ON MORSE-BOTT DEGENERATIONS WITH $\text{Crit}(\pi) \simeq \mathbb{C}P^k$ AND FLOER HOMOLOGY

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ABSTRACT. Morse-Bott fibrations generalize the class of Lefschetz fibrations, allowing for non-isolated singularities. A Morse-Bott degeneration is a Morse-Bott fibration $\pi : X \rightarrow \mathbb{D}$ whose only singular value is $0 \in \mathbb{D}$. We show that Morse-Bott degenerations with $\text{Crit}(\pi) = \mathbb{C}P^k$ for $0 < k$ admit restrictions which are unique to the non-isolated case. These restrictions are obtained as an application of methods of Floer homology for monotone Lagrangian submanifolds.

1. INTRODUCTION AND SUMMARY OF MAIN RESULTS

The geometry of Lefschetz fibrations is a fundamental subject of study in both algebraic and symplectic geometry. Recently, the class of Morse-Bott fibrations was introduced, generalizing the classical Lefschetz fibrations (see for instance [16, 18]). In contrast to Lefschetz fibrations, Morse-Bott degenerations allow for non-isolated singularities in the singular fibers. Examples of Lefschetz fibrations exist in abundance. For instance, any smooth very ample divisor Σ of a projective manifold X can be given as the fiber of some Lefschetz fibration. Such a fibration is obtained by blowing up the base locus of a Lefschetz pencil in the linear system $|\Sigma|$ passing through Σ [19]. In symplectic geometry, a central result of Donaldson [7] states that any symplectic 4-manifold gives rise to a symplectic Lefschetz fibration, via a far-reaching generalization of the algebraic construction. However, in the non-isolated Morse-Bott case, only few non-trivial examples are currently known. In particular, no widely applicable construction of Morse-Bott fibrations with compact fibers, as in the Lefschetz case, is known. We will consider the following holomorphic local model of Morse-Bott fibrations:

Definition 1.1. A proper holomorphic map $\pi : X \rightarrow \mathbb{D}$ from a Kähler manifold (X, Ω) to the unit disc $\mathbb{D} \subset \mathbb{C}$ is called a Morse-Bott degeneration of Σ if:

- (1) The only critical value of π is $0 \in \mathbb{D}$ and $\text{Crit}(\pi)$ is a Kähler submanifold of X .
- (2) The holomorphic Hessian of π is a non-degenerate quadratic form when restricted to the normal bundle of $\text{Crit}(\pi)$ in X .
- (3) Σ is bi-holomorphic to one of the smooth fibers $\Sigma_z = \pi^{-1}(z)$ for $z \neq 0$.

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Our aim in this work is to show that Morse-Bott degenerations admit restrictions which are unique to the non-isolated case. We focus on the case of Morse-Bott degenerations with $\text{Crit}(\pi) = \mathbb{C}P^k$. For example, we show:

Theorem 1.2. *Let $\Sigma = \Sigma_d^n \subset \mathbb{C}P^{n+1}$ be a hypersurface of degree $d \leq \frac{n}{2}$. Then Σ does not admit a Morse-Bott degeneration with $\text{Crit}(\pi) = \mathbb{C}P^k$ for $1 \leq k \leq \min(n - 2d + 1, n - 2)$.*

In particular, Theorem 1.2 implies that $\Sigma = \mathbb{C}P^n$ does not admit any Morse-Bott degeneration with critical locus $\mathbb{C}P^k$ for $1 \leq k \leq n - 2$. It should be noted that $\Sigma = \mathbb{C}P^n$ does not admit an isolated degeneration via results of [3, 17]. On the other hand, any $\Sigma = \Sigma_d^n$ with $2 \leq d$ is classically the fiber of a Lefschetz fibration with isolated singularities.

Theorem 1.3. *Let $\Sigma = \text{Gr}(m, r)$ be the Grassmannian of r -dimensional planes in $\mathbb{C}^m$ with $2 \leq r$. If $r^2 - m - 2 < k < m - r^2 + 1$, then Σ admits no Morse-Bott degeneration with $\text{Crit}(\pi) \simeq \mathbb{C}P^k$.*

Both Theorem 1.2 and Theorem 1.3 follow from a more general result for the class of Fano manifolds with $b_2(\Sigma) = 1$. Recall that a manifold Σ is said to be Fano if its anticanonical class $-K_\Sigma \in \text{Pic}(\Sigma)$ is ample. A fundamental invariant of Fano manifolds is the Fano index $r_\Sigma \in \mathbb{N}$, which is the minimal positive integer by which K_Σ is divisible in $\text{Pic}(\Sigma)$. It is shown by Kobayashi-Ochiai [11] that $r_\Sigma \leq n + 1$, where $n = \dim_{\mathbb{C}} \Sigma$, with equality if and only if $\Sigma = \mathbb{C}P^n$. We show:

Theorem 1.4. *Let Σ be a Fano manifold of $\dim_{\mathbb{C}} \Sigma = n$ with $b_2(\Sigma) = 1$. Suppose there exists a Morse-Bott degeneration $\pi : X \rightarrow \mathbb{D}$ of Σ with $\text{Crit}(\pi) = \mathbb{C}P^k$ such that $1 \leq k \leq n - 2$. Then one of the following holds:*

- (i) $r_\Sigma \mid k + 1$.
- (ii) $2r_\Sigma \mid n - k + 1$.
- (iii) $2r_\Sigma \mid n + k + 2$.

