ON NON-PURE FORMS ON ALMOST COMPLEX MANIFOLDS

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ABSTRACT. In 2009 T.-J. Li and W. Zhang defined an almost complex structure J on a manifold X to be $\mathcal{C}^\infty$ -*pure-and-full* if the second de Rham cohomology group can be decomposed as a direct sum of the subgroups whose elements are cohomology classes admitting J -invariant and J -anti-invariant representatives. It turns out (see T. Draghici, T.-J. Li and W. Zhang (2010)) that any almost complex structure on a 4-dimensional compact manifold is $\mathcal{C}^\infty$ -pure-and-full. We study the J -invariant and J -anti-invariant cohomology subgroups on almost complex manifolds, possibly non-compact. In particular, we prove an analytic continuation result for anti-invariant forms on almost complex manifolds.

INTRODUCTION

Let (X, J, ω) be a compact Kähler manifold of complex dimension n . Then the celebrated Hodge decomposition theorem states that the (p, q) -decomposition on forms transfers at the cohomology level, namely, $H^k(X; \mathbb{C}) \cong \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(X)$, where $H_{\bar{\partial}}^{p,q}(X)$ denotes the (p, q) -Dolbeault group. In order to generalize the previous decomposition to arbitrary compact almost complex manifolds, in [16] T.-J. Li and W. Zhang introduced the notion of $\mathcal{C}^\infty$ -pure-and-full almost complex structures. An almost complex structure J on a $2n$ -dimensional manifold X is said to be $\mathcal{C}^\infty$ -*pure-and-full* if $H^2(X; \mathbb{R}) = H_J^{(1,1)}(X)_{\mathbb{R}} \oplus H_J^{(2,0),(0,2)}(X)_{\mathbb{R}}$, where $H_J^{(1,1)}(X)_{\mathbb{R}}$ and $H_J^{(2,0),(0,2)}(X)_{\mathbb{R}}$ are the subgroups of the 2-nd de Rham cohomology group of X , whose elements are cohomology classes represented by real pure forms of types $(1, 1)$ and $(2, 0) + (0, 2)$, respectively, namely, the J -invariant, J -anti-invariant cohomology subgroups. More generally, J is said to be $\mathcal{C}^\infty$ -*pure* if $H_J^{(1,1)}(X)_{\mathbb{R}} \cap H_J^{(2,0),(0,2)}(X)_{\mathbb{R}} = \{0\}$ and $\mathcal{C}^\infty$ -*full* if $H^2(X; \mathbb{R}) = H_J^{(1,1)}(X)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(X)_{\mathbb{R}}$. Similar notions of $\mathcal{C}^\infty$ -pure and $\mathcal{C}^\infty$ -full almost complex structures at stage k can be defined. By taking the complex of (p, q) -currents one can define *pure* and *full* almost complex structures.

In [9], T. Drăghici, T.-J. Li and W. Zhang showed that an almost complex structure J on a 4-dimensional compact manifold X is $\mathcal{C}^\infty$ -pure-and-full, namely

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$H^2(X; \mathbb{R}) = H_J^{(1,1)}(X)_{\mathbb{R}} \oplus H_J^{(2,0),(0,2)}(X)_{\mathbb{R}}$. The last decomposition can be considered as a Hodge decomposition for 4-dimensional almost complex compact manifolds. Furthermore, by setting

$$\mathcal{K}_J^t = \{[\omega] \in H^2(X; \mathbb{R}) \mid \omega \text{ is symplectic and } J \text{ is } \omega\text{-tamed}\}$$

as the *tamed symplectic cone* and

$$\mathcal{K}_J^c = \{[\omega] \in H^2(X; \mathbb{R}) \mid \omega \text{ is symplectic and } J \text{ is } \omega\text{-compatible}\}$$

as the *compatible symplectic cone*, if J is a complex structure on a compact manifold M which admits a Kähler metric, then

$$\mathcal{K}_J^t = \mathcal{K}_J^c + \left(\left(H_{\bar{\partial}}^{2,0}(X) \oplus H_{\bar{\partial}}^{0,2}(X) \right) \cap H^2(X; \mathbb{R}) \right)$$

(see [16]). For a non-integrable almost complex structure J , in [16] it is proved that if J is a $\mathcal{C}^\infty$ -full almost complex structure which carries an almost Kähler metric, then

$$\mathcal{K}_J^t = \mathcal{K}_J^c + H_J^{(2,0),(0,2)}(X)_{\mathbb{R}}.$$

Hence the subgroup $H_J^{(2,0),(0,2)}(X)_{\mathbb{R}} \subset H^2(M; \mathbb{R})$ can be thought of as a generalization of the real Dolbeault group $\left(H_{\bar{\partial}}^{2,0}(X) \oplus H_{\bar{\partial}}^{0,2}(X) \right) \cap H^2(X; \mathbb{R})$ to the non-integrable case.

For other results on $\mathcal{C}^\infty$ -pure-and-full almost complex structures, see e.g. [10], [1], [3].

In dimension higher than four, there are examples of non- $\mathcal{C}^\infty$ -pure and non- $\mathcal{C}^\infty$ -full almost complex structures on compact manifolds. Nevertheless, in [9] and [12] it is proved that if (X, ω) is a compact symplectic manifold of dimension $2n$, then every ω -compatible almost complex structure J is $\mathcal{C}^\infty$ -pure.

In this paper we are interested in studying the J -invariant and J -anti-invariant respectively cohomology subgroups on almost complex manifolds X^{2n} , possibly non-compact. In particular we will search for pairs (ω, J) where ω is either a J -invariant or J -anti-invariant closed 2-form. We are also interested in determining whether ω can also be taken to be symplectic. We note that symplectic forms always have compatible almost complex structures, which are invariant in our language. Therefore in the symplectic case we will focus on finding anti-invariant almost complex structures.

After some preliminaries in section 1, in section 2 we start with the observation that if n is odd and (ω, J) is an anti-invariant pair, then ω^n is identically zero; see Proposition 2.1. In particular ω cannot be symplectic. However it is not true that lower powers of ω must vanish, even in cohomology. In Example 2.2 we provide an example of a 2-cohomology class on a 6-dimensional compact nilmanifold N endowed with a left-invariant almost complex structure J which belongs to $H_J^{(1,1)}(N)_{\mathbb{R}} \cap H_J^{(2,0),(0,2)}(N)_{\mathbb{R}}$ and whose square is non-zero.

