ASYMPTOTIC EXPANSION OF THE MODIFIED LOMMEL POLYNOMIALS $h_{n,\nu}(x)$ AND THEIR ZEROS

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ABSTRACT. The modified Lommel polynomials satisfy the second-order linear difference equation

$$h_{n+1,\nu}(x) - 2(n + \nu)x h_{n,\nu}(x) + h_{n-1,\nu}(x) = 0, \quad n \geq 0,$$

with initial values $h_{-1,\nu}(x) = 0$ and $h_{0,\nu}(x) = 1$, where x is a real variable and ν is a fixed positive parameter. An asymptotic expansion, as $n \rightarrow \infty$, is derived for these polynomials by using a turning-point theory for three-term recurrence relations developed by Wang and Wong [Numer. Math. **91** (2002) and **94** (2003)]. The result holds uniformly in the infinite interval $0 \leq x < \infty$, containing the critical value $x = 1/N$, where $N = n + \nu$. Behavior of the zeros of these polynomials is also studied.

1. INTRODUCTION

The Lommel polynomials $R_{n,\nu}(x)$ arise in the theory of Bessel functions. The modified Lommel polynomials $h_{n,\nu}(x)$ are defined by

$$(1.1) \quad h_{n,\nu}(x) = R_{n,\nu}(1/x);$$

see [8, §§9.6-9.73]. The modified Lommel polynomials satisfy the three-term recurrence relation

$$(1.2) \quad h_{n+1,\nu}(x) - 2(n + \nu)x h_{n,\nu}(x) + h_{n-1,\nu}(x) = 0, \quad n \geq 0,$$

with initial values $h_{-1,\nu}(x) = 0$, $h_{0,\nu}(x) = 1$. The polynomials have the hypergeometric representation (for $n > 1$)

$$(1.3) \quad h_{n,\nu}(x) = (\nu)_n (2x)^n {}_2F_3(-n/2, (-n+1)/2; \nu, -n, 1 - \nu - n; -1/x^2),$$

where $(\nu)_n = \Gamma(\nu + n)/\Gamma(\nu)$. Equation (1.3) implies the reflection formula of modified Lommel polynomials

$$(1.4) \quad h_{n,\nu}(-x) = (-1)^n h_{n,\nu}(x).$$

Furthermore, the correct orthogonality relation is

$$(1.5) \quad \sum_{k=1}^{\infty} h_{m,\nu}(\pm x_k) h_{n,\nu}(\pm x_k) x_k^2 = \frac{1}{2(n + \nu)} \delta_{mn},$$

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where $x_k = j_{\nu-1,k}^{-1}$ and $\{j_{\nu-1,k}\}$ are the zeros of the Bessel function $J_{\nu-1}(x)$, for $k = 1, 2, \dots$; see [3, p. 197, (6.5.17)]. (Here we wish to point out that the orthogonality relations given in Dickinson [2] and Chihara [1] are incorrect.) From some formulas involving Lommel polynomials (cf., Watson [8, §9.6]), we may define the modified Lommel polynomials $h_{n,\nu}(x)$ in terms of Bessel functions by

$$(1.6) \quad J_{\nu+n}(1/x) = J_{\nu}(1/x)h_{n,\nu}(x) - J_{\nu-1}(1/x)h_{n-1,\nu+1}(x),$$

or, more explicitly,

$$(1.7) \quad \pi^{-1}(2x \sin \pi \nu)h_{n,\nu}(x) = J_{\nu+n}(1/x)J_{-\nu+1}(1/x) + (-1)^n J_{-\nu-n}(1/x)J_{\nu-1}(1/x),$$

where n is an integer. The behavior of $h_{n,\nu}(x)$ has been investigated by Hurwitz (see Watson [8, §9.65]), and he used an elementary approach to show that

$$(1.8) \quad \lim_{n \rightarrow \infty} \frac{(2x)^{1-\nu-n}h_{n,\nu}(x)}{\Gamma(n+\nu)} = J_{\nu-1}(1/x)$$

uniformly for all x in any closed and bounded annular region centered at the origin.

The purpose of this paper is to present an asymptotic expansion for $h_{n,\nu}(t/N)$, which holds uniformly for t in $[0, \infty)$, where $N = n + \nu$. Our approach is to use a turning-point theory recently introduced by Wang and Wong [6, 7] for three-term recurrence relations. Asymptotic formulas are also obtained for the zeros of the modified Lommel polynomials. It should be mentioned that our result for $h_{n,\nu}(x)$ can also be obtained from the uniform asymptotic expansions of Bessel functions. But, the derivations of these expansions are themselves very sophisticated. Our approach is more direct from the viewpoint of difference equations.