In section 3 we prove a stronger version of this theorem. On the other hand, consider the following example:

Example 1.5. Let $\pi : X \rightarrow \mathbb{D}$ be a Morse-Bott degeneration with an *isolated* singularity and $p : E \rightarrow X$ be a vector bundle on X . The map $\tilde{\pi} : \mathbb{P}(E) \rightarrow \mathbb{D}$ given by $\tilde{\pi} = \pi \circ p$ is a Morse-Bott degeneration with $\text{Crit}(\tilde{\pi}) = \mathbb{C}P^k$ where $k = \text{rank}(E) - 1$. For $z \neq 0$, the fiber $\tilde{\Sigma}_z = \tilde{\pi}^{-1}(z)$ is, by definition, a projective bundle over $\Sigma_z = \pi^{-1}(z)$. In particular, assuming the original fiber Σ to be Fano, one can find a bundle p such that $\tilde{\Sigma}$ is also Fano. However, as projective bundles such examples always have $b_2(\tilde{\Sigma}) \geq 2$.

Let us conclude the introduction by giving a brief description of the proof of Theorem 1.4. Recall that a Lagrangian correspondence from a symplectic manifold (M_1, ω_1) to a symplectic manifold (M_2, ω_2) is a Lagrangian submanifold L of $(M_1 \times M_2, (-\omega_1) \oplus \omega_2)$; see [20]. Lagrangian correspondences arising in the context of Morse-Bott degenerations were studied by Seidel and Smith in [18] and by Perutz in [16]. To a Morse-Bott degeneration $\pi : X \rightarrow \mathbb{D}$ of Σ with $\text{Crit}(\pi) \simeq \mathbb{C}P^k$ we associate a Lagrangian correspondence L from Σ to $\mathbb{C}^{k+1}$ which is topologically a sphere bundle over S^{2k+1} . The correspondence L is defined as the geometric composition of a vanishing cycle of π (see for instance [16]) and the dual of the graph of the Hopf map.

The restrictions of Theorem 1.4 are obtained as an application of Lagrangian Floer theory to the correspondence L . Lagrangian Floer theory was introduced by Floer, for some class of Lagrangian submanifolds, in order to study problems of Lagrangian intersections; see e.g. [9]. The theory was later extended by Oh [14, 15] for the class of so-called monotone Lagrangian submanifolds. An essential feature of Lagrangian Floer homology is that it can be computed via a spectral sequence, discovered by Oh in [15], relating the singular homology $H_*(L; \mathbb{Z}_2)$ of a monotone Lagrangian submanifold $L \subset (M, \omega)$ to its Floer homology $HF(L, L)$. In our case, the assumption that Σ is a Fano manifold with $b_2(\Sigma) = 1$ (actually $\rho(\Sigma) = 1$) assures that L is a monotone Lagrangian correspondence. In particular, L admits a well-defined Floer homology $HF(L, L)$. Moreover, $L \subset \Sigma \times \mathbb{C}^{k+1}$ is Hamiltonian displaceable (due to the $\mathbb{C}^{k+1}$ factor) which implies $HF(L, L) = 0$. By the properties of Oh's spectral sequence, the vanishing of $HF(L, L)$ implies restrictions on the singular homology $H_*(L; \mathbb{Z}_2)$. These restrictions, in turn, imply the numerical conditions of Theorem 1.4.

The rest of the paper is organized as follows: In section 2 we review relevant facts from Lagrangian Floer theory. In section 3 we prove Theorem 1.4 and its applications.

2. RELEVANT FACTS ON LAGRANGIAN FLOER HOMOLOGY

Let (M, ω) be a connected tame symplectic manifold; see [1]. Denote by $\mathcal{J}$ the space of ω -compatible almost complex structures J on M for which (M, ω, g_J) is geometrically bounded, where g_J is the associated Riemannian metric defined by

$$(2.1) \quad g_J(\cdot, \cdot) = \omega(\cdot, J\cdot).$$

Let $L \subset M$ be a closed connected Lagrangian manifold. Denote by $H_2^D(M, L) \subset H_2(M, L)$ the image of the Hurewicz homomorphism $\pi_2(M, L) \rightarrow H_2(M, L)$. One has two homomorphisms

$$(2.2) \quad \omega : H_2^D(M, L) \rightarrow \mathbb{Z}, \quad \mu : H_2^D(M, L) \rightarrow \mathbb{R}$$

given respectively by integration of ω and by the Maslov index. One has the following essential definition:

Definition 2.1. The Lagrangian $L \subset M$ is said to be monotone if

$$(2.3) \quad \omega(A) = \lambda \cdot \mu(A) \text{ for all } A \in H_2^D(M, L)$$

for some $\lambda > 0$.