In searching for anti-invariant forms with ω symplectic, we must assume that n is even. We consider the case of open manifolds. Theorem 3.1 says that on an open symplectic 4 manifold (X^4, ω) there exists an almost complex structure J such that ω is J -anti-invariant if and only if the tangent bundle is symplectically trivial. However Proposition 3.3 shows that this is no longer true in higher dimensions. If we do not fix the symplectic form, Theorem 3.5 says that given any open manifold X^{2n} with n even and TX trivial, and any cohomology class $\mathfrak{a} \in H^2(X, \mathbb{R})$, there exists an anti-invariant pair (ω, J) with $[\omega] = \mathfrak{a}$.

In the remainder of the paper we are motivated by the problem of finding anti-invariant pairs (ω, J) in a given cohomology class without the restrictive symplectic assumption. As in Proposition 2.1 it is not hard to show, for example, that on a 6-manifold such an ω must everywhere have rank 0 or 4. There are no other pointwise restrictions. Still, this seems restrictive, but we show in section 5 that for many codimension 2 submanifolds W we can find closed forms ω supported in arbitrarily small tubular neighborhoods such that ω has everywhere rank 0 or 4 and $[\omega]$ is Poincaré dual to W . Given this, it is natural to ask whether anti-invariant pairs can be constructed using cut and paste methods.

Section 4 shows that this is impossible. Indeed, despite the rank condition, the compactly supported forms constructed in section 5 have no anti-invariant almost complex structures. This is a consequence of the unique continuation theorem; see Theorem 4.1. This says that given an anti-invariant pair (ω, J) , if ω vanishes on an open subset, then it must vanish everywhere on a connected X^{2n} .

1. PRELIMINARIES

1.1. $\mathcal{C}^\infty$ -pure-and-full almost complex structures. Let X be a $2n$ -dimensional manifold (without boundary) and J be a smooth almost complex structure on X . Let $(T^{1,0}X)^*$ and $(T^{0,1}X)^*$ be the bundle of complex 1-forms on X of type $(1,0)$ and $(0,1)$, respectively. Denote by $\Lambda_{\mathbb{C}}^k(X)$ the bundle of $\mathcal{C}^\infty$ complex k -forms on X and by the same symbol $\Lambda_{\mathbb{C}}^k(X) = \Gamma(M, \Lambda_{\mathbb{C}}^k(X))$ the space of sections of $\Lambda_{\mathbb{C}}^k(X)$. Then, as usual, setting $\Lambda_J^{p,q}(X) = \Lambda^p((T^{1,0}X)^*) \wedge \Lambda^q((T^{0,1}X)^*)$, the space of complex k -forms decomposes as

$$\Lambda_{\mathbb{C}}^k(X) = \bigoplus_{p+q=k} \Lambda_J^{p,q}(X),$$

where $\Lambda_J^{p,q}(X) = \Gamma(M, \Lambda_J^{p,q}(X))$.

According to the above decomposition, the space $\Lambda_{\mathbb{R}}^k(X)$ of real smooth differential k -forms has a type decomposition:

$$\Lambda_{\mathbb{R}}^k(X) = \bigoplus_{p+q=k} \Lambda_J^{(p,q),(q,p)}(X)_{\mathbb{R}},$$

where

$$\Lambda_J^{(p,q),(q,p)}(X)_{\mathbb{R}} = \{\alpha \in \Lambda_J^{p,q}(X) \oplus \Lambda_J^{q,p}(X) \mid \alpha = \bar{\alpha}\}.$$

In particular, since J acts as the identity on $\Lambda_J^{(1,1)}(X)_{\mathbb{R}}$ and as minus the identity on $\Lambda_J^{(2,0),(0,2)}(X)_{\mathbb{R}}$, sometimes we will set

$$\Lambda_J^+ = \Lambda_J^{(1,1)}(X)_{\mathbb{R}}, \quad \Lambda_J^- = \Lambda_J^{(2,0),(0,2)}(X)_{\mathbb{R}},$$

and we will refer to them as the spaces of J -invariant and J -anti-invariant forms, respectively.

For a finite set S of pairs of integers, let

$$\mathcal{Z}_J^S = \bigoplus_{(p,q) \in S} \mathcal{Z}_J^{(p,q),(q,p)}, \quad \mathcal{B}_J^S = \bigoplus_{(p,q) \in S} \mathcal{B}_J^{(p,q),(q,p)},$$

where

$$\mathcal{Z}_J^{(p,q),(q,p)} = \{\alpha \in \Lambda_J^{(p,q),(q,p)}(X)_{\mathbb{R}} \mid d\alpha = 0\}$$

and

$$\mathcal{B}_J^{(p,q),(q,p)} = \{\beta \in \Lambda_J^{(p,q),(q,p)}(X)_{\mathbb{R}} \mid \text{there exists } \gamma \text{ such that } \beta = d\gamma\}.$$

Denoting by $\mathcal{B}$ the space of d -exact forms, we have that

$$\frac{\mathcal{Z}_J^S}{\mathcal{B}_J^S} = \frac{\mathcal{Z}_J^S}{\mathcal{B} \cap \mathcal{B}_J^S} = \frac{\mathcal{Z}_J^S}{\mathcal{B}}.$$

Hence, there is a natural inclusion

$$\rho_S : \mathcal{Z}_J^S / \mathcal{B}_J^S \rightarrow \mathcal{Z} / \mathcal{B}.$$

As in [16] we will write $\rho_S(\mathcal{Z}_J^S / \mathcal{B}_J^S)$ simply as $\mathcal{Z}_J^S / \mathcal{B}_J^S$ and we may define the cohomology subgroups

$$H_J^S(X)_{\mathbb{R}} = \{[\alpha] \mid \alpha \in \mathcal{Z}_J^S\} = \frac{\mathcal{Z}_J^S}{\mathcal{B}}.$$

In particular, according to the above definition, there is a natural inclusion

$$H_J^{(1,1)}(X)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(X)_{\mathbb{R}} \subseteq H^2(X, \mathbb{R}).$$