2. DIFFERENCE EQUATION

Equation (1.2) is already in the canonical form considered in [7],

$$(2.1) \quad h_{n+1,\nu}(x) - (A_n x + B_n)h_{n,\nu}(x) + h_{n-1,\nu}(x) = 0,$$

with

$$(2.2) \quad A_n = 2(n + \nu) \quad \text{and} \quad B_n = 0.$$

In terms of the notation

$$(2.3) \quad A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s} \quad \text{and} \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s} \quad (\text{as } n \rightarrow \infty)$$

used in [7], we have

$$(2.4) \quad \theta = -1, \quad \alpha_0 = 2, \quad \alpha_1 = 2\nu, \quad \alpha_n = 0 \quad \text{for } n = 2, 3, \dots,$$

and $\beta_0 = \beta_1 = \beta_2 = \dots = 0$. If these expansions are recast in the form

$$(2.5) \quad A_n \sim N^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha'_s}{N^s} \quad \text{and} \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta'_s}{N^s},$$

where $N = n + \tau_0$ and τ_0 is some fixed real number to be determined, it is easily found that

$$(2.6) \quad \alpha'_0 = 2, \quad \alpha'_1 = 2\nu - 2\tau_0, \dots,$$

and $\beta'_0 = \beta'_1 = \beta'_2 = \cdots = 0$. To apply the result in [7], we first choose τ_0 so that $\alpha'_1 = 0$. From (2.6), it is obvious that the choice is

$$(2.7) \quad \tau_0 = \nu.$$

According to the equation in [7, (2.4)], the characteristic equation is

$$(2.8) \quad \lambda^2 - 2t\lambda + 1 = 0,$$

where t is the rescaled variable $t = Nx$. The two roots of this equation are

$$(2.9) \quad \lambda_{\pm} = t \pm \sqrt{t^2 - 1}.$$

The points $t_{\pm} = \pm 1$, where the two roots coincide, are called the *turning points* of equation (2.1).

We now define the function $\zeta(t)$ introduced in [7, (4.10)]. With $t_+ = 1$, $\theta = -1$, $\alpha'_0 = 2$ and $\beta'_0 = 0$, this function is given by

$$\frac{2}{3}[\zeta(t)]^{3/2} := \log(t + \sqrt{t^2 - 1}) - \frac{1}{t} \int_1^t \frac{s}{\sqrt{s^2 - 1}} ds, \quad t \geq 1,$$

and

$$\frac{2}{3}[-\zeta(t)]^{3/2} := t^{-1} \int_t^1 \frac{s}{\sqrt{1 - s^2}} ds - \cos^{-1} t, \quad 0 < t < 1.$$

By direct calculation, one obtains

$$(2.10) \quad \frac{2}{3}[\zeta(t)]^{3/2} = \log(t + \sqrt{t^2 - 1}) - \frac{1}{t}(t^2 - 1)^{\frac{1}{2}}$$

for $t \geq 1$ and

$$(2.11) \quad \frac{2}{3}[-\zeta(t)]^{3/2} = \frac{1}{t}(1 - t^2)^{\frac{1}{2}} - \cos^{-1} t$$

for $0 < t < 1$. Note that as $t \rightarrow 1$, we have $\zeta(t) \rightarrow 0$. In fact, near $t = 1$, we have the Taylor expansion

$$(2.12) \quad \zeta(t) = 2^{\frac{1}{3}}(t - 1) \left[1 - \frac{7}{10}(t - 1) + \frac{102}{175}(t - 1)^2 + \cdots \right].$$

We also define the functions $H_0(\zeta)$ and $\Phi(\zeta)$ introduced in [7, (4.19) and (4.28)]. In the present situation, these functions are given by

$$(2.13) \quad H_0(\zeta) := -\sqrt{\frac{t^2 - 1}{\zeta}} \quad \text{and} \quad \Phi(\zeta) = 0,$$

where ζ is the function defined in (2.10) and (2.11). Note that in our special case, $\alpha'_1 = \beta'_1 = 0$. Hence, according to the definition of $\Phi(\zeta)$ given in [7, (4.28)], the second equation in (2.13) holds for $0 < t < \infty$ instead of $t \geq \delta$, $0 < \delta < 1$.

With this preliminary work done, we can now apply the main result in [7] to conclude that there are constants $C_1(x)$ and $C_2(x)$ such that the polynomials $h_{n,\nu}(x)$ in (2.1) can be expressed as

$$(2.14) \quad h_{n,\nu}(x) = C_1(x)P_n(x) + C_2(x)Q_n(x),$$

where, with $x = N^{-1}t$, $P_n(x)$ and $Q_n(x)$ have the asymptotic expansions

$$(2.15) \quad P_n(x) = \left(\frac{\zeta}{t^2 - 1} \right)^{\frac{1}{4}} \left[\text{Ai} \left(N^{\frac{2}{3}} \zeta \right) \sum_{s=0}^{p-1} \frac{\tilde{A}_s(\zeta)}{N^{s-\frac{1}{6}}} + \text{Ai}' \left(N^{\frac{2}{3}} \zeta \right) \sum_{s=0}^{p-1} \frac{\tilde{B}_s(\zeta)}{N^{s+\frac{1}{6}}} + \epsilon_p(N, t) \right]$$

and

(2.16)

