ON SUPPORT POINTS OF THE CLASS $S^0(B^n)$

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ABSTRACT. We consider support points of the class $S^0(B^n)$ introduced by G. Kohr and prove that, given a normalized Loewner chain $f(z, t)$ such that $f(\cdot, 0)$ is a support point of $S^0(B^n)$, all elements of the chain are support points of $S^0(B^n)$. Also, we prove a similar result for Loewner chains that come from the Roper–Suffridge extension operator.

1. INTRODUCTION AND RESULTS

Let $D \subset \mathbb{C}^n$ be a domain. We denote by $\mathcal{H}(D, \mathbb{C}^n)$ the set of all holomorphic functions which map from D to $\mathbb{C}^n$. Furthermore we denote by $\|z\|$ the Euclidean norm of $z \in \mathbb{C}^n$. Let $B^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ be the unit ball and let $S(B^n)$ be the set of all normalized biholomorphic mappings on B^n , i.e.,

$$S(B^n) := \{f \in \mathcal{H}(B^n, \mathbb{C}^n) \mid f \text{ biholomorphic with } f(0) = 0, Df(0) = I_n\}.$$

Compared to the case $n \geq 2$, the class $S := S(B^1)$ is well understood, and a powerful tool to describe this class is the so-called Loewner theory: every element f of S can be embedded in a Loewner chain, which can be described by a differential equation. This equation, in turn, can be used to study properties of f . In higher dimensions, the situation is more complicated and G. Kohr introduced the subclass $S^0(B^n) \subseteq S(B^n)$ in [Koh01] by the method of Loewner chains in order to get a class having similar properties as S .

We recall the definition and some properties of $S^0(B^n)$ (see [GK03] or [GHKK12]). For $f, g \in \mathcal{H}(B^n, \mathbb{C}^n)$, we write $f \prec g$ if there exists a Schwarz mapping $v \in \mathcal{H}(B^n, \mathbb{C}^n)$ (i.e., $\|v(z)\| \leq \|z\|$ for all $z \in B^n$) such that $f = g \circ v$.

A mapping $f : B^n \times [0, \infty) \rightarrow \mathbb{C}^n$ is called a *normalized Loewner chain* if $e^{-t}f(\cdot, t) \in S(B^n)$ for all $t \geq 0$ and $f(\cdot, s) \prec f(\cdot, t)$ for all s, t with $0 \leq s \leq t$.

A mapping $f \in S(B^n)$ is said to have *parametric representation* if there exists a normalized Loewner chain $f(z, t)$ with $f = f(\cdot, 0)$ such that $\{e^{-t}f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n . $S^0(B^n)$ is the set of all biholomorphic mappings on B^n which have parametric representation.

When $n = 1$, the normalization in those Loewner chains is not really restrictive and we have $S^0(B^1) = S$. For $n \geq 2$, however, we always have $S^0(B^n) \subsetneq S(B^n)$. On the other hand, $S^0(B^n)$ has some nice properties similar to S ; e.g., it is compact in contrast to $S(B^n)$ and the elements satisfy a growth condition similar to the growth theorem for S (see Corollary 8.3.9 in [GK03]). Besides, the class $S^*(B^n) \subset S(B^n)$ of normalized starlike mappings is contained in $S^0(B^n)$ because $f \in S^*(B^n)$ if and

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only if $e^t f(z)$ is a normalized Loewner chain. We denote by $\text{supp } S^0(B^n)$ the set of all support points of $S^0(B^n)$, i.e., those functions that maximize over $S^0(B^n)$ the real part of a continuous linear functional $L : \mathcal{H}(B^n, \mathbb{C}^n) \rightarrow \mathbb{C}$, which is nonconstant on $S^0(B^n)$. We will prove the following theorem, which is already known to be true for the case $n = 1$, i.e., for the class S .