Recall that the minimal Maslov number $N_L \in \mathbb{N}$ of a Lagrangian submanifold $L \subset M$ is the non-negative generator of the the group of integers $\mu(H_2^D(M, L)) \subset \mathbb{Z}$. For instance, recall that a symplectic manifold (M, ω) is said to be monotone if $c_1(TM) = \lambda[\omega]$ for some $\lambda > 0$. Let $H_2^s(M)$ be the image of the Hurewicz map $\pi_2(M) \rightarrow H_2(M; \mathbb{Z})$. The minimal Chern number $C_M \in \mathbb{N}$ of M is the non-negative generator of the group of integers $c_1(H_2^s(M)) \subset \mathbb{Z}$, where $c_1 : H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$ is the map given by $A \mapsto \langle c_1(TM), A \rangle$. Assume M is a monotone symplectic manifold. If $L \subset M$ is a simply connected Lagrangian, L is also monotone, with minimal Maslov number given by $N_L = 2C_M$.

From now on we assume L is monotone with minimal Maslov number $N_L \geq 2$. Denote the Floer homology of a monotone Lagrangian submanifold L with $N_L \geq 2$ by $HF(L, L)$; see [14]. A Lagrangian submanifold $L \subset M$ is said to be *displaceable*

if there exists a Hamiltonian diffeomorphism $\varphi \in \text{Ham}(M, \omega)$ such that $\varphi(L) \cap L = \emptyset$. Floer homology admits the following fundamental geometric property: *If L is displaceable, then $HF(L, L) = 0$.*

In [15] Oh shows that Floer homology $HF(L, L)$ can be computed as a finitely generated $\mathbb{Z}_2$ -module, via a spectral sequence, whose first page is given in terms of $H_*(L; \mathbb{Z}_2)$, the singular homology of L . A class of spectral sequences that are especially convenient to work with are spectral sequences associated to filtered complexes, see [13]. It is shown by Biran and Cornea in [4, 5] that a variant of Oh's spectral sequence can be naturally realized as the spectral sequence of a filtered complex.

Let $f : L \rightarrow \mathbb{R}$ be a Morse function and ρ a Riemannian metric on L . Denote by $\text{Crit}(f)$ the set of critical points of f and denote by $|x|$ the Morse index of $x \in \text{Crit}(f)$. Let $\Lambda = \mathbb{Z}_2[t, t^{-1}]$ be the ring of Laurent polynomials graded so that $\deg(t) = -N_L$. Let $J \in \mathcal{J}$ be a generic almost complex structure. Set

$$(2.4) \quad \mathcal{C}_*(f, \rho, J) = \mathbb{Z}_2 \langle \text{Crit}(f) \rangle \otimes \Lambda$$

where the grading is the one induced by the Morse indices on $\text{Crit}(f)$ and the grading of Λ mentioned above. In [4, 5] Biran and Cornea define the pearl differential $d : \mathcal{C}_*(f, \rho, J) \rightarrow \mathcal{C}_{*-1}(f, \rho, J)$ by counting the 0-dimensional components of moduli spaces of objects, called pearl trajectories. Although the the pearl complex $(\mathcal{C}_*(f, \rho, J), d)$ is essentially different from the original definition of the Floer complex in [14], it is shown in [4] that, as rings, the homology of the pearl complex and $HF(L, L)$ are isomorphic. In particular, the pearl complex admits the following decreasing $\mathbb{Z}$ -filtration:

$$(2.5) \quad F_p \mathcal{C}_*(f, \rho, J) = \bigoplus_{j \geq -p} \mathcal{C}_{*+jN_L}(f, \rho) \otimes t^j,$$

where $\mathcal{C}_*(f, \rho)$ is the corresponding Morse complex. Moreover, the pearl differential is compatible with the filtration. The required spectral sequence $(E_{p,q}^r, d^r)_{r \geq 0}$ is thus the spectral sequence associated to this filtration. The main properties of the spectral sequence, relevant for our applications, are the following:

- (1) $E_{p,q}^1 \simeq H_{p+q-pN_L}(L; \mathbb{Z}_2) \cdot t^{-p}$ for every p, q .
- (2) $E_{p+1,q}^r \simeq E_{p,q+1-N_L}^r t^{-1}$ for every p, q, r .
- (3) $\{E_{p,q}^r, d^r\}$ collapses after a finite number of pages.
- (4) $\bigoplus_{q \in \mathbb{Z}} E_{p,q}^\infty \simeq HF(L, L)$ for any $p \in \mathbb{Z}$.

In particular, if L is displaceable, then $E_{p,q}^\infty = 0$ for every p, q .

