1}$.

If $\tau \neq 0$, then $H \neq 0$ and formula (2.3) gives

$$|A|^2 = -\frac{nH}{\tau} l_a. \tag{3.1}$$

Now we use (3.1) to conclude that

$$\frac{\tau}{H}\, l_a = -\frac{\tau^2}{nH^2}\, |\Phi|^2 - \tau^2,$$

where $\Phi = A - HI$ was previously defined. Therefore, the height function l_a satisfies $|l_a| \geq \beta$ for some positive constant β. We can assume, without loss of generality, that l_a is a strictly positive function on M^n.

We claim that the height function l_a is upper bounded. Indeed, since the mean curvature H of M^n is constant and $a^\top = \nabla l_a$, we obtain from (3.1) the following equality:

$$a^\top \left(|A|^2\right) = -\frac{nH}{\tau} |\nabla l_a|^2. \tag{3.2}$$

Now we choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on a neighborhood $\mathcal{U} \subset M^n$ which is geodesic at a point $p \in \mathcal{U}$. Thus, since A is a symmetric operator and $A(a^\top) = 0$, the Codazzi equation gives

$$a^\top \left(|A|^2\right) = 2 \sum_{i=1}^{n} \langle \nabla_{a^\top} A e_i \,,\, A e_i \rangle = -2 \sum_{i=1}^{n} \langle A^2(\nabla_{e_i} a^\top) \,,\, e_i \rangle. \tag{3.3}$$

On the other hand, since $\nabla_{e_i} a^\top = \tau A e_i + l_a e_i$ at p, we compare (3.1), (3.2) and (3.3) to deduce

$$nH \operatorname{tr}(A^3) = |A|^4 + \frac{n^2 H^2}{2\tau^2} |\nabla l_a|^2. \tag{3.4}$$

In [10], Nomizu and Smyth obtained the following Simons-type formula:

$$\frac{1}{2} \Delta |A|^2 = -n|A|^2 - |A|^4 + n^2 H^2 + nH \operatorname{tr}(A^3) + |\nabla A|^2. \tag{3.5}$$

Returning to our context, since H is constant and $f_a = \tau$, it follows from (2.2) and (3.1) that $\Delta |A|^2 = \Delta |\Phi|^2 = n|\Phi|^2$. Therefore, we can use (3.4) and (3.5) to deduce

$$|\Phi|^2 = \frac{nH^2}{3\tau^2} |\nabla l_a|^2 + \frac{1}{3n} |\nabla A|^2. \tag{3.6}$$

Using that l_a is a strictly positive function and the identity $|\nabla\, l_a|^2 + \tau^2 - l_a^2 = \langle a \,,\, a \rangle$, we obtain from (3.6) the following expression:

$$\left(\frac{H^2}{3\tau^2} \langle a \,,\, a \rangle + \frac{2H^2}{3} \right) \frac{1}{l_a} + \frac{H^2}{3\tau^2}\, l_a \leq -\frac{H}{\tau}. \tag{3.7}$$

Now we are in a position to prove that the height function l_a is bounded. Suppose by contradiction that there exists a sequence of points $(q_k)_{k \geq 1}$ in M^n such that $l_a(q_k) \to +\infty$ when $k \to \infty$. But, from the above inequality we obtain, after a straightforward computation, that

$$\lim_{k} l_a(q_k) \leq -\frac{3\tau}{H}, \tag{3.8}$$

which gives a contradiction. Consequently, l_a is bounded and we finish our claim.