We recall the following (see [16, Definitions 2.2 and 2.3, Lemma 2.2])

Definition 1.1. A smooth almost complex structure J on X is said to be

- $\mathcal{C}^\infty$ -pure if

$$H_J^{(1,1)}(X)_{\mathbb{R}} \cap H_J^{(2,0),(0,2)}(X)_{\mathbb{R}} = \{0\};$$

- $\mathcal{C}^\infty$ -full if

$$H^2(X; \mathbb{R}) = H_J^{(1,1)}(X)_{\mathbb{R}} + H_J^{(2,0),(0,2)}(X)_{\mathbb{R}};$$

- $\mathcal{C}^\infty$ -pure-and-full if

$$H^2(X; \mathbb{R}) = H_J^{(1,1)}(X)_{\mathbb{R}} \oplus H_J^{(2,0),(0,2)}(X)_{\mathbb{R}}.$$

1.2. Pure-and-full almost complex structures. Let (X, J) be a compact $2n$ -dimensional almost complex manifold. Denote by $\mathcal{D}_k(X)$ the space of *currents* of dimension k (or degree $(2n - k)$), i.e., the topological dual of the space $\Lambda^k(X)$ of r -forms on X (see e.g. [8], [7]). The exterior differential d on $\Lambda^\bullet(X)$ induces a differential on $\mathcal{D}_\bullet(X)$, still denoted by d . Then the de Rham homology $H_\bullet(X; \mathbb{R})$ is the cohomology of the differential complex $(\mathcal{D}_\bullet(X), d)$ and $H_{dR}^k(X; \mathbb{R}) \simeq H_{2n-k}(X; \mathbb{R})$. We will denote by $\mathcal{P} : H_{2n-k}(X; \mathbb{R}) \rightarrow H_{dR}^k(X; \mathbb{R})$ this isomorphism.

As in the case of complex forms, the almost complex structure J induces a bi-grading on the space $\mathcal{D}_k^{\mathbb{C}}(X)$ of complex currents of dimension k . Hence, the spaces of currents $\mathcal{D}_{p,q}(X)$ of bidimension (p, q) are defined.

Accordingly, let $H_{(2,0),(0,2)}^J(X)_{\mathbb{R}}$ (respectively, $H_{(1,1)}^J(X)_{\mathbb{R}}$) be the subspace of $H_2(X; \mathbb{R})$ given by the homology classes represented by a real current of bidimension $(2, 0) + (0, 2)$ (respectively, $(1, 1)$). Then we have the following definition due to T.-J. Li and W. Zhang.

Definition 1.2 ([16, Definition 2.5, Lemma 2.7]). An almost complex structure J on X is said to be:

- *pure* if

$$H_{(2,0),(0,2)}^J(X)_{\mathbb{R}} \cap H_{(1,1)}^J(X)_{\mathbb{R}} = \{0\};$$

- *full* if

$$H_{(2,0),(0,2)}^J(X)_{\mathbb{R}} + H_{(1,1)}^J(X)_{\mathbb{R}} = H_2(X; \mathbb{R});$$

- *pure-and-full* if it is both pure and full, i.e. if the following decomposition holds:

$$H_{(2,0),(0,2)}^J(X)_{\mathbb{R}} \oplus H_{(1,1)}^J(X)_{\mathbb{R}} = H_2(X; \mathbb{R}).$$

The notions of $\mathcal{C}^\infty$ -pure-and-full and pure-and-full almost complex structures can be given in a similar way for k -forms and k -currents, respectively.

The relations between being $\mathcal{C}^\infty$ -pure-and-full and being pure-and-full are summarized in the following.

Proposition 1.3 (see [16, Proposition 2.5], [1, Theorem 2.1]). *The following relations between $\mathcal{C}^\infty$ -pure-and-full and pure-and-full concepts hold:*

$$\begin{array}{ccc} \mathcal{C}^\infty\text{-full at the } k\text{-th stage} & \implies & \text{pure at the } k\text{-th stage} \\ \Downarrow & & \Downarrow \\ \text{full at the } (2n-k)\text{-th stage} & \implies & \mathcal{C}^\infty\text{-pure at the } (2n-k)\text{-th stage.} \end{array}$$

In the sequel we will use the following notation:

$$\begin{aligned} \mathcal{Z}_J^- &= \mathcal{Z}_J^{(2,0),(0,2)}, \\ \mathcal{Z}_J^+ &= \mathcal{Z}_J^{(1,1)}, \\ \mathcal{B}_J^- &= \mathcal{B}_J^{(2,0),(0,2)}, \\ \mathcal{B}_J^+ &= \mathcal{B}_J^{(1,1)}, \\ H_J^-(X) &= H_J^{(2,0),(0,2)}(X)_{\mathbb{R}}, \\ H_J^+(X) &= H_J^{(1,1)}(X)_{\mathbb{R}}, \\ H_-^J(X) &= H_{(2,0),(0,2)}^J(X)_{\mathbb{R}}, \\ H_+^J(X) &= H_{(1,1)}^J(X)_{\mathbb{R}}. \end{aligned}$$

Remark 1.4. Let $SU(2) \times \mathbb{T}^3$ be the product of the special unitary group with the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Let $\{e^1, \dots, e^6\}$ be the global parallelism on $SU(2) \times \mathbb{T}^3$ satisfying the Maurer-Cartan equations

$$\begin{cases} de^1 = e^2 \wedge e^3, \\ de^2 = -e^1 \wedge e^3, \\ de^3 = e^1 \wedge e^2, \\ de^4 = de^5 = de^6 = 0. \end{cases}$$

Define the almost complex structure J on $SU(2) \times \mathbb{T}^3$ by giving the following complex $(1, 0)$ -forms:

$$\varphi^1 = e^1 + ie^4, \quad \varphi^2 = e^2 + ie^5, \quad \varphi^3 = e^3 + ie^6;$$

then (see [2, example 5.4]) we have that J is a $\mathcal{C}^\infty$ -full non- $\mathcal{C}^\infty$ -pure almost complex structure. Hence, according to Proposition 1.3, J is a pure non-full almost complex structure on $SU(2) \times \mathbb{T}^3$.