$$Q_n(x) = \left(\frac{\zeta}{t^2 - 1} \right)^{\frac{1}{4}} \left[\text{Bi} \left(N^{\frac{2}{3}} \zeta \right) \sum_{s=0}^{p-1} \frac{\tilde{A}_s(\zeta)}{N^{s-\frac{1}{6}}} + \text{Bi}' \left(N^{\frac{2}{3}} \zeta \right) \sum_{s=0}^{p-1} \frac{\tilde{B}_s(\zeta)}{N^{s+\frac{1}{6}}} + \delta_p(N, t) \right]$$

for large values of n (or, equivalently, N). In (2.15) and (2.16), $\text{Ai}(\cdot)$ and $\text{Bi}(\cdot)$ are the Airy functions, the leading coefficients are given by

$$(2.17) \quad \tilde{A}_0(\zeta) = 1 \quad \text{and} \quad \tilde{B}_0(\zeta) = 0,$$

and there exists a positive constant M_p such that

$$(2.18) \quad |\epsilon_p(N, t)| \leq \frac{M_p}{N^{p-\frac{1}{6}}} |\tilde{\text{Ai}}(N^{\frac{2}{3}} \zeta)| \quad \text{and} \quad |\delta_p(N, t)| \leq \frac{M_p}{N^{p-\frac{1}{6}}} |\tilde{\text{Bi}}(N^{\frac{2}{3}} \zeta)|,$$

where $\tilde{\text{Ai}}$ and $\tilde{\text{Bi}}$ are the modulus functions given in [7, (7.10) and (7.11)].

3. DETERMINATION OF $C_1(x)$ AND $C_2(x)$

First, we examine the behavior of $h_{n,\nu}(x)$ as $n \rightarrow \infty$. Here we do not use the result of (1.8) since it will provide only the value of $C_2(x)$ and not that of $C_1(x)$. We first note that when N is large, we have from the well-known asymptotic formulas of $J_\nu(1/x)$ and $Y_\nu(1/x)$:

$$(3.1) \quad J_{\nu+n} \left(\frac{1}{x} \right) = J_N \left(\frac{1}{x} \right) \sim \frac{1}{\sqrt{2\pi N}} \left(\frac{e}{2N} \right)^N,$$

$$(3.2) \quad Y_{\nu+n} \left(\frac{1}{x} \right) = Y_N \left(\frac{1}{x} \right) \sim -\sqrt{\frac{2}{\pi N}} \left(\frac{e}{2N} \right)^{-N};$$

see [5, p. 436]. For a negative order of the Bessel function, we know from [5, p. 243] that

$$(3.3) \quad J_{-\nu-n} \left(\frac{1}{x} \right) = J_{-N} \left(\frac{1}{x} \right) = \cos(N\pi) J_N(1/x) - \sin(N\pi) Y_N(1/x)$$

and

$$(3.4) \quad Y_{\nu-1}(z) = \frac{1}{\sin(\nu\pi)} (J_{\nu-1}(z) \cos(\nu\pi) + J_{-\nu+1}(z)).$$

Coupling (1.7), (3.1), (3.2), (3.3) and (3.4), we have

$$(3.5) \quad h_{n,\nu}(x) \sim \frac{e^N (2xN)^{-N}}{\sqrt{2\pi N}} \frac{\pi}{2x} Y_{\nu-1}(1/x) + (2x)^{N-1} \sqrt{2\pi} N^{N-\frac{1}{2}} e^{-N} J_{\nu-1}(1/x).$$

Next, we recall the well-known asymptotic formulas

$$(3.6) \quad \text{Ai}(\eta) \sim \frac{\eta^{-\frac{1}{4}}}{2\sqrt{\pi}} \exp \left(-\frac{2}{3} \eta^{\frac{3}{2}} \right), \quad \text{Ai}'(\eta) \sim -\frac{\eta^{\frac{1}{4}}}{2\sqrt{\pi}} \exp \left(-\frac{2}{3} \eta^{\frac{3}{2}} \right),$$

and

$$(3.7) \quad \text{Bi}(\eta) \sim \frac{\eta^{-\frac{1}{4}}}{\sqrt{\pi}} \exp \left(\frac{2}{3} \eta^{\frac{3}{2}} \right), \quad \text{Bi}'(\eta) \sim \frac{\eta^{\frac{1}{4}}}{\sqrt{\pi}} \exp \left(\frac{2}{3} \eta^{\frac{3}{2}} \right),$$

as $\eta \rightarrow +\infty$; see [5, pp. 392-393]. Furthermore, from (2.10) we have

$$(3.8) \quad \frac{2}{3} [\zeta(t)]^{3/2} \sim \log(2t) - 1 = \log(2xN) - 1.$$

Since $N = n + \nu$, it is readily seen from (3.6), (3.7) and (3.8) that

$$P_n(x) \sim \frac{1}{2\sqrt{\pi x N}} \frac{1}{(2xN)^N} e^N, \quad Q_n(x) \sim \frac{2^N}{\sqrt{\pi}} (xN)^{N-\frac{1}{2}} e^{-N}.$$

Comparing the two sides of (2.14) gives

$$(3.9) \quad C_1(x) = \frac{\pi}{\sqrt{2x}} Y_{\nu-1}(1/x), \quad C_2(x) = \frac{\pi}{\sqrt{2x}} J_{\nu-1}(1/x).$$

