ON SHARP CONSTANTS IN MARCINKIEWICZ-ZYGMUND AND PLANCHEREL-POLYA INEQUALITIES

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ABSTRACT. The Plancherel-Polya inequalities assert that for $1 < p < \infty$, and entire functions f of exponential type at most π ,

$$A_p \sum_{j=-\infty}^{\infty} |f(j)|^p \leq \int_{-\infty}^{\infty} |f|^p \leq B_p \sum_{j=-\infty}^{\infty} |f(j)|^p.$$

The Marcinkiewicz-Zygmund inequalities assert that for $n \geq 1$ and polynomials P of degree $\leq n-1$,

$$\frac{A'_p}{n} \sum_{j=1}^n \left| P(e^{2\pi i j/n}) \right|^p \leq \int_0^1 |P(e^{2\pi i t})|^p dt \leq \frac{B'_p}{n} \sum_{j=1}^n \left| P(e^{2\pi i j/n}) \right|^p.$$

We show that the sharp constants in both inequalities are the same; that is, $A_p = A'_p$ and $B_p = B'_p$. Moreover, the two inequalities are equivalent. We also discuss the case $p \leq 1$.

1. INTRODUCTION

The Plancherel-Polya inequalities [5, p. 152] assert that for $1 < p < \infty$ and entire functions f of exponential type at most π ,

$$(1.1) \quad A_p \sum_{j=-\infty}^{\infty} |f(j)|^p \leq \int_{-\infty}^{\infty} |f|^p \leq B_p \sum_{j=-\infty}^{\infty} |f(j)|^p,$$

provided either the series or integral is finite. For $0 < p \leq 1$, the left-hand inequality is still true, but the right-hand inequality requires additional restrictions [1], [3], [9]. Of course, A_p, B_p are independent of f . Moreover, a dilation of the variable yields an analogous inequality for f of any given finite type. These inequalities play an important role in sampling theory and applications of Paley-Wiener spaces [5].

The Marcinkiewicz-Zygmund inequalities assert [11, Vol. II, p. 30] that for $p > 1$, $n \geq 1$, and polynomials P of degree $\leq n-1$,

$$(1.2) \quad \frac{A'_p}{n} \sum_{j=1}^n \left| P(e^{2\pi i j/n}) \right|^p \leq \int_0^1 |P(e^{2\pi i t})|^p dt \leq \frac{B'_p}{n} \sum_{j=1}^n \left| P(e^{2\pi i j/n}) \right|^p.$$

Here too, A'_p and B'_p are independent of n and P , and the left-hand inequality is also true for $0 < p \leq 1$ [7]. These inequalities are useful in studying convergence

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of Fourier series, Lagrange interpolation, in number theory, and weighted approximation. They have been extended to many settings, and there are a great many methods to prove them [4], [7], [8].

To the best of this author's knowledge, the sharp constants in (1.1) and (1.2) are unknown, except for the case $p = 2$, where of course $A_2 = B_2 = A'_2 = B'_2 = 1$ [5, p. 150]. Throughout, we assume that A_p, B_p, A'_p, B'_p are the sharp constants, so that A_p and A'_p are as large as possible, while B_p and B'_p are as small as possible. The main result of this paper is:

Theorem 1.1. *For $0 < p < \infty$,*

$$(1.3) \quad A_p = A'_p$$

and for $1 < p < \infty$,

$$(1.4) \quad B_p = B'_p.$$

This theorem can be seen as a further example of the long-standing connection between asymptotics for polynomials and entire functions of exponential type [2], [10, Chapters 4, 5]. Indeed, the main idea in the proof is that scaling limits transform polynomials into entire functions of exponential type. We also prove a duality inequality. Let σ_p denote the norm of the Fourier partial sum projection in L_p , that is

$$(1.5) \quad \sigma_p = \sup \left\{ \int_0^1 \left| \sum_{j=0}^n c_j e^{2\pi i j t} \right|^p dt : n \geq 1, \{c_j\} \subset \mathbb{C} \right. \\ \left. \text{and } \int_0^1 \left| \sum_{j=-\infty}^{\infty} c_j e^{2\pi i j t} \right|^p dt \leq 1 \right\}.$$