Theorem 1.1. *Let $f \in \text{supp } S^0(B^n)$ and let $f(z, t)$ be a normalized Loewner chain with $f(\cdot, 0) = f$ such that $\{e^{-t} f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n . Then $e^{-t} f(\cdot, t) \in \text{supp } S^0(B^n)$ for all $t \geq 0$.*

In order to get (nontrivial) examples for biholomorphic mappings for $n \geq 2$, one can use extension operators, which are mappings of the form $\Theta : S(B^k) \rightarrow S(B^{k+m})$, $k, m \in \mathbb{N}$. Such an operator is said to *preserve normalized Loewner chains* if for any normalized Loewner chain $f(z, t)$ on B^k the mapping $e^t \Theta(e^{-t} f(z, t))$ is a normalized Loewner chain on B^{k+m} . One example of such an operator is the classical Roper–Suffridge extension operator $\Psi_n : S \rightarrow S(B^n)$ with $n \geq 2$ and

$$(1.1) \quad \Psi_n(f)(z_1, \dots, z_n) = (f(z_1), z_2 \sqrt{f'(z_1)}, \dots, z_n \sqrt{f'(z_1)})$$

(see Section 11.3 in [GK03]). Here, the branch of the square root is chosen such that $\sqrt{1} = 1$. We have $\Psi_n(S) \subseteq S^0(B^n)$. Ψ_n has other nice properties; e.g., it maps starlike/convex/Bloch mappings again onto starlike/convex/Bloch mappings (see Section 11.1 in [GK03]).

We will prove Conjecture 3.1 in [GKP07], which says that all support points of $\Psi_n(S)$ are unbounded mappings. This implies the following result.

Theorem 1.2. *Let $f \in S$ and $F = \Psi_n(f)$. Also let $f(z_1, t)$ be a normalized Loewner chain with $f(\cdot, 0) = f$ and let $F(z, t) = e^t \Psi_n(e^{-t} f(z, t))$. If $F \in \text{supp } \Psi_n(S)$, then $e^{-t} F(\cdot, t) \in \text{supp } \Psi_n(S)$ for all $t \geq 0$.*

2. PROOF OF THEOREM 1.1

We will prove Theorem 1.1 by generalizing ideas from a proof for the case $n = 1$, which is described in [HM84]. First, we need to take a look at Runge domains. Let $D \subseteq \mathbb{C}^n$ be a domain of holomorphy. We let $\mathcal{P}(D, \mathbb{C}^n) \subset \mathcal{H}(D, \mathbb{C}^n)$ be the set of all polynomials with the topology induced by locally uniform convergence in D .

For $n = 1$ (a version of) the polynomial Runge theorem says that $\mathcal{P}(D, \mathbb{C})$ is dense in $\mathcal{H}(D, \mathbb{C})$ whenever D is simply connected. In higher dimensions this is no longer true and one calls D a *Runge domain* if $\mathcal{P}(D, \mathbb{C}^n)$ is dense in $\mathcal{H}(D, \mathbb{C}^n)$. Furthermore, if $E \subseteq \mathbb{C}^n$ is a domain with $D \subseteq E$, then the pair (D, E) is called a *Runge pair* if $\mathcal{H}(E, \mathbb{C}^n)$ is dense in $\mathcal{H}(D, \mathbb{C}^n)$. So D is a Runge domain if and only if $(D, \mathbb{C}^n)$ is a Runge pair. The unit ball B^n is a simple example of a Runge domain. However, the Runge property is not invariant with respect to biholomorphic mappings, as an example of J. Wermer shows; see [Wer59].

First, we rephrase the property that B^n is mapped biholomorphically onto a Runge domain by f in terms of f .

Lemma 2.1. *Let $f \in S(B^n)$. The following statements are equivalent:*

- a) $f(B^n)$ is a Runge domain.
- b) $f^{-1} \in \overline{\mathcal{P}(f(B^n), \mathbb{C}^n)}$.
- c) For every $g \in \mathcal{H}(B^n, \mathbb{C}^n)$ there exists a sequence p_k of polynomials with $g(z) = \lim_{k \rightarrow \infty} p_k(f(z))$ locally uniformly in B^n .

Proof. “ $a \Rightarrow b$ ”: If $f(B^n)$ is Runge, then $f^{-1} \in \mathcal{H}(f(B^n), \mathbb{C}^n) = \overline{\mathcal{P}(f(B^n), \mathbb{C}^n)}$.

“ $b \Rightarrow c$ ”: Let h_k be a sequence of polynomials which approximate f^{-1} locally uniformly on $f(B^n)$ and let $g \in \mathcal{H}(B^n, \mathbb{C}^n)$. As B^n is Runge, there exists a sequence q_k of polynomials which approximate g locally uniformly on B^n . Then $q_k \circ h_k \circ f$ approximates locally uniformly the map $g \circ f^{-1} \circ f = g$.