2. COMPACT MANIFOLDS

2.1. J -Invariant cohomology classes. For general 2-cohomology classes on arbitrary manifolds, we can show

Proposition 2.1. *Let X be a $2n$ -dimensional manifold. Let J be an almost complex structure on X and $[\alpha] \in H^2(X; \mathbb{R})$ such that $[\alpha]^n \neq 0$. Then*

- i) $[\alpha] \notin H_J^-(X)_{\mathbb{R}} \cap H_J^+(X)_{\mathbb{R}}$.
- ii) *Moreover, if n is odd, then $[\alpha] \notin H_J^-(X)$.*

Proof. i) By contradiction. Let $[\alpha] \in H_J^-(X) \cap H_J^+(X)$; then $[\alpha] = [\alpha_1] = [\alpha_2]$, where α_1 is J -anti-invariant and α_2 is J -invariant. Then $\alpha_1 \wedge \alpha_2^{n-1}$ is a J -anti-invariant $2n$ -form which does not vanish identically. Therefore, $J(\alpha_1 \wedge \alpha_2^{n-1}) = -\alpha_1 \wedge \alpha_2^{n-1}$. Take $\{e_1, \dots, e_n, Je_1, \dots, Je_n\}$ a basis of $T_p X$. Then

$$\begin{aligned} J(\alpha_1 \wedge \alpha_2^{n-1})(e_1, \dots, e_n, Je_1, \dots, Je_n) &= \alpha_1 \wedge \alpha_2^{n-1}(Je_1, \dots, Je_n, -e_1, \dots, -e_n) \\ &= \alpha_1 \wedge \alpha_2^{n-1}(e_1, \dots, e_n, Je_1, \dots, Je_n), \end{aligned}$$

i.e., $J(\alpha_1 \wedge \alpha_2^{n-1}) = \alpha_1 \wedge \alpha_2^{n-1}$. This is absurd.

ii) We observe that $\alpha^n \neq 0$ at some point (otherwise, $[\alpha]^n = 0$). Assume by contradiction that $[\alpha] \in H_J^-(X)$. Then there exists $\varphi \in \Lambda_J^-(X)$ such that $[\alpha] = [\varphi]$; furthermore, there exists a point $p \in X$ such that $\varphi^n(p) \neq 0$. Since $\varphi \in \Lambda_J^-(X)$ and n is odd, we have that $J\varphi^n = -\varphi^n$. Arguing as in the proof of i), we obtain that $\alpha^n = 0$. This is absurd. $\square$

As a direct consequence of the last proposition, in the compact case we have the following.

Corollary 2.2. *Let (X, ω) be a $2n$ -dimensional compact symplectic manifold, with n odd. Let J be an almost complex structure on M . Then $[\omega] \notin H_J^-(X)$.*

The following provides an example of a 2-cohomology class $\mathfrak{a}$ on a 6-dimensional compact nilmanifold N endowed with a left-invariant almost complex structure J , which belongs to $H_J^+(N) \cap H_J^-(N)$ and whose square is non-zero; nevertheless, $\mathfrak{a}^3 = 0$, according to Proposition 2.1.

2.2. Example. Let $\mathfrak{g}$ be the 6-dimensional nilpotent Lie algebra whose dual vector space admits a basis $\{e^1, \dots, e^6\}$ satisfying the following Maurer-Cartan equations:

$$de^1 = 0, \quad de^2 = 0, \quad de^3 = e^{12}, \quad de^4 = e^{13}, \quad de^5 = e^{14}, \quad de^6 = e^{23},$$

where $e^{ij} = e^i \wedge e^j$, and so on. Let G be the connected and simply-connected Lie group having $\mathfrak{g}$ as a Lie algebra. Then, G admits a uniform discrete subgroup Γ and $N = \Gamma \backslash G$ is a 6-dimensional nilmanifold. Let J be the almost complex structure on N defined by the following global $(1, 0)$ -forms:

$$\psi^1 = e^1 + ie^2, \quad \psi^2 = e^4 + ie^6, \quad \psi^3 = e^3 + ie^5.$$

By the Nomizu Theorem, it follows that $H^\bullet(\mathfrak{g}) \simeq H^\bullet(N; \mathbb{R})$ and, consequently, $b_1(N) = 2$, $b_2(N) = 4$ and the $b_3(N) = 6$. By a straightforward computation it follows that $0 \neq [e^{26}] \in H^2(N; \mathbb{R})$ and $0 \neq [e^{15}] \in H^2(N; \mathbb{R})$. Furthermore,

$$[e^{26}] = [e^{26} - e^{14}] = [-\Re(\psi^1 \wedge \psi^2)], \quad [e^{26}] = [e^{26} + e^{14}] = [\Re(\psi^1 \wedge \overline{\psi^2})]$$

and

$$[e^{15}] = [e^{15} + e^{23}] = [\Im(\psi^1 \wedge \psi^3)], \quad [e^{15}] = [e^{15} - e^{23}] = [\Im(\overline{\psi^1} \wedge \psi^3)].$$

Therefore, $0 \neq [e^{26}] \in H_J^+(N) \cap H_J^-(N)$ and $0 \neq [e^{15}] \in H_J^+(N) \cap H_J^-(N)$. Set $\mathfrak{a} = [e^{26} - e^{15}] \in H_J^+(N) \cap H_J^-(N)$ and $\mathfrak{a}^2 = 2[e^{1256}] \neq 0$.

3. NON-COMPACT MANIFOLDS

In this section we consider open manifolds (without boundary), namely manifolds without compact connected components. Starting with a symplectic 4-manifold, we have

Theorem 3.1. *Let (X, ω) be a 4-dimensional open symplectic manifold. Then there exists an almost complex structure J on X such that $\omega \in \mathcal{Z}_J^-$, that is, ω is J -anti-invariant, if and only if the tangent bundle TX is symplectically trivial.*

Proof. ($\Leftarrow$) Assume that $\{V_1, V_2, W_1, W_2\}$ is a global symplectic frame on X , namely

$$\omega(V_i, W_j) = \delta_{ij}, \quad \omega(V_i, V_j) = \omega(W_i, W_j) = 0, \quad i, j = 1, \dots, 4.$$

Define

$$JV_1 = V_2, \quad JW_1 = -W_2.$$

Then $\omega = V^1 \wedge W^1 + V^2 \wedge W^2$, where $\{V^1, V^2, W^1, W^2\}$ is the dual frame of $\{V_1, V_2, W_1, W_2\}$, and it is immediate to check that $J\omega = -\omega$.