Theorem 1.2. (a) *Let $1 < p < \infty$, $p \neq 2$, and $q = \frac{p}{p-1}$. Then*

$$(1.6) \quad \left(B_q^{1/q} \right)^{-1} \leq A_p^{1/p} \leq \left(B_q^{1/q} \right)^{-1} \sigma_p^{1/p}.$$

(b) *Moreover, for all $p > 1$ except $p = 2$,*

$$(1.7) \quad B_p > 1,$$

while for $0 < p < \infty$ except $p = 2$,

$$(1.8) \quad A_p < 1.$$

In [1, p. 101, Thm. 6.7.15], it is proven that $B_p \leq \frac{4}{\pi} e^{p\pi/2}$.

2. PROOFS

Throughout $C, C_1, C_2, \dots$ denote constants independent of n, m, x, t . We shall use the sinc kernel $S(z) = \frac{\sin \pi z}{\pi z}$, an entire function of exponential type π . We shall also use the fact that for $\lambda > 1$,

$$(2.1) \quad \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=-\infty}^{\infty} \left| S\left(\frac{k+x}{m}\right) \right|^\lambda = \int_{-\infty}^{\infty} |S(t)|^\lambda dt,$$

uniformly for $x \in [0, 1]$. This is easily established: if $r \geq 2$,

$$\begin{aligned} & \frac{1}{m} \sup_{x \in [0, 1]} \sum_{k: |k| \geq mr} \left| S\left(\frac{k+x}{m}\right) \right|^\lambda + \int_{|t| \geq r} |S(t)|^\lambda dt \\ & \leq C \left(\frac{1}{m} \sum_{k: |k| \geq mr} \left(\frac{m}{|k|} \right)^\lambda + \int_{|t| \geq r} \frac{1}{|t|^\lambda} dt \right) \leq Cr^{1-\lambda}, \end{aligned}$$

so that the tails of both the series and the integral are uniformly small for large r while S is uniformly continuous in $\mathbb{R}$, then the theory of Riemann sums yields uniformly for $x \in [0, 1]$,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k: |k| \leq mr} \left| S\left(\frac{k+x}{m}\right) \right|^\lambda = \int_{-r}^r |S(t)|^\lambda dt.$$

We start with:

Proof that for $p > 0$, $A_p \leq A'_p$. Let $n \geq 1$ and P be a polynomial of degree $\leq n-1$. Choose positive integers m, J such that $Jp > 1$ and $m > J$. Let

$$f(z) = e^{-i\pi(1-\frac{1}{n})z} P\left(e^{2\pi iz/n}\right) S\left(\frac{z}{mn}\right)^J.$$

It is easily seen that f is entire of exponential type $\leq \pi$. Moreover,

$$\begin{aligned} \sum_{j=-\infty}^{\infty} |f(j)|^p &= \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{n-1} |f(kn + \ell)|^p \\ &= \sum_{\ell=0}^{n-1} \left| P\left(e^{2\pi i \ell/n}\right) \right|^p \sum_{k=-\infty}^{\infty} \left| S\left(\frac{k + \ell/n}{m}\right) \right|^{Jp}, \end{aligned}$$

while

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^p dx &= n \int_{-\infty}^{\infty} |P(e^{2\pi i t})|^p \left| S\left(\frac{t}{m}\right) \right|^{Jp} dt \\ &= n \int_0^1 |P(e^{2\pi i s})|^p \sum_{k=-\infty}^{\infty} \left| S\left(\frac{k+s}{m}\right) \right|^{Jp} ds. \end{aligned}$$