“ $c \Rightarrow a$ ”: Note that the linear map $F : \mathcal{H}(f(B^n), \mathbb{C}^n) \rightarrow \mathcal{H}(B^n, \mathbb{C}^n)$, $F(g) = g \circ f$, is an isomorphism, and hence locally uniform convergence of a sequence h_k in $f(B^n)$ is equivalent to locally uniform convergence of the sequence $h_k \circ f$ in B^n . $\square$

Let $f \in S^0(B^n)$ and $f(z, t)$ be a normalized Loewner chain with $f = f(\cdot, 0)$ such that $\{e^{-t}f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n . For every s, t with $0 \leq s \leq t$ we have $f(\cdot, s) \prec f(\cdot, t)$ and we can write $f(\cdot, s) = f(v(\cdot, s, t), t)$ with a unique Schwarz mapping $v(\cdot, s, t)$, which is also called *transition mapping*. For $s = 0$ we will just write $v_t := v(\cdot, 0, t)$. For a fixed s , the family $\{v(\cdot, s, t)_{t \geq s}\}$ satisfies the so-called Loewner ordinary differential equation (see [GHKK12]).

In [ABFW], a remarkable fact concerning solutions to this equation is shown.

Theorem 2.2 (Proposition 5.1 in [ABFW]). *$v(B^n, s, t)$ is a Runge domain for all s, t with $0 \leq s \leq t$.*

We note a simple consequence of the proof of Theorem 2.2.

Theorem 2.3. *Every $f \in S^0(B^n)$ maps B^n onto a Runge domain.*

Proof. Let $f \in S^0(B^n)$ and $f(z, t)$ be a normalized Loewner chain with $f = f(\cdot, 0)$ such that $\{e^{-t}f(\cdot, t)\}_{t \geq 0}$ is a normal family on B^n . For every s, t with $0 \leq s \leq t$ we have $f(\cdot, s) \prec f(\cdot, t)$, and Theorem 4.2 in [ABFW] immediately implies that $(f(B^n, s), f(B^n, t))$ is a Runge pair. Let R be the Loewner range of the Loewner chain $f(z, t)$, i.e., $R = \cup_{t \geq 0} f(B^n, t)$. One can easily show that $(f(B^n, t), R)$ is also a Runge pair for every $t \geq 0$. Now, the Koebe distortion theorem for the class $S^0(B^n)$, which is Corollary 8.3.9 in [GK03], implies

$$R = \bigcup_{t \geq 0} f(B^n, t) \supseteq \bigcup_{t \geq 0} \{z \in \mathbb{C}^n \mid \|z\| < \frac{e^t}{4}\} = \mathbb{C}^n,$$

and consequently $R = \mathbb{C}^n$. Hence $(f(B^n, t), \mathbb{C}^n)$ is a Runge pair for every $t \geq 0$. In particular, $f(B^n) = f(B^n, 0)$ is a Runge domain. $\square$

Corollary 2.4. *For $n \geq 2$, every $f \in S^0(B^n)$ can be approximated locally uniformly by automorphisms of $\mathbb{C}^n$.*

Proof. We just have to apply Theorem 2.1 in [AL92], which states that every bi-holomorphic map from a starlike domain to a Runge domain can be approximated by automorphisms of $\mathbb{C}^n$ when $n \geq 2$. $\square$

It would be nice to find out more about those $f \in S^0(B^n)$ that extend to automorphisms of $\mathbb{C}^n$. When $n = 1$, the only normalized univalent function that is also univalent in $\mathbb{C}$ is the identity mapping. Corollary 2.4 suggests that the case $n \geq 2$ should be more interesting.

Remark 2.5. Theorem 2.3 can also be shown without using the Loewner range, namely by proving Corollary 2.4 first: Every $f \in S^0(B^n)$ can be calculated by v_t via

$$f(z) = \lim_{t \rightarrow \infty} e^t v_t(z)$$

locally uniformly on B^n ; see Theorem 8.1.9 in [GK03]. By combining Theorem 2.2 with Theorem 2.1 in [AL92], we see that f can be approximated locally uniformly by automorphisms of $\mathbb{C}^n$. This fact already implies that $f(B^n)$ is a Runge domain; see the remark at the end of page 372 in [AL92].

Next, we note the important fact that, given a transition mapping v_t and a $G \in S^0(B^n)$, then $e^t G(v_t)$ is also in $S^0(B^n)$, which is mentioned in the proof of Theorem 2.1 in [GHKK12].