3. RESTRICTIONS ON DEGENERATIONS WITH $\text{Crit}(\pi) = \mathbb{C}P^k$

Recall that a Lagrangian correspondence L from a symplectic manifold (M, ω_1) to another symplectic manifold (M, ω_2) is a Lagrangian submanifold of $(M_1 \times M_2, (-\omega_1) \oplus \omega_2)$. We refer the reader to [20] for the general theory of Lagrangian correspondences. We show:

Proposition 3.1. *Let Σ be a manifold of $\dim_{\mathbb{C}} \Sigma = n$ and let $\pi : X \rightarrow \mathbb{D}$ be a Morse-Bott degeneration of Σ with $\text{Crit}(\pi) \simeq \mathbb{C}P^k$. There exists a Lagrangian correspondence L from Σ to $\mathbb{C}^{k+1}$ which is diffeomorphic to a S^{n-k} -bundle over S^{2k+1} .*

Lagrangian correspondences arising in the context of Morse-Bott degenerations were studied in the works of Seidel and Smith in [18] and Perutz in [16]. In [16]

Perutz shows that a path $\gamma : [0, 1] \rightarrow \mathbb{D}$ such that $\gamma(0) = z$ and $\gamma(1) = 0$ gives rise to a Lagrangian correspondence L_γ from Σ_z to $\text{Crit}(\pi)$. In particular, these correspondences are viewed as a natural generalization of the classical “vanishing cycles”, arising in Picard-Lefschetz theory, to the Morse-Bott case.

We give a short description of the existence part of this result. Given a Morse-Bott degeneration $\pi : X \rightarrow \mathbb{D}$ consider the function $F = \text{Re}(\pi)$. Let ϕ_t be the negative gradient flow of F with respect to the metric $g_J(\cdot, \cdot) = \Omega(\cdot, J\cdot)$, where J is the complex structure of X . Let $S \subset X$ be the stable submanifold of $\text{Crit}(\pi)$ under this flow and consider the end point map $\Theta : S \rightarrow \text{Crit}(\pi)$ given by $\Theta(x) := \lim_{t \rightarrow \infty} \phi_t(x)$; see [10].

Lemma 3.2. *Let $\pi : X \rightarrow \mathbb{D}$ be a Morse-Bott degeneration and denote $\Sigma_t = \pi^{-1}(t)$ for $t \in (0, 1]$.*

(1) $\tilde{S} = \{(x, \Theta(x)) \mid x \in S\}$ is a Lagrangian correspondence from X to $\text{Crit}(\pi)$.

(2) $\tilde{S}_t = \{(x, \Theta(x)) \mid x \in S \cap \Sigma_t\}$ is a Lagrangian correspondence from Σ_t to $\text{Crit}(\pi)$. Moreover, for $t > 0$ small enough $\tilde{S}_t$ is a sphere bundle over $\text{Crit}(\pi)$.

Proof. (1) Since π is holomorphic ϕ_t is the same as the Hamiltonian flow of $H = \text{Im}(\pi)$ with respect to the Kähler form Ω . In particular, $\tilde{S}$ is a Lagrangian correspondence, as the graph of the symplectomorphism $\Theta : S \rightarrow \text{Crit}(\pi)$; see [20].

(2) Denote $S_t = F^{-1}(t) \cap S$. It is known by Morse-Bott theory that for $t > 0$ small enough, S_t is a sphere bundle over $\text{Crit}(\pi)$, of dimension $\dim_{\mathbb{R}} S_t = \dim_{\mathbb{R}} S - 1$, see [2]. As ϕ_t is the Hamiltonian flow of H , the function H is constant along the flow lines of ϕ_t . We deduce that $S \subset H^{-1}(0)$. Consequently $S_t = S \cap (F^{-1}(t) \cap H^{-1}(0)) = S \cap \Sigma_t$. Hence, $\tilde{S}_t = \{(x, \Theta(x)) \mid x \in S_t\} \subset \Sigma_t \times \text{Crit}(\pi)$ is a Lagrangian correspondence from Σ_t to $\text{Crit}(\pi)$. $\square$

Remark 3.3. Note that by the Moser argument all fibers $\Sigma_z = \pi^{-1}(z)$ for $z \neq 0$ are symplectomorphic. In particular, the correspondence $\tilde{S}_t$ does not depend on the choice of fiber Σ_t for $t > 0$ small enough.

Remark 3.4. In [18] Seidel and Smith show that if $\pi : X \rightarrow \mathbb{D}$ is a Morse-Bott degeneration of Σ and $L \subset \text{Crit}(\pi)$ is a Lagrangian submanifold, then Σ admits a Lagrangian submanifold $N(L) \subset \Sigma$ which is topologically a sphere bundle over L . Note that such a Lagrangian is given as the geometric composition of $\tilde{S}_t$ with the Lagrangian correspondence L from $\text{Crit}(\pi)$ to a point.