In summary, we have from (2.14), (2.15) and (2.16) that

$$(3.10) \quad \begin{aligned} h_{n,\nu}(x) = & \frac{\pi}{\sqrt{2x}} Y_{\nu-1}(1/x) \left(\frac{\zeta}{t^2-1} \right)^{\frac{1}{4}} N^{\frac{1}{6}} \left[\text{Ai} \left(N^{\frac{2}{3}} \zeta \right) + O \left(\frac{1}{N} \right) \right] \\ & + \frac{\pi}{\sqrt{2x}} J_{\nu-1}(1/x) \left(\frac{\zeta}{t^2-1} \right)^{\frac{1}{4}} N^{\frac{1}{6}} \left[\text{Bi} \left(N^{\frac{2}{3}} \zeta \right) + O \left(\frac{1}{N} \right) \right], \end{aligned}$$

where $x = N^{-1}t$ and $N = n + \nu$. This result holds uniformly for $0 \leq t < \infty$.

4. ZEROS

It is well known that the zeros of the modified Lommel polynomials $h_{n,\nu}(x)$ all lie in the interval $-j_{\nu-1,1}^{-1} \leq x \leq j_{\nu-1,1}^{-1}$. This can be seen from (1.5). After the rescaling $x = N^{-1}t$, the zeros of $h_{n,\nu}(t/N)$ all lie in the interval $-N/j_{\nu-1,1}^{-1} \leq t \leq N/j_{\nu-1,1}^{-1}$. In this section, we shall use the asymptotic approximation given in (3.10) to derive asymptotic formulas for the zeros of $h_{n,\nu}(t/N)$ as $n \rightarrow \infty$. We divide our discussion into three cases: (i) large zeros in $1 < t < N/j_{\nu-1,1}^{-1}$; (ii) small zeros in $0 < t < 1$; and (iii) zeros near the turning point $t = 1$ and on two sides of the point. Several tables are provided to give a comparison of the numerical values of the asymptotic zeros and the true zeros.

Case (i). Let $t_{n,m}$ denote the m -th zero of $h_{n,\nu}(t/N)$, arranged in ascending order:

$$(4.1) \quad -\frac{N}{j_{\nu-1,1}} \leq t_{n,1} < t_{n,2} < \cdots < t_{n,n-1} < t_{n,n} \leq \frac{N}{j_{\nu-1,1}}.$$

For large zeros, we have the following result.

Theorem 4.1. *The asymptotic behavior of the large zeros $t_{n,m}$ is given by*

$$(4.2) \quad t_{n,n-k} = \frac{N}{j_{\nu-1,k+1}} \left[1 + O \left(\frac{1}{N} \right) \right], \quad n \rightarrow \infty,$$

for fixed $k = 0, 1, 2, \dots$, where $N = n + \nu$.

Proof. Returning to (3.10), we have

$$(4.3) \quad \begin{aligned} h_{n,\nu} \left(\frac{t}{N} \right) = & \frac{\pi}{\sqrt{2x}} \left(\frac{\zeta}{t^2-1} \right)^{\frac{1}{4}} \left[N^{\frac{1}{6}} \left(Y_{\nu-1} \left(\frac{1}{x} \right) \text{Ai} \left(N^{\frac{2}{3}} \zeta \right) \right. \right. \\ & \left. \left. + J_{\nu-1} \left(\frac{1}{x} \right) \text{Bi} \left(N^{\frac{2}{3}} \zeta \right) \right) + O \left(\frac{1}{N} \right) \right]. \end{aligned}$$

Since $N = n + \nu$ and $\zeta > 0$ for $t > 1$, the terms involving $\text{Ai}(N^{2/3}\zeta)$ and $\text{Ai}'(N^{2/3}\zeta)$ are exponentially small, while the terms involving $\text{Bi}(N^{2/3}\zeta)$ and $\text{Bi}'(N^{2/3}\zeta)$ are

exponentially large; see (3.6) and (3.7). Thus, (4.3) reduces to

$$h_{n,\nu}\left(\frac{t}{N}\right) = \frac{\pi}{\sqrt{2x}} \left(\frac{\zeta}{t^2-1}\right)^{\frac{1}{4}} \left[N^{\frac{1}{6}} J_{\nu-1}\left(\frac{1}{x}\right) \text{Bi}(N^{\frac{2}{3}}\zeta) + O\left\{N^{-1} \exp\left(\frac{2}{3}N\zeta^{\frac{3}{2}}\right)\right\}\right].$$

If t is a zero of $h_{n,\nu}(t/N)$, the left-hand side of the equation is equal to zero, and we have

$$(4.4) \quad J_{\nu-1}\left(\frac{1}{x}\right) = O\left(\frac{1}{N}\right)$$

since $\text{Bi}(N^{2/3}\zeta)$ has no zero for $\zeta > 0$. In view of (4.1), the solutions of (4.4) are given in (4.2). $\square$

For $n = 100$ and $\nu = 1$, the numerical values of the large zeros are given in Table 1. Note that in this case, the polynomial $h_{n,\nu}(t/N)$ has 100 zeros; half of them lie on the positive real axis, and the other half lie on the negative real axis.