Lemma 2.6. *Let $G \in S^0(B^n)$ and $t \geq 0$. Furthermore, let $f \in S^0(B^n)$ and $f(z, s)$ be a normalized Loewner chain with $f = f(\cdot, 0)$ such that $\{e^{-s} f(\cdot, s)\}$ is a normal family. Write $f = f(v_t, t)$. Then $e^t G(v_t) \in S^0(B^n)$.*

Proof. Let $G(z, s)$ be a normalized Loewner chain with $G(\cdot, 0) = G$ such that $\{e^{-s} G(\cdot, s)\}_{s \geq 0}$ is a normal family and let $F(z, s) : B^n \times [0, \infty) \rightarrow \mathbb{C}^n$ be the mapping

$$F(z, s) = \begin{cases} e^t G(v(z, s, t)), & 0 \leq s \leq t, \\ e^t G(z, s - t), & s > t. \end{cases}$$

Then $F(z, s)$ is a normalized Loewner chain, $F(\cdot, 0) = e^t G(v_t)$ and $\{e^{-s} F(\cdot, s)\}_{s \geq 0}$ is a normal family. Thus $e^t G(v_t) \in S^0(B^n)$. $\square$

Choosing $G(z) = z$ in Lemma 2.6 shows that $e^t v_t \in S^0(B^n)$. Support points of the class S map B^1 onto $\mathbb{C}$ minus a slit. In particular, they are unbounded mappings. It is not known whether support points of $S^0(B^n)$ or $S(B^n)$ can be bounded when $n \geq 2$. Now we will prove Conjecture 2.6 in [GHKK12], which says that $e^t v_t$ is not a support point of $S^0(B^n)$ for any $t \geq 0$.

Lemma 2.7. *Let v_t be defined as in Lemma 2.6 and let $h = e^t v_t \in S^0(B^n)$. Furthermore, let $P \in \mathcal{P}(B^n, \mathbb{C}^n)$ with $P(0) = 0$, $DP = 0$. Then there exists $\delta > 0$ such that $h + \varepsilon e^t P(e^{-t} h) \in S^0(B^n)$ for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$.*

Proof. Let $g_\varepsilon(z) = z + \varepsilon P(z)$. Obviously we have $g_\varepsilon(0) = 0$, $Dg_\varepsilon(0) = I_n$. Now $\det(Dg_\varepsilon(z)) \rightarrow 1$ for $\varepsilon \rightarrow 0$ uniformly on $\overline{B^n}$, so g_ε is locally biholomorphic for ε small enough. In this case, for every $z \in \overline{B^n}$, we have:

$$[Dg_\varepsilon(z)]^{-1} = [I_n + \varepsilon DP(z)]^{-1} = I_n - \varepsilon DP(z) + \varepsilon^2 DP(z)^2 + \dots = I_n - \underbrace{\varepsilon (DP(z) + \dots)}_{:= U(z) \in \mathbb{C}^{n \times n}}.$$

Write $[Dg_\varepsilon(z)]^{-1} g_\varepsilon(z) = z + \varepsilon P(z) - \varepsilon U(z)z - \varepsilon^2 U(z)P(z) = (I_n + \varepsilon M(z))z$, with a matrix-valued function $M(z)$. Consequently

$$\langle [Dg_\varepsilon(z)]^{-1} g_\varepsilon(z), z \rangle = \langle (I_n + \varepsilon M(z))z, z \rangle,$$

and there is an $\delta > 0$ such that $I_n + \varepsilon M(z)$ has only eigenvalues with positive real part for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \delta$. In this case we have

$$\operatorname{Re} \langle [Dg_\varepsilon(z)]^{-1} g_\varepsilon(z), z \rangle > 0 \quad \forall z \in B^n \setminus \{0\},$$

and thus $g_\varepsilon \in S^*(B^n) \subset S^0(B^n)$ by Theorem 6.2.2 in [GK03]. From Lemma 2.6 it follows that $e^t g_\varepsilon(v_t) = e^t g_\varepsilon(e^{-t} h) = h + \varepsilon e^t P(e^{-t} h) \in S^0(B^n)$. $\square$

Proposition 2.8. *Let v_t be defined as in Lemma 2.6 and let $h = e^t v_t \in S^0(B^n)$. Then h is not a support point of $S^0(B^n)$.*