We can then recast the left inequality in (1.1) for this function f as

$$\begin{aligned} & A_p \frac{1}{n} \sum_{\ell=0}^{n-1} \left| P\left(e^{2\pi i \ell/n}\right) \right|^p \left(\frac{1}{m} \sum_{k=-\infty}^{\infty} \left| S\left(\frac{k + \ell/n}{m}\right) \right|^{Jp} \right) \\ & \leq \int_0^1 |P(e^{2\pi i s})|^p \left(\frac{1}{m} \sum_{k=-\infty}^{\infty} \left| S\left(\frac{k+s}{m}\right) \right|^{Jp} \right) ds. \end{aligned}$$

We now let $m \rightarrow \infty$ and use the uniform convergence in (2.1) to obtain

$$A_p \frac{1}{n} \sum_{\ell=0}^{n-1} \left| P\left(e^{2\pi i \ell/n}\right) \right|^p \leq \int_0^1 |P(e^{2\pi i s})|^p ds.$$

Since this holds for any $n \geq 1$ and any polynomial P of degree $\leq n-1$, it follows that $A'_p \geq A_p$. $\square$

Proof that for $p > 1$, $B_p \geq B'_p$. Let P have degree $\leq n - 1$. We proceed as above, but use the right inequality in (1.1), to obtain

$$\begin{aligned} & \int_0^1 |P(e^{2\pi is})|^p \left(\frac{1}{m} \sum_{k=-\infty}^{\infty} \left| S\left(\frac{k+s}{m}\right) \right|^{Jp} \right) ds \\ & \leq B_p \frac{1}{n} \sum_{\ell=0}^{n-1} \left| P\left(e^{2\pi i\ell/n}\right) \right|^p \left(\frac{1}{m} \sum_{k=-\infty}^{\infty} \left| S\left(\frac{k+\ell/n}{m}\right) \right|^{Jp} \right). \end{aligned}$$

Letting $m \rightarrow \infty$ yields

$$\int_0^1 |P(e^{2\pi is})|^p ds \leq B_p \frac{1}{n} \sum_{\ell=0}^{n-1} \left| P\left(e^{2\pi i\ell/n}\right) \right|^p.$$

As above, it follows that $B_p \geq B'_p$. $\square$

The converse inequalities are more difficult:

Proof that for $p > 1$, $B_p \leq B'_p$ and hence $B_p = B'_p$. Let f be an entire function of exponential type $\leq \pi$, for which the integral in (1.1) is convergent. Then f admits the sampling series expansion [5, p. 152]

$$(2.2) \quad f(z) = \sum_{k=-\infty}^{\infty} f(k) S(z-k).$$

It converges uniformly in compact subsets of the plane. Moreover,

$$(2.3) \quad \lim_{L \rightarrow \infty} \int_{-\infty}^{\infty} \left| f(x) - \sum_{|k| \leq L} f(k) S(x-k) \right|^p dx = 0.$$

For $|j| \leq [n/2]$, let

$$(2.4) \quad \ell_{jn}(z) = \frac{1}{n} \frac{z^n - 1}{ze^{-2\pi ij/n} - 1}$$

denote the j th fundamental polynomial of Lagrange interpolation at the n th roots of unity. If n is even, we consider only $j = n/2$, not $j = -n/2$. Let us fix $L \geq 1$, and for $n > 2L$, define

$$P_n(z) = \sum_{|j| \leq L} f(j) (-1)^j \ell_{jn}(z),$$

a polynomial of degree $\leq n - 1$. Then

$$(2.5) \quad \sum_{j=1}^n \left| P_n\left(e^{2\pi ij/n}\right) \right|^p = \sum_{|j| \leq L} |f(j)|^p.$$

A straightforward calculation shows that for fixed j ,

$$\lim_{n \rightarrow \infty} \ell_{jn}\left(e^{2\pi it/n}\right) = e^{i\pi t} (-1)^j S(t-j),$$

uniformly for t in compact sets. Moreover, for $\frac{1}{3} \geq |s| \geq 2|j|/n$,

$$(2.6) \quad \left| \ell_{jn}\left(e^{2\pi is}\right) \right| \leq \frac{1}{n |\sin \pi(s-j/n)|} \leq \frac{1}{2n |s-j/n|} \leq \frac{1}{n |s|}.$$