TABLE 1. True values and approximate values of large zeros when $n = 100$ and $\nu = 1$ evaluated with Maple.

m -th	true	(4.2)	error %
$t_{100,1,91}$	3.2969	3.2969	-1.3785×10^{-8}
$t_{100,1,92}$	3.6736	3.6736	1.6330×10^{-8}
$t_{100,1,93}$	4.1474	4.1474	-6.1147×10^{-9}
$t_{100,1,94}$	4.7615	4.7615	6.8524×10^{-9}
$t_{100,1,95}$	5.5890	5.5890	-5.8648×10^{-9}
$t_{100,1,96}$	6.7645	6.7645	-1.3906×10^{-8}
$t_{100,1,97}$	8.5655	8.5655	-1.4199×10^{-8}
$t_{100,1,98}$	11.6713	11.6713	3.5866×10^{-8}
$t_{100,1,99}$	18.2968	18.2968	3.0196×10^{-8}
$t_{100,1,100}$	41.9989	41.9989	-3.7573×10^{-9}

Case (ii). For the small zeros in $0 < t < 1$, we recall the asymptotic formulas

$$(4.5) \quad \text{Ai}(-\eta) = \frac{\eta^{-\frac{1}{4}}}{\sqrt{\pi}} \left[\cos\left(\frac{2}{3}\eta^{\frac{3}{2}} - \frac{1}{4}\pi\right) + O(\eta^{-\frac{3}{2}}) \right],$$

$$(4.6) \quad \text{Bi}(-\eta) = \frac{\eta^{-\frac{1}{4}}}{\sqrt{\pi}} \left[-\sin\left(\frac{2}{3}\eta^{\frac{3}{2}} - \frac{1}{4}\pi\right) + O(\eta^{-\frac{3}{2}}) \right],$$

$$(4.7) \quad J_{\nu}(\eta) = \sqrt{\frac{2}{\pi\eta}} \left[\cos\left(\eta - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(\eta^{-1}) \right],$$

$$(4.8) \quad Y_{\nu}(\eta) = \sqrt{\frac{2}{\pi\eta}} \left[\sin\left(\eta - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(\eta^{-1}) \right],$$

and the corresponding formulas of $\text{Ai}'(-\eta)$ and $\text{Bi}'(-\eta)$; see [5, pp. 392-393]. From these formulas, it is evident that if t is a zero of the modified Lommel polynomials $h_{n,\nu}(t/N)$, then equation (4.3) can be reduced to

$$(4.9) \quad \frac{\pi}{\sqrt{2x}} N^{\frac{1}{6}} \left[Y_{\nu-1}\left(\frac{1}{x}\right) \text{Ai}(N^{\frac{2}{3}}\zeta) + J_{\nu-1}\left(\frac{1}{x}\right) \text{Bi}(N^{\frac{2}{3}}\zeta) \right] = O\left(\frac{1}{N}\right).$$

Let $\tilde{t}_{n,m}$ denote the m -th zero of $h_{n,\nu}(t/N)$, arranged in ascending order:

$$(4.10) \quad 0 \leq \tilde{t}_{n,1} < \tilde{t}_{n,2} < \cdots < 1.$$

Since $\zeta(t) < 0$ for $0 < t < 1$, it is readily seen from (4.5) and (4.6) that the left-hand side of (4.9) can be written as

$$\begin{aligned} & \frac{\pi}{\sqrt{2x}} N^{\frac{1}{6}} Y_{\nu-1} \left(\frac{1}{x} \right) \text{Ai}(N^{\frac{2}{3}} \zeta) + \frac{\pi}{\sqrt{2x}} N^{\frac{1}{6}} J_{\nu-1} \left(\frac{1}{x} \right) \text{Bi}(N^{\frac{2}{3}} \zeta) \\ &= \frac{1}{|\zeta(t)|^{1/4}} \left[\sin \left(\frac{N}{t} - \frac{\nu\pi}{2} + \frac{\pi}{4} \right) \cos \left(\frac{2}{3} N |\zeta(t)|^{\frac{3}{2}} - \frac{\pi}{4} \right) \right. \\ & \quad \left. - \cos \left(\frac{N}{t} - \frac{\nu\pi}{2} + \frac{\pi}{4} \right) \sin \left(\frac{2}{3} N |\zeta(t)|^{\frac{3}{2}} - \frac{\pi}{4} \right) + O \left(\frac{1}{N} \right) \right] \\ &= \frac{1}{|\zeta(t)|^{1/4}} \left[\sin \left(\frac{N}{t} - \frac{\nu\pi}{2} + \frac{\pi}{4} - \frac{2}{3} N |\zeta(t)|^{\frac{3}{2}} + \frac{\pi}{4} \right) + O \left(\frac{1}{N} \right) \right]. \end{aligned}$$

Therefore, from (4.9) we have

$$(4.11) \quad \sin \left(\frac{N}{t} - \frac{\nu\pi}{2} + \frac{\pi}{4} - \frac{2}{3} N |\zeta(t)|^{\frac{3}{2}} + \frac{\pi}{4} \right) = O \left(\frac{1}{N} \right).$$