Proof. Let $h = e^t v_t$ be a support point of $S^0(B^n)$; i.e., there is a continuous linear functional $L : \mathcal{H}(B^n, \mathbb{C}^n) \rightarrow \mathbb{C}$ such that $\operatorname{Re} L$ is nonconstant on $S^0(B^n)$ and

$$\operatorname{Re} L(h) = \max_{g \in S^0(B^n)} \operatorname{Re} L(g).$$

Let P be a polynomial with $P(0) = 0$ and $DP(0) = 0$. Then $h + \varepsilon e^t P(e^{-t} h) \in S^0(B^n)$ for all $\varepsilon \in \mathbb{C}$ small enough by Lemma 2.7. We conclude $\operatorname{Re} L(P(e^{-t} h)) = \operatorname{Re} L(P(v_t)) = 0$; otherwise we could choose ε such that $\operatorname{Re} L(h + \varepsilon e^t P(e^{-t} h)) > \operatorname{Re} L(h)$. Now $v_t(B^n)$ is a Runge domain by Theorem 2.2. Hence we can write any analytic function g in the unit ball B^n with $g(0) = 0$ and $Dg(0) = 0$ as $g = \lim_{k \rightarrow \infty} P_k(v_t)$, where every P_k is a polynomial with $P_k(0) = 0$ and $DP_k(0) = 0$, according to Lemma 2.1 c). The continuity of L implies $\operatorname{Re} L(g) = 0$. Hence $\operatorname{Re} L$ is constant on $S(B^n)$, a contradiction. $\square$

Proof of Theorem 1.1. Let L be a continuous linear functional on $\mathcal{H}(B^n, \mathbb{C}^n)$ such that $\operatorname{Re} L$ is nonconstant on $S^0(B^n)$ with

$$\operatorname{Re} L(f) = \max_{g \in S^0(B^n)} \operatorname{Re} L(g).$$

Fix $t \geq 0$; then $f(z) = f(v_t(z), t)$ for all $z \in B^n$. Define the continuous linear functional

$$J(g) := L(e^t \cdot g \circ v_t) \quad \text{for } g \in \mathcal{H}(B^n, \mathbb{C}^n).$$

Now we have

$$J(e^{-t} f(\cdot, t)) = L(f) \quad \text{and} \quad \operatorname{Re} J(g) \leq \operatorname{Re} J(e^{-t} f(\cdot, t)) \quad \text{for all } g \in \mathcal{H}(B^n, \mathbb{C}^n).$$

Furthermore, $\operatorname{Re} J$ is not constant on $S^0(B^n)$: as $e^t v_t$ is not a support point of $S^0(B^n)$, we have $\operatorname{Re} J(\operatorname{id}) = \operatorname{Re} L(e^t v_t) < \operatorname{Re} L(f) = \operatorname{Re} J(e^{-t} f(\cdot, t))$. $\square$

3. PROOF OF THEOREM 1.2

In order to simplify notation, we look at only the Roper-Suffridge-Operator $\Psi := \Psi_2$:

$$\Psi(f)(z_1, z_2) = \left(f(z_1), z_2 \sqrt{f'(z_1)} \right).$$

Ψ maps S into $S^0(B^2)$. Of course, $\Psi(\operatorname{supp} S) \subseteq \operatorname{supp} \Psi(S)$, because the first coordinate of $\Psi(f)(z_1, z_2)$ is just $f(z_1)$.

Now we prove Conjecture 3.1 in [GKP07].

Proposition 3.1. *Let $f \in S$ be bounded. Then $\Psi(f)$ is not a support point of $\Psi(S)$.*

Proof. Let $f \in S$ be bounded and let $L : \mathcal{H}(B^n, \mathbb{C}^n) \rightarrow \mathbb{C}$ be a continuous linear functional such that $\operatorname{Re} L$ is nonconstant on $\Psi(S)$ and

$$\operatorname{Re} L(\Psi(f)) = \max_{G \in \Psi(S)} \operatorname{Re} L(G).$$

For every $n \geq 1$ there is an $\varepsilon_0 > 0$ such that $f + \frac{\varepsilon}{n+1}f^{n+1} \in S$, for all $\varepsilon \in \mathbb{C}$ with $|\varepsilon| < \varepsilon_0$, as f is bounded. It follows that

$$\begin{aligned} \Psi(f + \frac{\varepsilon}{n+1}f^{n+1})(z_1, z_2) &= \left(f(z_1) + \frac{\varepsilon}{n+1}f(z_1)^{n+1}, z_2\sqrt{f'(z_1)}\sqrt{1+\varepsilon f(z_1)^n} \right) \\ &= \left(f(z_1) + \frac{\varepsilon}{n+1}f(z_1)^{n+1}, z_2\sqrt{f'(z_1)} + \varepsilon z_2\sqrt{f'(z_1)}(1/2 \cdot f(z_1)^n + \dots) \right) \\ &= \left(f(z_1), z_2\sqrt{f'(z_1)} \right) + \underbrace{\varepsilon \left(\frac{1}{n+1}f(z_1)^{n+1}, z_2/2\sqrt{f'(z_1)}f(z_1)^n \right)}_{=: P_n} + (0, \mathcal{O}(|\varepsilon|^2)) \in \Psi(S). \end{aligned}$$