($\Rightarrow$) Assume that there exists an almost complex structure J such that $J\omega = -\omega$. As X is open we can choose a never vanishing vector field V_1 on X and let K be an ω -calibrated almost complex structure on X . Since ω is J -anti-invariant, we have that

$$\omega(V_1, JV_1) = 0, \quad \omega(KV_1, JKV_1) = 0;$$

furthermore, $\omega(V_1, KV_1) > 0$, with K being ω -calibrated, and, consequently, $\omega(JV_1, JKV_1) < 0$. Then, by using the standard symplectic Gram-Schmidt process, we can extract a global symplectic basis from $\{V_1, JV_1, KV_1, JKV_1\}$. $\square$

Remark 3.2. We observe that the proof of ($\Leftarrow$) holds for every $2n$ -dimensional symplectic manifold with a symplectically trivial tangent bundle, provided that n is even. However a corollary of the proof of Proposition 2.1 says that symplectic forms on $2n$ -dimensional manifolds with n odd are never J -anti-invariant.

The next result shows that the ($\Rightarrow$) direction of Theorem 3.1 holds only in the 4-dimensional case. Namely,

Proposition 3.3. *Let (X^{2n}, J) be an almost complex manifold such that the Pontrjagin class $p_1(TX) \neq 0$ and it admits a J -anti-invariant symplectic form α . Then the open almost complex $(2n + 4k)$ -dimensional manifold $X \times \mathbb{C}^{2k}$ has a symplectic anti-invariant form but $T(X \times \mathbb{C}^{4k})$ is non-trivial.*

Proof. By assumption, there exists a symplectic form $\alpha \in \mathcal{Z}_J^-$. Let $(z_1, \dots, z_k, w_1, \dots, w_k)$ be coordinates on $\mathbb{C}^{2k}$ and set $\omega = \sum_{j=1}^k dz_j \wedge dw_j$. Then $\alpha + \omega \in \mathcal{Z}_{\hat{J}}^-(X \times \mathbb{C}^{2k})$, where $\hat{J}$ is the product almost complex structure on $X \times \mathbb{C}^{2k}$.

We have $p_1(T(X \times \mathbb{C}^{2k})|_{X \times \{0\}}) \in H^4(X)$ and $p_1(TX \oplus \mathbb{C}^{2k}) = p_1(TX) \neq 0$ by assumption, and so the tangent bundle is non-trivial. $\square$

Example 3.4. As an application of Proposition 3.3, we take the *Fermat quartic* $X = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3(\mathbb{C}) \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$. It turns out that the signature $\tau(X) = -16$ (see e.g. [14, p. 156]). Then, according to the Hirzebruch signature theorem, $p_1(TX) = 3\tau(X) = -48$. Furthermore, it is well known that X has a nowhere vanishing holomorphic $(2, 0)$ -form Ω . Hence, setting $\alpha = \frac{1}{2}(\Omega + \overline{\Omega})$,

we have that α is a J -anti-invariant symplectic form on X . Therefore, $\alpha + \omega \in \mathcal{Z}_{\hat{J}}^-(X \times \mathbb{C}^{2k})$, where $\hat{J}$ is the product almost complex structure on $X \times \mathbb{C}^{2k}$ and $T(X \times \mathbb{C}^{2k})$ is non-trivial.

Theorem 3.5. *Let X be an open $2n$ -dimensional manifold with trivial tangent bundle and n even. Then for any cohomology class $\mathfrak{a} \in H^2(X)$ there exists an almost complex structure J and an anti-invariant symplectic form ω on X such that $\mathfrak{a} = [\omega]$.*

Proof. Let $\{V_1, \dots, V_n, W_1, \dots, W_n\}$ be a global frame on X , J_0 be the almost complex structure on X defined as $J_0 V_i = W_i$, $i = 1, \dots, n$, and g be the Hermitian metric defined by requiring that $\{V_1, \dots, V_n, W_1, \dots, W_n\}$ is an orthonormal frame. Then the fundamental form ω_0 of g is an almost symplectic form with symplectically trivial tangent bundle. Denote by

$$\begin{aligned} \mathbb{S}_{\text{symp}}^{\mathfrak{a}} &= \{\text{symplectic forms } \omega \text{ on } X \text{ such that } \mathfrak{a} = [\omega]\}, \\ \mathbb{S}_{\text{symp}} &= \{\text{symplectic forms on } X\}, \\ \mathcal{S}_{\text{symp}} &= \{\text{almost symplectic forms on } X\}. \end{aligned}$$

Then $\mathbb{S}_{\text{symp}}^{\mathfrak{a}} \subset \mathbb{S}_{\text{symp}} \subset \mathcal{S}_{\text{symp}}$, and according to the h -principle (see [13] or [11, 10.2.2]) the inclusion $\mathbb{S}_{\text{symp}}^{\mathfrak{a}} \hookrightarrow \mathcal{S}_{\text{symp}}$ is a homotopy equivalence. In other words there exists a family ω_t of almost symplectic forms on X with ω_1 symplectic and $\mathfrak{a} = [\omega_1]$.

Denote by $E \rightarrow X \times [0, 1]$ the bundle over $X \times [0, 1]$ whose fibre is given by

$$E_x = \{\text{ordered symplectic basis of } T_x X \text{ with respect to } \omega_t\}.$$

Since (X, ω_0) is symplectically trivial, then there exists a section of E over $X \times \{0\}$. By the homotopy lifting property there is a section of E over $X \times [0, 1]$. In particular (X, ω_1) is symplectically trivial. Hence, in view of Remark 3.2, it follows that there exists an almost complex structure J such that ω_1 is a J -anti-invariant form. $\square$

Concerning invariant cohomology classes, we have

Proposition 3.6. *Let (X, J) be an open almost complex manifold. Then given any non-zero class $\mathfrak{a} \in H^2(X; \mathbb{R})$, there exists an almost complex structure $\hat{J}$ on X which is homotopic to J and such that $\mathfrak{a} \in H_{\hat{J}}^+(X)$ and $\mathfrak{a} \notin H_{\hat{J}}^-(X)$.*

Proof. Let $0 \neq \mathfrak{a} \in H^2(X; \mathbb{R})$; then by the h -principle as above (see the remark in [11] following Theorem 10.2.2), there exists a symplectic form ω and a homotopy equivalence of almost complex structures $\{\mathcal{J}_t\}$ for $t \in [0, 1]$ such that

- $J \in \{\mathcal{J}_t\}$;
- $\mathfrak{a} = [\omega]$ and $\mathcal{J}_1$ is compatible with ω .