Thus, uniformly for t in compact sets,

$$\lim_{n \rightarrow \infty} P_n \left(e^{2\pi i t/n} \right) = e^{i\pi t} \sum_{|k| \leq L} f(k) S(t-k),$$

while for $|s| \geq 2L/n$,

$$|P_n(e^{2\pi i s})| \leq \frac{1}{n|s|} \sum_{|k| \leq L} |f(k)| =: \frac{C_L}{n|s|}.$$

Then, given $r > 2L$,

$$\begin{aligned} n \int_0^1 |P_n(e^{2\pi i s})|^p ds &= n \int_{-1/2}^{1/2} |P_n(e^{2\pi i s})|^p ds \\ &= \int_{-r}^r |P_n(e^{2\pi i t/n})|^p dt + O\left(C_L^p n \int_{|s| \geq r/n} \frac{ds}{(n|s|)^p}\right) \\ &= \int_{-r}^r \left| \sum_{|k| \leq L} f(k) S(t-k) \right|^p dt + O(C_L^p r^{1-p}). \end{aligned}$$

Using this and (2.5), we may now recast the right-hand inequality in (1.2) as

$$\int_{-r}^r \left| \sum_{|k| \leq L} f(k) S(t-k) \right|^p dt + O(C_L^p r^{1-p}) \leq B'_p \sum_{|j| \leq L} |f(j)|^p.$$

Letting $r \rightarrow \infty$ gives

$$\int_{-\infty}^{\infty} \left| \sum_{|k| \leq L} f(k) S(t-k) \right|^p dt \leq B'_p \sum_{|j| \leq L} |f(j)|^p.$$

We may now let $L \rightarrow \infty$, and use (2.3) to deduce

$$\int_{-\infty}^{\infty} |f(t)|^p dt \leq B'_p \sum_{j=-\infty}^{\infty} |f(j)|^p.$$

As f was an arbitrary entire function of exponential type $\leq \pi$, we deduce that $B_p \leq B'_p$. Together with the previous proof, this shows that $B_p = B'_p$. $\square$

Proof that for $p > 1$, $A_p \geq A'_p$ and hence $A_p = A'_p$. Here, we proceed as above, but use the left-hand inequality in (1.2), leading to

$$A'_p \sum_{|j| \leq L} |f(j)|^p \leq \int_{-r}^r \left| \sum_{|k| \leq L} f(k) S(t-k) \right|^p dt + O(C_L^p r^{1-p}).$$

Letting $r \rightarrow \infty$ gives for $M \leq L$

$$A'_p \sum_{|j| \leq M} |f(j)|^p \leq \int_{-\infty}^{\infty} \left| \sum_{|k| \leq L} f(k) S(t-k) \right|^p dt.$$

We now let $L \rightarrow \infty$ in the right-hand side, and use (2.3), and then finally let $M \rightarrow \infty$ in the left-hand side, giving

$$A'_p \sum_{j=-\infty}^{\infty} |f(j)|^p \leq \int_{-\infty}^{\infty} |f(t)|^p dt.$$

As this holds for any such f , we obtain $A_p \geq A'_p$. The converse inequality $A_p \leq A'_p$ was proved above. $\square$

The case $p \leq 1$ is more difficult:

Proof that for $0 < p \leq 1$, $A_p \leq A'_p$ and hence $A_p = A'_p$. Let f be entire of exponential type $\leq \pi$, with the integral in (1.1) finite. We note that the left-hand inequality in (1.1) (which is valid even for $p \leq 1$), shows that $\sup_j |f(j)| < \infty$, and hence

$$(2.7) \quad \sum_{j=-\infty}^{\infty} |f(j)|^\lambda < \infty \text{ for all } \lambda > p.$$