If $\operatorname{Re}L(P_n) \neq 0$, then we can choose ε such that $\operatorname{Re}L(\Psi(f + \frac{\varepsilon}{n+1}f^{n+1})) > \operatorname{Re}L(\Psi(f))$, a contradiction. Hence,

$$(3.1) \quad \operatorname{Re}L\left(\frac{1}{n+1}f(z_1)^{n+1}, z_2/2 \cdot \sqrt{f'(z_1)}f(z_1)^n\right) = 0 \quad \forall n \geq 1.$$

Now we can repeat this argument for the ε -terms of higher order, because all coefficients of the expansion $\sqrt{1+x} = 1 + \frac{1}{2}x \mp \dots$ are $\neq 0$, and we get

$$(3.2) \quad \operatorname{Re}L\left(0, z_2\sqrt{f'(z_1)}f(z_1)^{jn}\right) = 0 \quad \forall j \geq 2, n \geq 1.$$

Now consider an arbitrary function of the form $(0, z_2g(z_1))$ with $g(0) = g'(0) = 0$. Write $z_2g(z_1) = z_2\sqrt{f'(z_1)} \cdot \frac{g(z_1)}{\sqrt{f'(z_1)}}$ and approximate the second factor by a sequence of polynomials in f (see Lemma 2.1 c)):

$$z_2g(z_1) = z_2\sqrt{f'(z)} \cdot \sum_{k \geq 2} a_k f(z_1)^k = \sum_{k \geq 2} a_k z_2\sqrt{f'(z)}f(z_1)^k.$$

By using (3.2), it follows that

$$(3.3) \quad \operatorname{Re}L(0, z_2g(z_1)) = 0 \quad \text{for all } g \in \mathcal{H}(B^1, \mathbb{C}) \text{ with } g(0) = g'(0) = 0.$$

We can apply this to (3.1) for $n \geq 2$ to get $\operatorname{Re}L(f(z_1)^n, 0) = 0$ for all $n \geq 3$, and by Runge approximation it follows that

$$(3.4) \quad \operatorname{Re}L(g(z_1), 0) = 0 \quad \text{for all } g \in \mathcal{H}(B^1, \mathbb{C}) \text{ with } g(0) = g'(0) = g''(0) = 0.$$

Now let $H(z_1, z_2) = (h(z_1), z_2\sqrt{h'(z_1)}) \in \Psi(S)$. If $h(z) = z + a_2z^2 + \dots$, then $\sqrt{h'(z)} = \sqrt{1 + 2a_2z + \dots} = 1 + a_2z + \dots$. Thus $(h(z_1), z_2\sqrt{h'(z_1)}) = (z_1 + a_2z_1^2 + \dots, z_2 + a_2z_2z_1 + \dots)$ and with (3.3) and (3.4):

$$\operatorname{Re}L(H) = d_1 + d_2 \cdot a_2 + d_3 + d_4 \cdot a_2$$

with $d_1 = \operatorname{Re}L(z_1, 0)$, $d_2 = \operatorname{Re}L(z_1^2, 0)$, $d_3 = \operatorname{Re}L(0, z_2)$, $d_4 = \operatorname{Re}L(0, z_2z_1)$. Finally, use (3.1) with $n = 1$ to get

$$\operatorname{Re}L(z_1^2, z_2z_1) = 0 \iff d_2 + d_4 = 0,$$

which implies $\operatorname{Re}L(H) = d_1 + d_3$. Hence, $\operatorname{Re}L$ is constant on $\Psi(S)$ and $\Psi(f)$ cannot be a support point of $\Psi(S)$. $\square$

Proof of Theorem 1.2. The proof is now quite the same as the proof of Theorem 1.1. We just have to replace $S^0(B^n)$ by $\Psi(S)$; see remark 3.1 in [GKP07]. $\square$

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