Hence, $\mathfrak{a} \in H_{\mathcal{J}_1}^+(X)$. We claim that $\mathfrak{a} \notin H_{\mathcal{J}_1}^-(X)$. On the contrary, $\mathfrak{a} \in H_{\mathcal{J}_1}^+(X) \cap H_{\mathcal{J}_1}^-(X)$; then $\mathfrak{a} = [\omega] = [\alpha_-]$, for suitable, $\alpha_- \in \mathcal{Z}^-$. Hence, we have

$$J(\alpha_- \wedge \omega^{n-1}) = -\alpha_- \wedge \omega^{n-1},$$

and thus the volume form

$$\alpha_- \wedge \omega^{n-1} = 0.$$

Therefore, as ω is non-degenerate, α_- vanishes identically, and, consequently, $\mathfrak{a} = 0$. This is absurd. $\square$

4. ANALYTIC CONTINUATION OF ANTI-INVARIANT FORMS

In this section we prove an analytic continuation result for anti-invariant closed forms on almost complex manifolds.

Theorem 4.1. *Let (X, J) be a connected $2n$ -dimensional almost complex manifold. Let α be a closed J -anti-invariant form on X such that $\alpha = 0$ on an open set $\mathcal{U} \subset X$. Then $\alpha = 0$ on X .*

The proof in the 4-dimensional case is a direct consequence of Hodge theory.

Proof (4-dimensional case). We may assume that $X = \mathbb{R}^4$. Take a J -Hermitian metric g on $\mathbb{R}^4$; let $*_g$ be the Hodge operator and $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$ be the fundamental form of g . Denote as usual by Λ_g^+ the space of self-dual forms on $(\mathbb{R}^4, g)$. Then,

$$\Lambda_g^+ = \mathbb{R}\omega \oplus \Lambda_J^-.$$

Hence, if $\alpha \in \mathcal{Z}^-$, then $d*\alpha = d\alpha = 0$. Therefore $\Delta\alpha = 0$, and by assumption $\alpha = 0$ on $\mathcal{U}$. In view of the Unique Continuation Theorem by Aronszajn (see [4] and [15] or [6]) we obtain that $\alpha = 0$ on $\mathbb{R}^4$. $\square$

The general case is more subtle but eventually reduces again to Aronszajn's Theorem.

Proof (general case). Let $\mathbb{D}$ be a J -holomorphic disk in X . Then following [17], Lemma 2.4.2, we can find a sufficiently small neighborhood U of $\mathbb{D}$ and a smooth chart $\phi : U \rightarrow \mathbb{D} \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$ such that $\phi(z) = (z, 0, \dots, 0)$ and $d\phi(z)$ is holomorphic for all $z \in \mathbb{D}$. The second condition means that the pushed forward almost-complex structure ϕ_*J agrees with the standard product structure i on $\mathbb{C}^n$ along $\mathbb{D} \times \{0\} \subset \mathbb{D} \times \mathbb{C}^{n-1}$. Hence, by working in this chart we can smoothly identify a small neighborhood of $\mathbb{D} \subset X$ with the total space of the complex vector bundle $\mathbb{D} \times \mathbb{R}^{2(n-1)} \equiv \mathbb{D} \times \mathbb{C}^{n-1} \rightarrow \mathbb{D}$ such that under the identification, $J|_{\mathbb{D} \times \{0\}}$ coincides with the standard product structure.

Let $t : (z, w) \mapsto (z, tw)$ denote the scaling in the fibers.

We linearize J on $\mathbb{D} \times \mathbb{C}^{n-1}$ by replacing J by

$$J_0 = \lim_{t \rightarrow 0} \left(\frac{1}{t}\right)^* \circ J \circ t_*.$$

Then J_0 is an almost complex structure on $\mathbb{D} \times \mathbb{C}^{n-1}$ invariant under scaling and restricting to the standard complex structure on the fibres $\{x\} \times \mathbb{C}^{n-1}$.

We can also replace α by α_0 defined by

$$\alpha_0(U, V) = \lim_{t \rightarrow 0} \left(\frac{1}{t}\right) \alpha(tU, tV).$$

Claim. α_0 is a closed, smooth J_0 -anti-invariant form on $\mathbb{D} \times \mathbb{C}^{n-1}$. It vanishes on the fibres $\{x\} \times \mathbb{C}^{n-1}$.

To justify the claim we compute the almost complex structures and anti-invariant forms explicitly in local coordinates.

Let $(x_1, \dots, x_{2n})$ be standard local coordinates in a neighborhood of a point $z \in \mathbb{D} \times \{0\} \subset \mathbb{D} \times \mathbb{R}^{2(n-1)}$. Then the scaling is given by

$$t(x_1, \dots, x_{2n}) = (x_1, x_2, tx_3, \dots, tx_{2n}).$$

Denote the coordinate vector fields by $e_i = \frac{\partial}{\partial x_i}$. Then $t_*e_i = e_i$ for $i = 1, 2$ and $t_*e_i = te_i$ for $i > 2$. We can write

$$\alpha = \sum_{i < j} a_{ij} dx_i \wedge dx_j$$

and

$$Je_i = \sum_j b_{ij} e_j,$$

where a_{ij} and b_{ij} are functions of $(x_1, \dots, x_{2n})$. Now, by our choice of coordinates we have that $b_{1j}(z) = \delta_{2j}$, $b_{2j}(z) = -\delta_{1j}$ and $b_{i1}(z) = b_{i2}(z) = 0$ for all $i > 2$ at all points $z = (x_1, x_2, 0, \dots, 0)$. Indeed, the matrix $b_{ij}(z)$ gives the standard complex structure at such points. Next, as anti-holomorphic forms necessarily vanish on complex planes, we have that $a_{12}(z) = 0$.