In particular, f satisfies (1.1) with $p = 2$, and consequently, we still have the sampling series expansion (2.2). Choose a positive integer J such that $Jp \geq 2$, and let $\varepsilon \in (0, 1/2)$. Let $L > 1$,

$$U_k(z) = \frac{1}{k} \frac{z^k - 1}{z - 1},$$

and with $[x]$ denoting the greatest integer $\leq x$, let

$$P_n(z) = \left(\sum_{|j| \leq L} f(j) (-1)^j \ell_{j, n-[\varepsilon n]}(z) \right) U_{[\frac{\varepsilon}{J}n]}(z)^J,$$

a polynomial of degree $\leq n-1$. A straightforward calculation shows for as $n \rightarrow \infty$,

$$\ell_{j, n-[\varepsilon n]}(e^{2\pi i t/n}) = (-1)^j e^{i\pi t(1-\varepsilon)} S((1-\varepsilon)t - j) + o(1),$$

uniformly for t in compact sets and any fixed j (cf. [6, Lemma 2.2]). In a similar way, uniformly for t in compact sets,

$$U_{[\frac{\varepsilon}{J}n]}(e^{2\pi i t/n}) = e^{i\pi \frac{\varepsilon t}{J}} S\left(\frac{\varepsilon t}{J}\right) + o(1),$$

so uniformly for such t , as $n \rightarrow \infty$,

$$\begin{aligned} \left| P_n(e^{2\pi i t/n}) \right| &= \left| e^{i\pi t} \left(\sum_{|j| \leq L} f(j) S((1-\varepsilon)t - j) \right) S\left(\frac{\varepsilon t}{J}\right)^J + o(1) \right| \\ &\leq \left| \sum_{|j| \leq L} f(j) S((1-\varepsilon)t - j) \right| \left| S\left(\frac{\varepsilon t}{J}\right)^J \right| + o(1) \\ &\leq |f(t(1-\varepsilon))| + O\left(\sum_{|j| > L} |f(j)| \right) \min\left\{1, \frac{1}{|t|}\right\}^J + o(1), \end{aligned}$$

recall that $|S(s)| \leq \min \left\{ 1, \frac{1}{\pi|s|} \right\}$, and (2.7). Then for fixed $r > 4L$, we obtain, as $n \rightarrow \infty$,

$$n \int_{-r/n}^{r/n} |P_n(e^{2\pi i s})|^p ds \leq \frac{1}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p dt + C \left(\sum_{|j|>L} |f(j)| \right)^p + o(1).$$

Here, as $Jp > 1$, C is independent of r, n, L but does depend on ε and J . Next, for $\frac{1}{3} \geq |s| \geq r/n$, (2.6) gives

$$|P_n(e^{2\pi i s})| \leq \left(\frac{C}{n|s|} \sum_{|j| \leq L} |f(j)| \right) \left(\frac{1}{\frac{\varepsilon}{J} n|s|} \right)^J \leq \frac{C}{r} \left(\frac{1}{n|s|} \right)^J$$

where C is independent of L, s, r, n . Then

$$n \int_{r/n \leq |s| \leq 1/2} |P_n(e^{2\pi i s})|^p ds \leq \frac{Cn}{r^p} \int_{r/n \leq |s| \leq 1/2} \left(\frac{1}{n|s|} \right)^{Jp} ds \leq Cr^{-p+1-Jp}.$$

Thus, as $n \rightarrow \infty$,

$$\begin{aligned} n \int_{-1/2}^{1/2} |P_n(e^{2\pi i s})|^p ds \\ \leq \frac{1}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p dt + C \left(\sum_{|j| \geq L} |f(j)| \right)^p + Cr^{-p+1-Jp} + o(1). \end{aligned}$$

Here C is independent of r, n, L . Next, for each fixed k , as $n \rightarrow \infty$,

$$\left| P_n(e^{2\pi i k/n}) \right| = \left| \sum_{|j| \leq L} f(j) S((1-\varepsilon)k-j) \right| \left| S\left(\frac{\varepsilon k}{J}\right) \right|^J + o(1).$$