Note that near $t = 0$, $|\zeta(t)|^{\frac{3}{2}}$ has the expansion

$$(4.12) \quad \frac{2}{3} |\zeta(t)|^{\frac{3}{2}} = \frac{1}{t} - \frac{\pi}{2} + \frac{1}{2}t + \frac{1}{24}t^3 + \frac{1}{80}t^5 + O(t^6).$$

Upon solving (4.11) and (4.12), we have

$$(4.13) \quad \frac{n+1}{2}\pi - \frac{1}{2}Nt - \frac{1}{24}Nt^3 - \frac{1}{80}Nt^5 + \cdots = k\pi + O \left(\frac{1}{N} \right).$$

As a first approximation, we take just the first two terms on the left-hand side of the equation and obtain

$$(4.14) \quad \frac{n+1}{2}\pi - \frac{1}{2}Nt = k\pi + O \left(\frac{1}{N} \right).$$

The solutions of equation (4.14) are presented in two separate cases. In the order arranged in (4.10), when n is even, the small zeros are given by

$$(4.15) \quad \tilde{t}_{n,m+1} = \frac{2}{N} \left[\frac{\pi}{2} + m\pi + O \left(\frac{1}{N} \right) \right]$$

for fixed $m = 0, 1, 2, \dots$. Note that $\tilde{t}_{n,1} \neq 0$. When n is odd, the small zeros are given by $\tilde{t}_{n,1} = 0$ and

$$(4.16) \quad \tilde{t}_{n,m+1} = \frac{2}{N} \left[m\pi + O \left(\frac{1}{N} \right) \right]$$

for fixed $m = 0, 1, 2, \dots$.

Theorem 4.2. *The asymptotic behavior of the small zeros of $h_{n,\nu}(t/N)$, arranged in ascending order in (4.10), are given by the formulas in (4.15)–(4.16).*

The above result can be improved if we pick three terms in (4.13), in which case we have

$$(4.17) \quad \frac{n+1}{2}\pi - \frac{1}{2}Nt - \frac{1}{24}Nt^3 = k\pi + O \left(\frac{1}{N} \right).$$

This is a cubic equation, and we are looking for only the real root. Recall that the real root of the cubic equation

$$(4.18) \quad t^3 + pt + q = 0$$

is given explicitly by

$$(4.19) \quad t = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}},$$

provided that

$$(4.20) \quad \frac{q^2}{4} + \frac{p^3}{27} > 0.$$

The solutions of (4.17) are again presented in two separate cases. Let $K_1 = \frac{\pi}{2} + m\pi + O(\frac{1}{N})$ and $K_2 = m\pi + O(\frac{1}{N})$. When n is even, the small zeros are given by

$$(4.21) \quad \tilde{t}_{n,m+1} = \left[\frac{12}{N}K_1 + 12\sqrt{\left(\frac{K_1}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}} + \left[\frac{12}{N}K_1 - 12\sqrt{\left(\frac{K_1}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}}$$

for fixed $m = 0, 1, 2, \dots$. Note that $\tilde{t}_{n,1} \neq 0$. When n is odd, the small zeros are given by $\tilde{t}_{n,1} = 0$ and

$$(4.22) \quad \tilde{t}_{n,m+1} = \left[\frac{12}{N}K_2 + 12\sqrt{\left(\frac{K_2}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}} + \left[\frac{12}{N}K_2 - 12\sqrt{\left(\frac{K_2}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}}$$

for fixed $m = 0, 1, 2, \dots$.

It can be shown that as $n \rightarrow \infty$, the quantities on the right-hand sides of equations (4.21) – (4.22) are asymptotically equal to the quantities on the right-hand sides of (4.15) – (4.16), respectively. More precisely, we have

$$(4.23) \quad \left[\frac{12}{N}K_1 + 12\sqrt{\left(\frac{K_1}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}} + \left[\frac{12}{N}K_1 - 12\sqrt{\left(\frac{K_1}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}} \\ \sim \frac{2}{N} \left[\frac{\pi}{2} + m\pi + O\left(\frac{1}{N}\right) \right]$$

and

$$(4.24) \quad \left[\frac{12}{N}K_2 + 12\sqrt{\left(\frac{K_2}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}} + \left[\frac{12}{N}K_2 - 12\sqrt{\left(\frac{K_2}{N}\right)^2 + \frac{4}{9}} \right]^{\frac{1}{3}} \\ \sim \frac{2}{N} \left[m\pi + O\left(\frac{1}{N}\right) \right],$$

respectively. Note that the O – terms in (4.23) and (4.24) are still $O(1/N)$. To improve the order estimate from $O(1/N)$ to $O(1/N^2)$, one should use (2.14) and (3.9) with $p = 2$ in (2.15) and (2.16), instead of (3.10).

Numerical computation shows that the solutions given in (4.21) and (4.22) are more accurate than those presented in (4.15) and (4.16); see the numerics in Table 2. Use of numerical solutions of equation (4.11) would produce even better results.