We compute at a fixed point x in the fiber over $z = (x_1, x_2) \in \mathbb{D}$:

$$\begin{aligned} J_t(e_1(x)) &:= \left(\frac{1}{t}\right)^* J(t_*e_1(x)) = \left(\frac{1}{t}\right)^* J(e_1(tx)) \\ &= \left(\frac{1}{t}\right)^* \sum_j b_{1j}(tx) e_j = b_{11}(tx) e_1 + b_{12}(tx) e_2 + \sum_{j>2} b_{1j}(tx) \frac{1}{t} e_j \\ &\rightarrow e_2 + \sum_{j,k>2} \frac{\partial b_{1j}}{\partial x_k}(z) x_k e_j \end{aligned}$$

as $t \rightarrow 0$.

Similarly $J_t(e_2(x)) := \left(\frac{1}{t}\right)^* J(t_*e_2(x))$ converges to

$$-e_1 + \sum_{j,k>2} \frac{\partial b_{2j}}{\partial x_k}(z) x_k e_j$$

as $t \rightarrow 0$.

For $i > 2$ we have

$$\begin{aligned} J_t(e_i(x)) &:= \left(\frac{1}{t}\right)^* J(t_*e_i(x)) = \left(\frac{1}{t}\right)^* J(tx)(te_i) \\ &= \left(\frac{1}{t}\right)^* \sum_j b_{ij}(tx) te_j = b_{i1}(tx) te_1 + b_{i2}(tx) te_2 + \sum_{j>2} b_{ij}(tx) e_j \\ &\rightarrow \sum_{j>2} b_{ij}(z) e_j \end{aligned}$$

as $t \rightarrow 0$. Hence in these coordinates the fibers converge to complex vector spaces as $t \rightarrow 0$.

For the forms, we have

$$\begin{aligned} \alpha_t(x)(e_1, e_2) &:= \frac{1}{t} \alpha(tx)(e_1, e_2) = \frac{1}{t} a_{12}(tx) \\ &\rightarrow \sum_{k>2} \frac{\partial a_{12}}{\partial x_k}(z) x_k \end{aligned}$$

as $t \rightarrow 0$.

For $i, j > 2$ we have

$$\alpha_t(x)(e_i, e_j) := \frac{1}{t} \alpha(tx)(te_i, te_j) = ta_{ij}(tx) \rightarrow 0$$

as $t \rightarrow 0$.

Finally for $i > 2$ we have

$$\alpha_t(x)(e_1, e_i) := \frac{1}{t} \alpha(tx)(e_1, te_i) = a_{1i}(tx) \rightarrow a_{1i}(z)$$

as $t \rightarrow 0$ and

$$\alpha_t(x)(e_2, e_i) := \frac{1}{t} \alpha(tx)(e_2, te_i) = a_{2i}(tx) \rightarrow a_{2i}(z)$$

as $t \rightarrow 0$. As the convergence here is smooth, α_0 is closed and J_0 -anti-invariant as claimed.

Now, by the computations above, to understand α_0 we are only interested in the values of a_{1i} and a_{2i} for $i > 2$ along the zero-section, and hence we will think of these only as functions of x_1 and x_2 . The function a_{12} , on the other hand, we must still consider as a function of $x_1, \dots, x_{2n}$, although to simplify notation it is enough to consider the case when it is linear when restricted to the fibres. Similarly we may think of the partial derivatives $\frac{\partial b_{ij}}{\partial x_k}$ for $k > 2$ as depending only on (x_1, x_2) .

There are relations amongst the functions above which we now detail. First of all, as a limit of closed forms, α_0 is closed. Calculating its $dx_1 \wedge dx_2 \wedge dx_i$ component for $i > 2$ we get

$$(1) \quad \frac{\partial a_{12}}{\partial x_i} - \frac{\partial a_{1i}}{\partial x_2} + \frac{\partial a_{2i}}{\partial x_1} = 0.$$

Now we consider the anti-invariance of α_0 at points along the zero-section and obtain

$$(2) \quad a_{1,(2i-1)} = -a_{2,2i}, \quad a_{1,2i} = a_{2,(2i-1)}$$

for $i > 1$. For instance, to prove the first of these, we use the formula $\alpha_0(z)(e_1, e_{2i-1}) = -\alpha_0(z)(e_2, e_{2i})$ for $z \in \mathbb{D}$.

Next, using our formulas for α_0 , we compute

$$\begin{aligned} 0 &= \alpha_0(e_1, J_0 e_1) = \alpha_0(e_1, e_2 + \sum_{j,k>2} \frac{\partial b_{1j}}{\partial x_k} x_k e_j) \\ &= \sum_{k>2} \frac{\partial a_{12}}{\partial x_k} x_k + \sum_{j,k>2} a_{1j} \frac{\partial b_{1j}}{\partial x_k} x_k. \end{aligned}$$

As this holds for all choices of x_k we obtain

$$(3) \quad \frac{\partial a_{12}}{\partial x_k} = - \sum_{j>2} a_{1j} \frac{\partial b_{1j}}{\partial x_k}.$$

Applying equations (2), (1) and then (3) we obtain

$$(4) \quad \frac{\partial a_{1,(2i-1)}}{\partial x_2} - \frac{\partial a_{1,2i}}{\partial x_1} = \frac{\partial a_{1,(2i-1)}}{\partial x_2} - \frac{\partial a_{2,(2i-1)}}{\partial x_1} = \frac{\partial a_{12}}{\partial x_{2i-1}} = - \sum_{j>2} a_{1j} \frac{\partial b_{1j}}{\partial x_{2i-1}}$$

for all $i > 1$, and similarly

$$(5) \quad \frac{\partial a_{1,2i}}{\partial x_2} + \frac{\partial a_{1,(2i-1)}}{\partial x_1} = - \sum_{j>2} a_{1j} \frac{\partial b_{1j}}{\partial x_{2i}}.$$