The left-hand inequality in (1.2) gives, for any fixed $L \geq M \geq 1$, as $n \rightarrow \infty$,

$$\begin{aligned} A'_p \sum_{|k| \leq M} \left| \sum_{|j| \leq L} f(j) S((1-\varepsilon)k-j) \right|^p \left| S\left(\frac{\varepsilon k}{J}\right) \right|^{Jp} \\ \leq \frac{1}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p dt + C \left(\sum_{|j| \geq L} |f(j)| \right)^p + Cr^{-p+1-Jp}. \end{aligned}$$

Here C is independent of L, r, n but depends on ε, J . We now let $r \rightarrow \infty$ (with ε still fixed), obtaining for $M \leq L$,

$$\begin{aligned} A'_p \sum_{|k| \leq M} \left| \sum_{|j| \leq L} f(j) S((1-\varepsilon)k-j) \right|^p \left| S\left(\frac{\varepsilon k}{J}\right) \right|^{Jp} \\ \leq \frac{1}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p dt + C \left(\sum_{|j| \geq L} |f(j)| \right)^p. \end{aligned}$$

Next, let $L \rightarrow \infty$, which is permissible as $\sum_j |f(j)| < \infty$. We obtain

$$A'_p \sum_{|k| \leq M} |f((1-\varepsilon)k)|^p \left| S\left(\frac{\varepsilon k}{J}\right) \right|^{Jp} \leq \frac{1}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p dt.$$

We now let $\varepsilon \rightarrow 0+$, and use the continuity of S , to obtain

$$A'_p \sum_{|k| \leq M} |f(k)|^p \leq \int_{-\infty}^{\infty} |f(t)|^p dt.$$

Letting $M \rightarrow \infty$ gives

$$A'_p \sum_{k=-\infty}^{\infty} |f(k)|^p \leq \int_{-\infty}^{\infty} |f(t)|^p dt.$$

As f is an arbitrary entire function of exponential type $\leq \pi$, we obtain $A_p \geq A'_p$ and hence $A_p = A'_p$ from earlier results. $\square$

Proof of Theorem 1.1. This has been proved above. $\square$

Proof of Theorem 1.2. (a) We use a duality argument: if P is a polynomial of degree $< n$, then

$$\left(\frac{1}{n} \sum_{j=1}^n |P(e^{2\pi i j/n})|^p \right)^{1/p} = \frac{1}{n} \left| \sum_{j=1}^n \overline{c_j} P(e^{2\pi i j/n}) \right|$$

for some $\{c_j\}$ with $\frac{1}{n} \sum_{j=1}^n |c_j|^q = 1$. Equivalently, if R is a polynomial of degree $\leq n-1$ with $R(e^{2\pi i j/n}) = c_j$, then for all j ,

$$\begin{aligned} \left(\frac{1}{n} \sum_{j=1}^n |P(e^{2\pi i j/n})|^p \right)^{1/p} &= \frac{1}{n} \left| \sum_{j=1}^n \overline{R(e^{2\pi i j/n})} P(e^{2\pi i j/n}) \right| \\ &= \left| \int_0^1 (\overline{R}P)(e^{2\pi i t}) dt \right|, \end{aligned}$$

by the simple sum

$$(2.8) \quad \frac{1}{n} \sum_{j=1}^n (e^{2\pi i j/n})^k = \int_0^1 e^{2\pi i k t} dt, \quad |k| < n.$$

We use Hölder's inequality to continue this as

$$\begin{aligned} \left(\frac{1}{n} \sum_{j=1}^n |P(e^{2\pi i j/n})|^p \right)^{1/p} &\leq \left(\int_0^1 |R(e^{2\pi i t})|^q dt \right)^{1/q} \left(\int_0^1 |P(e^{2\pi i t})|^p dt \right)^{1/p} \\ &\leq \left(B_q \frac{1}{n} \sum_{j=1}^n |R(e^{2\pi i j/n})|^q \right)^{1/q} \left(\int_0^1 |P(e^{2\pi i t})|^p dt \right)^{1/p}, \end{aligned}$$

by (1.2). As the sum involving $|R|$ is at most 1, we deduce that

$$B_q^{1-p/q} \frac{1}{n} \sum_{j=1}^n |P(e^{2\pi i j/n})|^p \leq \int_0^1 |P(e^{2\pi i t})|^p dt.$$