Therefore, we can use (3.1) to conclude that $|A|^2$ is also bounded. On the other hand, from the Gauss equation we have that the Ricci curvature tensor of M^n, denoted by Ric_M, is given by

$$Ric_M(X,Y) = -(n-1)\langle X,Y\rangle + nH\langle AX,Y\rangle - \langle AX,AY\rangle, \tag{3.9}$$

for all $X,Y \in \mathfrak{X}(M)$. Thus, using the Cauchy-Schwarz inequality, from (3.9) we have that

$$Ric_M(X,X) \geq \left(1-n-n|H||A|-|A|^2\right)|X|^2, \tag{3.10}$$

for all $X \in \mathfrak{X}(M)$. Since $|A|^2$ is bounded and H is constant, we conclude from (3.10) that Ric_M is bounded from below. So, we can apply Lemma 3.1 to pick out a sequence of points $(p_k)_{k\geq 1}$ in M^n such that

$$\lim_k |\Phi|^2(p_k) = \sup_M |\Phi|^2 \quad \text{and} \quad \limsup_k \Delta|\Phi|^2(p_k) \leq 0.$$

Since $\Delta|\Phi|^2 = n|\Phi|^2$, we get

$$0 \geq \limsup_k \Delta|\Phi|^2(p_k) = n\sup_M |\Phi|^2 \geq 0.$$

Hence, $\sup_M |\Phi|^2 = 0$ and, therefore, $|\Phi|^2 = 0$ on M^n, which means that M^n is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ which was to be proved. □

Before we present the proof of our second result, we will quote an extension of the classical Hopf maximum principle for an n-dimensional complete Riemannian manifold M^n due to Yau [16]. In what follows, $\mathcal{L}^1(M)$ stands for the space of Lebesgue integrable functions on M^n.

Lemma 3.2. *Let M^n be an n-dimensional complete Riemannian manifold and let $u : M^n \to \mathbb{R}$ be a smooth function. If u is a subharmonic (or superharmonic) function with $|\nabla u| \in \mathcal{L}^1(M)$, then u must actually be harmonic.*

Proof of Theorem 1.2. Let $x : M^n \to \mathbb{H}^{n+1}$ be a complete immersed hypersurface with constant mean curvature H. Initially, we observe that our hypothesis under the image of the Gauss mapping ν of M^n amounts to the fact that, for some vector $a \in \mathbb{L}^{n+2}$, the angle function $f_a = \langle \nu, a\rangle$ does not change sign on M^n.

Now a straightforward computation allows us to conclude that the Hessian of the height function $l_a = \langle x, a\rangle$ satisfies

$$|\nabla^2 l_a|^2 = |\Phi|^2 f_a^2 + \frac{1}{n}(\Delta l_a)^2, \tag{3.11}$$

where A is the Weingarten operator of M^n while Φ is its associated traceless operator.

Moreover, since the mean curvature of M^n is constant, we have from formulas (2.2) and (2.3) that

$$\Delta(f_a + Hl_a) = -|\Phi|^2 f_a.$$

Thus, $\Delta(f_a + Hl_a)$ does not change sign on M^n.

On the other hand, from (3.9) we have that the scalar curvature R of M^n satisfies

$$R = n(1-n) + n^2H^2 - |A|^2.$$

Since R is bounded from below, we get that $|A|^2$ is bounded on M^n. Consequently,

$$|\nabla(f_a + Hl_a)| = |-A(a^\top) + Ha^\top| \leq (|A| + |H|)|a^\top| \in \mathcal{L}^1(M).$$

Thus, from Lemma 3.2 we conclude that the function $f_a + Hl_a$ is harmonic and, hence, $|\Phi|^2 f_a = 0$ on M^n. On the other hand, since the Hilbert-Schmidt norm of $\nabla^2 l_a - \frac{1}{n}\Delta l_a g$ satisfies $|\nabla^2 l_a - \frac{1}{n}\Delta l_a g|^2 = |\nabla^2 l_a|^2 - \frac{1}{n}(\Delta l_a)^2$, we use (3.11) to conclude that

$$\nabla^2 l_a = \frac{1}{n}(\Delta\, l_a)g,$$

where g stands for the induced metric of M^n. Therefore, $\nabla l_a = a^\top$ is a conformal vector field on M^n and, since a cannot be a tangent vector to the hypersurface, we have from Lemma 2.1 that M^n is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$. This completes the proof of our theorem. □

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