TABLE 2. True values and approximate values of zeros evaluated with Maple, when $n = 100$ and $\nu = 1$.

m -th	true	(4.15), (4.16)	error %	(4.21), (4.22)	error %
$\tilde{t}_{100,1,1}$	0.0311	0.0311	0.0081	0.0311	0.0000
$\tilde{t}_{100,1,2}$	0.0932	0.0933	0.0726	0.0932	0.0001
$\tilde{t}_{100,1,3}$	0.1552	0.1555	0.2021	0.1552	0.0013
$\tilde{t}_{100,1,4}$	0.2169	0.2177	0.3973	0.2169	0.0053
$\tilde{t}_{100,1,5}$	0.2781	0.2799	0.6596	0.2782	0.0148

Case (iii). Now we consider the zeros near the turning point $t = 1$, that is, on both sides of $\sigma = 0$, where $\sigma = N^{2/3}\zeta(t)$ is the variable inside the Airy functions in (4.3). If t is a zero of $h_{n,\nu}(t/N)$ and σ is bounded, then equation (4.3) gives

$$(4.25) \quad \frac{\pi}{\sqrt{2}x} Y_{\nu-1} \left(\frac{1}{x} \right) \text{Ai}(\sigma) + \frac{\pi}{\sqrt{2}x} J_{\nu-1} \left(\frac{1}{x} \right) \text{Bi}(\sigma) = O \left(\frac{1}{N} \right),$$

which can be written as

$$(4.26) \quad \sin \left[\frac{N}{t} - \frac{\nu\pi}{2} + \frac{\pi}{4} + \arctan \frac{\text{Bi}(\sigma)}{\text{Ai}(\sigma)} \right] = O \left(\frac{1}{N} \right).$$

Since

$$(4.27) \quad \arctan \frac{\text{Bi}(\sigma)}{\text{Ai}(\sigma)} = \frac{1}{3}\pi + \frac{3}{4} \frac{3^{\frac{1}{3}}}{\pi} \left(\Gamma \left(\frac{2}{3} \right) \right)^3 \sigma - \frac{9}{16} \frac{3^{\frac{1}{6}}}{\pi^2} \left(\Gamma \left(\frac{2}{3} \right) \right)^4 \sigma^2 + O(\sigma^4)$$

for small σ , replacing $\arctan$ in (4.26) by its leading term given in (4.27) yields

$$(4.28) \quad \sin \left(\frac{N}{t} - \frac{\nu\pi}{2} + \frac{\pi}{4} + \frac{\pi}{3} \right) = O \left(\frac{1}{N} \right).$$

Recall that here we are only concerned with the zeros of the polynomials near $t = 1$ (i.e., σ close to 0). From (4.28), we obtain

$$(4.29) \quad \frac{N}{t\pi} - \frac{\nu}{2} + \frac{1}{4} + \frac{1}{3} = k + O \left(\frac{1}{N} \right),$$

where k is an integer. Recall that $\sigma = N^{2/3}\zeta(t)$. Hence, by Lagrange's inversion formula, we have from (2.12) that

$$(4.30) \quad t = 1 + \sum_{j=1}^{\infty} A_j (2^{-\frac{1}{3}} \sigma N^{-\frac{2}{3}})^j,$$

where $A_1 = 1$, $A_2 = \frac{7}{10}$, $A_3 = \frac{139}{350}$, $\dots$. Substituting (4.30) in (4.29) gives

$$(4.31) \quad \frac{N}{\pi} (1 - 2^{-\frac{1}{3}} \sigma N^{-\frac{2}{3}} + \dots) = k - \frac{7}{12} + \frac{\nu}{2} + O \left(\frac{1}{N} \right),$$

which in turn gives

$$(4.32) \quad 2^{-\frac{1}{3}} \frac{\sigma}{\pi} N^{\frac{1}{3}} = \frac{N}{\pi} - k + \frac{7}{12} - \frac{\nu}{2} + O \left(\frac{1}{N} \right).$$

Now, we choose k to be a large integer so that $\frac{N}{\pi} - k + \frac{7}{12} - \frac{\nu}{2}$ is a bounded quantity. Let us write (4.32) in the form

$$(4.33) \quad \sigma = \frac{t_1}{N^{1/3}} + \cdots$$

with

$$(4.34) \quad \frac{1}{2^{\frac{1}{3}}\pi}t_1 = \frac{N}{\pi} - k + \frac{7}{12} - \frac{\nu}{2}.$$

If we denote by N_ν the fractional part of $\frac{N}{\pi} + \frac{7}{12} - \frac{\nu}{2}$ (i.e., $N_\nu = \frac{N}{\pi} + \frac{7}{12} - \frac{\nu}{2} - [\frac{N}{\pi} + \frac{7}{12} - \frac{\nu}{2}]$), (4.34) can be expressed as

$$(4.35) \quad t_1 = 2^{\frac{1}{3}}\pi(m + N_\nu),$$

where m is an integer. Let $t_{n,m+1}^{(l)}$ and $t_{n,m+1}^{(r)}$, $m = 0, 1, 2, \dots$, denote, respectively, the $(m+1)$ th zero on the left-hand and on the right-hand side of the turning point $t = 1$, arranged in the order

$$(4.36) \quad \cdots < t_{n,2}^{(l)} < t_{n,1}^{(l)} < 1 < t_{n,1}^{(r)} < t_{n,2}^{(r)} < \cdots.$$