Proof of Theorem 3.1. Let ω_0 be the standard symplectic form on $\mathbb{C}^{k+1}$ and let ω_{FS} be the Fubini-Study form on $\mathbb{C}P^k$ normalized such that $\int_{\mathbb{C}P^1} \omega_{FS} = \pi$. Consider the Hopf map $h : S^{2k+1} \subset \mathbb{C}^{k+1} \rightarrow \mathbb{C}P^k$ given by $z \mapsto \mathbb{C} \cdot z$. As h is a symplectomorphism, its graph, $L_h = \{(z, h(z)) \mid z \in S^{2k+1}\}$ is a Lagrangian correspondence from $\mathbb{C}^{k+1}$ to $\mathbb{C}P^k$.

Denote by $\tilde{S}_\Sigma$ the Lagrangian correspondence from Σ to $\mathbb{C}P^k$ of Lemma 3.2 (see Remark 3.3 above). Set L be the geometric composition of $\tilde{S}_\Sigma$ with $L_h^\circ = \{(h(z), z) \mid z \in S^{2k+1}\}$, the dual of L_h ; see [20]. For $t > 0$ the correspondence L is diffeomorphic to

$$(3.1) \quad L_t = \tilde{S}_t \circ L_h^\circ = \{(x, z) \mid \Theta(x) = h(z)\} \subset \Sigma_t \times \mathbb{C}^{k+1}.$$

Note that, topologically, L_t is the pullback of $\tilde{S}_t \rightarrow \mathbb{C}P^k$ by the Hopf map, hence, a S^{n-k-1} -bundle over S^{2k+1} . $\square$

On the other hand, for the Floer homology we have:

Lemma 3.5. *Let Σ be a Fano manifold with $\rho(\Sigma) = 1$. Let L be a compact simply connected Lagrangian correspondence from Σ to $\mathbb{C}^{k+1}$. Then L is monotone with $N_L = 2C_\Sigma$. Moreover, $HF(L, L) = 0$.*

Proof. By the condition $\rho(\Sigma) = 1$ the symplectic manifold $(\Sigma, \Omega|_\Sigma)$ is monotone. As L is simply connected, the map $j : \pi_2(\Sigma \times \mathbb{C}^{k+1}) \rightarrow \pi_2(\Sigma \times \mathbb{C}^{k+1}, L)$ is surjective. Note that $\mu(h \circ j(A)) = 2 \langle c_1(TM), h(A) \rangle$ for classes $A \in \pi_2(\Sigma \times \mathbb{C}^{k+1})$. In particular, L is monotone and one has $N_L = 2C_{\Sigma \times \mathbb{C}^{k+1}} = 2C_\Sigma$. As L is monotone with $N_L \geq 2$ the Floer homology $HF(L, L)$ is well defined. Due to the compactness of L it can be displaced from itself by an affine translation in the $\mathbb{C}^{k+1}$ factor which implies $HF(L, L) = 0$. $\square$

We are now in position to prove the following result, which implies Theorem 1.4 of the introduction:

Theorem 3.6. *Let Σ be a Fano manifold with $\rho(\Sigma) = 1$ and $\dim_{\mathbb{C}} \Sigma = n$. Suppose there exists a Morse-Bott degeneration $\pi : X \rightarrow \mathbb{D}$ of Σ with $\text{Crit}(\pi) = \mathbb{C}P^k$ such that $1 \leq k \leq n - 2$. Then one of the following holds:*

- (i) $C_\Sigma \mid k + 1$.
- (ii) $2C_\Sigma \mid n - k + 1$.
- (iii) $2C_\Sigma \mid n + k + 2$. Furthermore, either $C_\Sigma \mid 2k + 1$ or $C_\Sigma \mid n - k$ holds.

Moreover, if $n = 3k + 1$, then (iii) cannot hold, and (i) and (ii) coincide.

Proof. Let L be the Lagrangian correspondence from Σ to $\mathbb{C}^{k+1}$ of Proposition 3.1. By Lemma 3.5 $HF(L, L) = 0$. Consider the spectral sequence described in section 2 applied to $HF(L, L)$. First, for the homology $H_*(L; \mathbb{Z}_2)$, the $\mathbb{Z}_2$ -coefficients Gysin sequence for the sphere bundle L gives

$$\begin{aligned} \cdots \rightarrow H^j(S^{2k+1}; \mathbb{Z}_2) \rightarrow H^j(L; \mathbb{Z}_2) \rightarrow H^{j-(n-k)}(S^{2k+1}; \mathbb{Z}_2) \rightarrow \\ H^{j+1}(S^{2k+1}; \mathbb{Z}_2) \rightarrow \cdots \end{aligned}$$

which implies that

$$H^j(L; \mathbb{Z}_2) \simeq H_{n+k+1-j}(L, \mathbb{Z}_2) \neq 0 \Rightarrow j = 0, 2k + 1, n - k, n + k + 1.$$

For $m \in \mathbb{Z}$ consider the line $l_m := \{(p, q) \mid q - pN_L + p = m\} \subset \mathbb{Z}^2$. The term $E_{p,q}^r$ could hence be non-trivial only for $(p, q) \in l_m$ such that $m = 0, 2k + 1, n - k, n + k + 1$.