Differentiating (4) with respect to x_2 gives

$$(6) \quad \frac{\partial^2 a_{1,(2i-1)}}{\partial x_2^2} - \frac{\partial^2 a_{1,2i}}{\partial x_1 \partial x_2} = - \sum_{j>2} \left(\frac{\partial a_{1j}}{\partial x_2} \frac{\partial b_{1j}}{\partial x_{2i-1}} + a_{1j} \frac{\partial^2 b_{1j}}{\partial x_2 \partial x_{2i-1}} \right),$$

and differentiating (5) with respect to x_1 gives

$$(7) \quad \frac{\partial^2 a_{1,2i}}{\partial x_1 \partial x_2} + \frac{\partial^2 a_{1,(2i-1)}}{\partial x_1^2} = - \sum_{j>2} \left(\frac{\partial a_{1j}}{\partial x_1} \frac{\partial b_{1j}}{\partial x_{2i}} + a_{1j} \frac{\partial^2 b_{1j}}{\partial x_1 \partial x_{2i}} \right).$$

Adding equations (6) and (7) gives a formula for $\Delta a_{1,(2i-1)}$ in terms of the a_{1j} for $j > 2$ and their first derivatives (recall that we think of a_{1j} and a_{2j} for $j > 2$ as functions just of (x_1, x_2)). In fact, if we think of the a_{1j} , a_{2k} as giving a function $a : \mathbb{D} \rightarrow \mathbb{R}^{4(n-1)}$, $(x_1, x_2) \mapsto (a_{13}, \dots, a_{1,2n}, a_{23}, \dots, a_{2,2n})$, then the above calculations give a bound on the Laplacian Δa in terms of bounds on the function and its first derivatives $\frac{\partial a}{\partial x_1}$ and $\frac{\partial a}{\partial x_2}$. Thus by Aronszajn's Theorem the function a satisfies a unique continuation theorem; in particular, if a is identically zero near a point, then it is zero everywhere. Hence if the function a vanishes near a point in $\mathbb{D}$, then it vanishes everywhere on $\mathbb{D}$. As a_{12} is identically zero along the zero-section $\mathbb{D}$ (as anti-invariant forms vanish on holomorphic curves), we conclude that if $\alpha = 0$ near a point of $\mathbb{D}$, then it is zero at all points of $\mathbb{D}$.

To conclude the proof we need the following lemma.

Lemma 4.2. *Fix a Riemannian metric on X defining a distance function d and let $K \subset X$ be compact. There exists an $\epsilon > 0$ such that for any $x, y \in K$ with $d(x, y) < \epsilon$ there exists a holomorphic disk $\mathbb{D}$ in X passing through x and y .*

This follows as in [18], Theorem 3.1.1 (i).

The lemma, together with the unique continuation theorem established along disks, implies that the set of points where α vanishes identically in a neighborhood is closed. As it is clearly open we can conclude the proof of Theorem 4.1. $\square$

5. CLOSED FORMS OF RANK 0 AND 4

In this section we construct closed forms which have everywhere rank 0 or 4, have compact support near a submanifold W^{2n-2} of X^{2n} , and are Poincaré dual to $[W] \in H_2(X)$. As explained in the introduction, although there are no pointwise obstructions, the unique continuation theorem, Theorem 4.1, shows that there do not exist corresponding anti-invariant almost complex structures. This is our main proposition.

Proposition 5.1. *Let $W \subset X^{2n}$ be a $(2n-2)$ -dimensional compact submanifold with trivial normal bundle $\nu(W)$ and σ be a 1-form on W such that $d\sigma$ never vanishes and $d(\sigma \wedge d\sigma) = 0$. Then there is a compactly supported 2-form ω on the total space of $\nu(W)$ which everywhere has rank either 4 or 0 and is cohomologous to the Thom class τ of $\nu(W)$.*

Proof. Let $\tau = r(x_1, y_1) dx_1 \wedge dy_1$ be the local expression of the Thom class of $\nu(W)$, where $r(x_1, y_1)$ is a bump function such that $r = 1$ on a neighborhood of W in TW . Define

$$\omega = \tau + d(r\sigma).$$

Then,

$$\omega^2 = \tau \wedge \tau + dr \wedge \sigma \wedge dr \wedge \sigma + r^2 d\sigma \wedge d\sigma + 2\tau \wedge dr \wedge \sigma + 2r\tau \wedge d\sigma + 2rdr \wedge \sigma \wedge d\sigma.$$

Therefore,

$$\omega^2 = 2r(\tau + dr \wedge \sigma) \wedge d\sigma$$

and, consequently,

$$\omega^2 \neq 0,$$

where $r \neq 0$. We have

$$\begin{aligned} \omega^3 &= 2r(\tau^2 \wedge d\sigma + dr \wedge \sigma \wedge \tau \wedge d\sigma + \tau \wedge d\sigma \wedge dr \wedge \sigma + dr \wedge \sigma \wedge d\sigma \wedge dr \wedge \sigma) \\ &\quad + 2r^2(\tau \wedge (d\sigma)^2 + dr \wedge \sigma \wedge (d\sigma)^2) = 0. \end{aligned}$$

Therefore, at each point, ω has rank 4 or $\Omega = 0$. $\square$

The condition in Proposition 5.1 that W has a 1-form with certain properties is often satisfied. Indeed, let $\mathbb{T}^6 = \mathbb{R}^6 \setminus \mathbb{Z}^6$ be the 6-dimensional standard torus. Denote by $(x_1, \dots, x_6)$ the coordinates on $\mathbb{R}^6$. Let $W = \{[(x^1, \dots, x^6)] \in \mathbb{T}^6 \mid x_5 = x_6 = 0\}$. Then the Poincaré dual $\mathfrak{a}$ of W is given by $dx_5 \wedge dx_6$. Set

$$\sigma = \cos(2\pi x_3)dx_1 + \sin(2\pi x_3)dx_2.$$

Then

$$d\sigma = 2\pi(\sin(2\pi x_3)dx_1 \wedge dx_3 - \cos(2\pi x_3)dx_2 \wedge dx_3)$$

is never vanishing on W . We immediately obtain

$$d(\sigma \wedge d\sigma) = 0.$$

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