We can now write the asymptotic formulas as $n \rightarrow \infty$

$$(4.37) \quad t_{n,m+1}^{(r)} = 1 + \frac{2^{\frac{1}{3}}\pi}{N}(m + N_\nu)$$

and

$$(4.38) \quad t_{n,m+1}^{(l)} = 1 + \frac{2^{\frac{1}{3}}\pi}{N}(-m - 1 + N_\nu)$$

for fixed $m = 0, 1, 2, \dots$, where N_ν denotes the fractional part of $\frac{N}{\pi} + \frac{7}{12} - \frac{\nu}{2}$.

Theorem 4.3. *The asymptotic behavior of the zeros of $h_{n,\nu}(t/N)$ on two sides of the turning point $t = 1$, as arranged in (4.36), are given in (4.37) and (4.38).*

For $n = 100$ and $\nu = 1$, the numerical values of the first two zeros on the two sides of $t = 1$ are given in Table 3.

TABLE 3. True values and approximate values of zeros near the turning point $t = 1$ evaluated with Maple, when $n = 100$ and $\nu = 1$.

m -th	true	approx.	error %
$t_{100,1,1}^{(l)}$	0.9703	0.9761	0.5993
	1.0	1.0	
$t_{100,1,1}^{(r)}$	1.0086	1.0072	-0.1391

Asymptotic zero distribution of the modified Lommel polynomials has been investigated by Kuijlaars and Van Assche [4]. They used the three-term recurrence relation (1.2) to establish the zero distribution of $h_{n,\nu}(t/N)$ with $n/N \rightarrow c$ as $n \rightarrow \infty$. The zero distribution obtained by Kuijlaars and Van Assche [4] is

$$(4.39) \quad \mu_c(t) = \begin{cases} \frac{1}{\pi c t^2} \left(1 - \sqrt{1 - t^2 c^2}\right), & |t| < \frac{1}{c}, \\ \frac{1}{\pi c t^2}, & |t| \geq \frac{1}{c}. \end{cases}$$

(There is a minor error in the statement of their result; the functions $\alpha(t)$ and $\beta(t)$ should be $\alpha(t) = -1/t$ and $\beta(t) = 1/t$ in [4, p. 195]; therefore the zero distribution of the modified Lommel polynomials $h_{n,\nu}(t/N)$ should be the function given in (4.39), instead of the one in [4, (4.18)].) Note that in our case $N = n + \nu$, the ratio $n/N \rightarrow 1$ when $n \rightarrow \infty$, and the corresponding asymptotic zero distribution is

$$(4.40) \quad \mu(t) = \begin{cases} \frac{1}{\pi t^2} (1 - \sqrt{1 - t^2}), & |t| < 1, \\ \frac{1}{\pi t^2}, & |t| \geq 1. \end{cases}$$

The graph of the distribution function (4.40) is shown in Figure 1. The area under the curve of $\mu(t)$ in an interval $[a, b]$ represents the ratio

$$(4.41) \quad \frac{\text{the number of zeros in } [a, b]}{\text{total number of zeros}}.$$

FIGURE 1. The zero distribution of $h_{n,\nu}(t/N)$.

As an illustration, we take $n = 100$ and $\nu = 1$. We find that the polynomial $h_{n,\nu}(t/N)$ has 18 zeros in the interval $0 < t < 1$ and that the value of the integral of $\mu(t)$ on the interval $[0, 1]$ is 0.1817.

REFERENCES

- [1] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, New York, 1978. Mathematics and its Applications, Vol. 13. MR0481884 (58 #1979)
- [2] David Dickinson, *On Lommel and Bessel polynomials*, Proc. Amer. Math. Soc. **5** (1954), 946–956. MR0086897 (19,263c)
- [3] Mourad E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*. With two chapters by Walter Van Assche and a foreword by Richard A. Askey, Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge University Press, Cambridge, 2005. MR2191786 (2007f:33001)

- [4] A. B. J. Kuijlaars and W. Van Assche, *The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients*, J. Approx. Theory **99** (1999), no. 1, 167–197, DOI 10.1006/jath.1999.3316. MR1696553 (2000h:42015)
- [5] F. W. J. Olver, *Asymptotics and special functions*. Computer Science and Applied Mathematics, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. MR0435697 (55 #8655)
- [6] Z. Wang and R. Wong, *Uniform asymptotic expansion of $J_\nu(\nu a)$ via a difference equation*, Numer. Math. **91** (2002), no. 1, 147–193, DOI 10.1007/s002110100316. MR1896091 (2003g:33008)
- [7] Z. Wang and R. Wong, *Asymptotic expansions for second-order linear difference equations with a turning point*, Numer. Math. **94** (2003), no. 1, 147–194, DOI 10.1007/s00211-002-0416-y. MR1971216 (2004c:39012)
- [8] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England, 1944. MR0010746 (6,64a)

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