On the other hand, for any $p, q \in \mathbb{Z}$ and $r \geq 1$ the term $E_{p,q}^{r+1}$ of the sequence is given as the homology of

$$\cdots \rightarrow E_{p+r, q-r+1}^r \rightarrow E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r \rightarrow \cdots$$

Hence $E_{p,q}^{r+1}$ could vanish only if either $E_{p+r, q-r+1}^r$ or $E_{p-r, q+r-1}^r$ is non-trivial. Moreover, note that by the properties of the sequence if $E_{p,q}^r = 0$ for $p, q \in l_m$, then $E_{p',q'}^r = 0$ for any $p', q' \in l_m$.

For $p, q \in l_0$ one has $E_{p,q}^1 \simeq H_0(L; \mathbb{Z}_2)t^{-p} \simeq \mathbb{Z}_2 t^{-p}$. On the other hand, $E_{p+r, q-r+1}^1 \simeq H_{1-rN_L}(L; \mathbb{Z}_2) = 0$ for any $r \geq 1$. Hence also $E_{p+r, q-r+1}^r = 0$ for any $r \geq 1$. Thus, in order for $E_{p,q}^{r+1}$ to vanish for some $r \geq 1$ we must have $E_{p-r, q+r-1}^r \neq 0$. This can happen only if $(p - r, q + r - 1) \in l_m$ for $m = 2k + 1, n - k, n + k + 1$, which can, in turn, happen only in one of the following cases:

- (i) $rN_L - 1 = 2k + 1$ for some $r \geq 1 \Rightarrow N_L \mid 2k + 2$.

- (ii) $rN_L - 1 = n - k$ for some $r \geq 1 \Rightarrow N_L \mid n - k + 1$.
- (iii) $rN_L - 1 = n + k + 1$ for some $r \geq 1 \Rightarrow N_L \mid n + k + 2$.

The further restrictions are obtained by considering the vanishing of $E_{p,q}^r$ for $p, q \in l_{2k+1}, l_{n-k}$. This requires further analysis of the Gysin sequence in the cases $j = 2k + 1, n - k$.

- (a) $n \neq 3k, 3k + 1$: In this case the sequence has the form

$$0 \rightarrow H^{n-k}(L; \mathbb{Z}_2) \rightarrow H^0(S^{2k+1}; \mathbb{Z}_2) \rightarrow 0 \quad ; \quad 0 \rightarrow H^{2k+1}(S^{2k+1}; \mathbb{Z}_2) \rightarrow H^{2k+1}(L; \mathbb{Z}_2) \rightarrow 0$$

which gives $H_{2k+1}(L; \mathbb{Z}_2) \simeq H_{n-k}(L; \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Restrictions are obtained in the following cases:

(a.1) $3k < n$: Assume there exists a non-trivial differential $d^r : E_{p,q}^r \rightarrow E_{p',q'}^r$ for $p, q \in l_{2k+1}$ and $p', q' \in l_{n-k}$ and $r \geq 1$. By similar considerations, we get the condition $2k + 1 - rN_L + 1 = n - k$ for some $r \geq 1$ which implies $N_L \mid 3k - n + 2$. Note that in this case there is also a non-trivial differential $d^{r'} : E_{p'',q''}^{r'} \rightarrow E_{p''',q'''}^{r'}$ for $p'', q'' \in l_0$ and $p''', q''' \in l_{n-k}$ and some $r' \geq 1$, which implies $N_L \mid n + k + 2$. As $3k - n + 2 = (n + k + 2) - (2n - 2k)$ we get $N_L \mid 2n - 2k$. Equivalently $C_\Sigma \mid n - k$ and $2C_\Sigma \mid n + k + 2$.

(a.2) $n < 3k + 1$: Assume there exists a non-trivial differential $d^r : E_{p,q}^r \rightarrow E_{p',q'}^r$ for $p, q \in l_{n-k}$ and $p', q' \in l_{2k+1}$ and $r \geq 1$. This holds if $n - k - rN_L + 1 = 2k + 1$ for some $r \geq 1$ which implies $N_L \mid 3k - n$. Similarly to the above, due to the non-trivial differential $d^{r'} : E_{p'',q''}^{r'} \rightarrow E_{p''',q'''}^{r'}$ for $p'', q'' \in l_0$ and $p''', q''' \in l_{n-k}$ and some $r' \geq 1$ one has $N_L \mid n + k + 2$. As $3k - n = 4k + 2 - (n + k + 2)$ we get $N_L \mid 4k + 2$. Equivalently $C_\Sigma \mid 2k + 1$ and $2C_\Sigma \mid n + k + 2$.