Thus $B_q'^{-p/q} \leq A_p'$, or

$$(2.9) \quad B_q'^{-1/q} \leq A_p'^{1/p}.$$

Similarly, for some measurable function g , with $\int_0^1 |g(e^{2\pi it})|^q dt \leq 1$,

$$\left(\int_0^1 |P(e^{2\pi it})|^p dt \right)^{1/p} = \int_0^1 (P\bar{g})(e^{2\pi it}) dt = \int_0^1 (P\bar{R})(e^{2\pi it}) dt,$$

where if g has Fourier series $\sum_{j=-\infty}^{\infty} c_j e^{ij t}$, $R(t) = \sum_{j=0}^{n-1} c_j e^{ij t}$, and we are using orthogonality. We now again use quadrature on the unit circle, followed by Hölder's inequality, to continue this as

$$\begin{aligned} &= \frac{1}{n} \left| \sum_{j=1}^n \overline{R(e^{2\pi ij/n})} P(e^{2\pi ij/n}) \right| \\ &\leq \left(\frac{1}{n} \sum_{j=1}^n |R(e^{2\pi ij/n})|^q \right)^{1/q} \left(\frac{1}{n} \sum_{j=1}^n |P(e^{2\pi ij/n})|^p \right)^{1/p} \\ &\leq \left(A_q'^{-1} \int_0^1 |R(e^{2\pi it})|^q dt \right)^{1/q} \left(\frac{1}{n} \sum_{j=1}^n |P(e^{2\pi ij/n})|^p \right)^{1/p} \\ &\leq \left(A_q'^{-1} \sigma_q \int_0^1 |g(e^{2\pi it})|^q dt \right)^{1/q} \left(\frac{1}{n} \sum_{j=1}^n |P(e^{2\pi ij/n})|^p \right)^{1/p}. \end{aligned}$$

In summary,

$$\int_0^1 |P(e^{2\pi it})|^p dt \leq (A_q'^{-1} \sigma_q)^{p/q} \frac{1}{n} \sum_{j=1}^n |P(e^{2\pi ij/n})|^p.$$

Thus $B_p' \leq (A_q'^{-1} \sigma_q)^{p/q}$, or swapping the roles of p, q , $B_q' \leq (A_p'^{-1} \sigma_p)^{q/p}$. Hence,

$$\left(B_q'^{1/q} \right)^{-1} \sigma_p^{1/p} \geq (A_p')^{1/p}.$$

This, (2.9), and Theorem 1.1 give (1.6).

(b) Fix n and let ℓ_{0n} be as in (2.4). For $s \in [0, 1]$ and $p > 0$, let

$$\chi_p(s) = \int_0^1 |\ell_{0n}(e^{2\pi i(t-s)})|^p dt - \frac{1}{n} \sum_{j=0}^{n-1} |\ell_{0n}(e^{2\pi i(j/n-s)})|^p$$

denote the quadrature error for the polynomial $\ell_{0n}(ze^{-2\pi is})$. Observe that

$$\chi_p(0) = \int_0^1 |\ell_{0n}(e^{2\pi it})|^p dt - \frac{1}{n}$$

and, in particular, $\chi_2(0) = 0$ by (2.8). Inasmuch as $|\ell_{0n}(z)| < 1$ unless $z = 1$, this shows that

$$\chi_p(0) \begin{cases} > 0, & 0 < p < 2 \\ < 0, & p > 2. \end{cases}$$

Thus for $p \neq 2$, $\chi_p(s)$ is a continuous, not identically vanishing function of s . Also by periodicity,

$$\int_0^1 \chi_p(s) ds = 0.$$

It then follows that for some values of s , $\chi_p(s) > 0$, and for others, $\chi_p(s) < 0$. The former inequality shows that $B_p > 1$, and the latter $A_p < 1$. $\square$

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