- (b) $n = 3k$: Note that in this case the Gysin sequence gives

$$0 \rightarrow H^{n-k}(L; \mathbb{Z}_2) \rightarrow H^0(S^{2k+1}; \mathbb{Z}_2) \rightarrow H^{2k+1}(S^{2k+1}; \mathbb{Z}_2) \rightarrow H^{2k+1}(L; \mathbb{Z}_2) \rightarrow 0.$$

In particular, in this case, it is not necessary that $H^{n-k}(L; \mathbb{Z}_2) \simeq H^{2k+1}(L; \mathbb{Z}_2) \simeq \mathbb{Z}_2$. However, as $n + k + 2 = 4k + 2$, if we assume $2C_\Sigma \mid n + k + 2$, then also $C_\Sigma \mid 2k + 1$, which is the same as (a.2).

- (c) $n = 3k + 1$: In this case the sequence has the form

$$0 \rightarrow H^{2k+1}(S^{2k+1}; \mathbb{Z}_2) \rightarrow H^{2k+1}(L; \mathbb{Z}_2) \rightarrow H^0(S^{2k+1}; \mathbb{Z}_2) \rightarrow 0.$$

Hence $H_{2k+1}(L; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Note that in this case (iii) cannot hold. Indeed, assume $d^r : E_{p,q}^r \rightarrow E_{p',q'}^r$ is a non-trivial differential for $(p, q) \in l_0$ and $(p', q') \in l_{n+k+1}$. Then $E_{p,q}^{r'} \neq 0$ for $r < r'$ if and only if $(p, q) \in l_{2k+1} = l_{n-k}$. In particular, there could be no non-trivial differential $d^{r'}$ annihilating $E_{p,q}^{r'}$ for $(p, q) \in l_{2k+1} = l_{n-k}$, and hence only (i) (which coincides, in this case, with (ii)) can hold. $\square$

Remark 3.7. Theorem 1.4 of the introduction is stated in terms of the Fano index r_Σ . We note that, under the assumption $b_2(\Sigma) = 1$, the minimal Chern number, C_Σ , is equal to the Fano index, r_Σ . Indeed, as Σ is Fano it is simply connected; see [12]. In particular, $H_2^s(\Sigma) = H_2(\Sigma; \mathbb{Z})$. On the other hand, $c_1(T\Sigma) = c_1(-K_\Sigma) = r_\Sigma H$ where $H \in H^2(\Sigma; \mathbb{Z}) \simeq \mathbb{Z}$ is the positive generator. Let $A \in H_2^s(\Sigma)$ be the class such that $\langle H, A \rangle = 1$. Then, $\langle c_1(T\Sigma), A \rangle = r_\Sigma$. And thus, in this case, $C_\Sigma = r_\Sigma$.

Let us mention the applications of Theorem 3.6 to the following specific sub-classes of toric Fano manifolds with $\rho(\Sigma) = 1$.

Corollary 3.8. *Let $\Sigma = \Sigma_d^n \subset \mathbb{C}P^{n+1}$ be a hypersurface of degree $d \leq \frac{n}{2}$. Then Σ does not admit a Morse-Bott degeneration with $\text{Crit}(\pi) = \mathbb{C}P^k$ for $1 \leq k \leq \min(n - 2d + 1, n - 2)$.*

Proof. Assume $k \leq n - 2$. The Fano index of $\Sigma_d^n \subset \mathbb{C}P^{n+1}$ is given by $r_d^n := r_{\Sigma_d^n} = n + 2 - d$. On the other hand, note that, as $2k + 2, n - k + 1 < n + k + 2$, if $n + k + 2 < 2r_{\Sigma_d^n} = 2n + 4 - 2d$, none of the restrictions (i)-(iii) of Theorem 3.6 can hold. Hence, if $k < n - 2d + 2$ there is no Morse-Bott degeneration of Σ_d^n with $\text{Crit}(\pi) \simeq \mathbb{C}P^k$. $\square$

Corollary 3.9. *Let $\Sigma = \text{Gr}(m, r)$ be the Grassmannian of r -dimensional planes in $\mathbb{C}^m$ with $2 \leq r$. If $r^2 - m - 2 < k < m - r^2 + 1$, then Σ admits no Morse-Bott degeneration with $\text{Crit}(\pi) \simeq \mathbb{C}P^k$.*

Proof. In this case $\dim_{\mathbb{C}} \Sigma = r(m - r)$ and $r_{\Sigma} = m$. Note that (i) of Theorem 3.6 cannot hold if $k < m - 1$. The restriction (ii) in this case is $2m \mid r(m - r) - k + 1$. In particular, if (ii) holds, then so does $m \mid -r^2 - k + 1$. Hence, if $k < m - r^2 + 1$ both (i) and (ii) do not hold. Restriction (iii) is given by $2m \mid r(m - r) + k + 2$ and hence $m \mid -r^2 + k + 2$. If $0 < -r^2 + k + 2$, then (iii) does not hold if $k < m - r^2 + 1 < m + r^2 - 2$. If $-r^2 + k + 2 < 0$, then (iii) does not hold if $r^2 - k - 2 < m$. Hence, if $r^2 - m - 2 < k < m - r^2 + 1$ none of the restrictions (i)-(iii) can hold; in particular, there is no Morse-Bott degeneration of Σ with $\text{Crit}(\pi) \simeq \mathbb{C}P^k$. $\square$

We conclude with the following remarks:

Remark 3.10. Let X be a smooth projective manifold and let $\Sigma \subset X$ be a smooth very ample hypersurface. Let $|\Sigma|$ be the linear system of Σ and denote by $D_{\Sigma} \subset |\Sigma|$ the discriminant variety of all singular divisors in $|\Sigma|$. Recall that a line $\ell \subset |\Sigma|$ is said to be a Lefschetz pencil if either D_{Σ} is a hypersurface and ℓ intersects D_{Σ} transversally or if D_{Σ} is not a hypersurface and $D_{\Sigma} \cap \ell = \emptyset$. To any line $\ell \subset |\Sigma|$ one can associate the variety

$$(3.2) \quad \tilde{X}_{\ell} = \{(x, \Sigma') \mid x \in \Sigma', \Sigma' \in \ell\} \subset X \times \ell$$

and the projection map $\pi_{\ell} : \tilde{X}_{\ell} \rightarrow \ell \simeq \mathbb{C}P^1$. The fundamental property of such maps states that π_{ℓ} is a Lefschetz fibration if and only if ℓ is a Lefschetz pencil; see [19]. Note that the discriminant D_{Σ} admits a natural stratification given by

$$(3.3) \quad D_{\Sigma}^k = \{\Sigma' \mid \dim(\text{Sing}(\Sigma')) = k\} \subset D_{\Sigma}$$

for $0 \leq k \leq \dim_{\mathbb{C}} \Sigma$. It is thus natural to ask whether more general pencils $\ell \subset |\Sigma|$, which intersect D_{Σ}^k for $1 \leq k < \dim_{\mathbb{C}} \Sigma$, give rise to Morse-Bott degenerations. However, a simple calculation shows that if ℓ is such a pencil, then $\tilde{X}_{\ell}$ must, in fact, be singular (contrary to the Lefschetz case). In particular, $\pi_{\ell} : \tilde{X}_{\ell} \rightarrow \ell$ would not be a Morse-Bott fibration in the non-isolated case. However, it is interesting to ask whether one can apply a resolution of singularities to $\tilde{X}_{\ell}$ to obtain a well-defined Morse-Bott degeneration. It is also interesting to note that such a resolution would involve a change in the topology of the fibers. In particular, the fibers of such a resolution will also have $\rho \geq 2$. Let us note that pencils with non-isolated singularities were studied by algebraic geometers; see for instance [6].

Remark 3.11. It is hard to assess the sharpness of the above results as, currently, there are actually no known non-trivial examples of Morse-Bott degenerations of Fano manifolds with $\rho(\Sigma) = 1$ and $\text{Crit}(\pi) = \mathbb{C}P^k$. We suggest that the restrictions are actually stronger. This could be justified as follows: Note that in Lemma 3.2 we prove the existence of the Lagrangian correspondence $\tilde{S}_t$ from Σ_t to $\mathbb{C}P^k$, which is given as $\{(x, \Theta(x)) \mid x \in S \cap \Sigma_t\}$. Given a hypersurface $\Sigma'_t \subset \Sigma_t$ one could similarly define $\tilde{S}'_t = \{(x, \Theta(x)) \mid x \in S \cap \Sigma'_t\}$ which would be a Lagrangian correspondence from Σ'_t , this time to $\Theta(S \cap \Sigma'_t) \subset \mathbb{C}P^k$. The topology of $\Theta(S \cap \Sigma'_t)$ would depend on the topology of Σ'_t . Note, however, that if $\Theta(S \cap \Sigma'_t)$ would be linear, similar restrictions to Theorem 3.6 would also apply to $\tilde{S}'_t$. This, together with Remark (1) above, leads us to suggest that Morse-Bott degenerations with $\text{Crit}(\pi) = \mathbb{C}P^k$ of Fano manifolds with $\rho(\Sigma) = 1$ such that the generator of $\text{Pic}(\Sigma)$ is very ample are rare, if they exist.

Remark 3.12. In this work we have considered Morse-Bott degenerations in the holomorphic setting. Note that one has the notion of a symplectic Fano manifold which is a monotone symplectic manifold (M, ω) normalized such that $c_1(TM) = [\omega]$. Clearly, an algebraic Fano manifold is also a symplectic Fano manifold. Recently, Fine and Panov discovered in [8] examples of non-algebraic symplectic Fano manifolds, starting from real dimension 12. Indeed, Morse-Bott degenerations themselves are naturally defined in the symplectic context; see [16, 18]. In particular, the results of this paper apply also to general symplectic Fano manifolds, the Fano index r_Σ being replaced with the minimal Chern number C_Σ .

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