SUPERSOLVABLE REFLECTION ARRANGEMENTS

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ABSTRACT. Let $\mathcal{A} = (\mathcal{A}, V)$ be a complex hyperplane arrangement and let $L(\mathcal{A})$ denote its intersection lattice. The arrangement $\mathcal{A}$ is called supersolvable, provided its lattice $L(\mathcal{A})$ is supersolvable, a notion due to Stanley. Jambu and Terao showed that every supersolvable arrangement is inductively free, a notion previously due to Terao. So this is a natural subclass of this particular class of free arrangements.

Suppose that W is a finite, unitary reflection group acting on the complex vector space V . Let $\mathcal{A} = (\mathcal{A}(W), V)$ be the associated hyperplane arrangement of W . In a forthcoming paper by the authors, we determine all inductively free reflection arrangements.

The aim of this paper is to classify all supersolvable reflection arrangements. Moreover, we characterize the irreducible arrangements in this class by the presence of modular elements of rank 2 in their intersection lattice.

1. INTRODUCTION

Let $\mathcal{A} = (\mathcal{A}, V)$ be a complex hyperplane arrangement and let $L(\mathcal{A})$ denote its intersection lattice. We say that $\mathcal{A}$ is *supersolvable*, provided $L(\mathcal{A})$ is supersolvable; see Definition 2.2. Jambu and Terao [JT84, Thm. 4.2] have shown that every supersolvable arrangement is *inductively free*, a notion due to Terao [Ter80]; see Definition 2.1.

Now suppose that W is a finite, unitary reflection group acting on the complex vector space V . Let $\mathcal{A} = (\mathcal{A}(W), V)$ be the associated hyperplane arrangement of W . Terao [Ter80] has shown that each reflection arrangement $\mathcal{A}$ is free and that the multiset of exponents $\exp \mathcal{A}$ of $\mathcal{A}$ is given by the coexponents of W ; cf. [OT92, §6]. There is the stronger notion of inductive freeness referred to above; cf. Definition 2.1. In [HR12], we classified all inductively free reflection arrangements; cf. Theorem 1.1. In this paper we determine the subclass of all supersolvable reflection arrangements: apart from the braid and rank 2 arrangements, these consist only of the reflection arrangements of the groups $G(r, p, \ell)$ for $r, \ell \geq 2$ and $p \neq r$; cf. Theorem 1.2. In addition, the irreducible arrangements of this nature are characterized by the presence of a modular rank 2 element in the intersection lattice; see Theorem 1.3.

Within the theory of hyperplane arrangements supersolvability is a rather strong condition, as it implies essentially every desirable property, such as factored, free,

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fiber-type, $K(\pi, 1)$, rational $K(\pi, 1)$, etc.; see [FR85, (2.8)] for details. We briefly discuss three relevant properties.

Firstly, the Poincaré polynomial $\pi(\mathcal{A}, t)$ of the lattice $L(\mathcal{A})$ of a supersolvable arrangement $\mathcal{A}$ factors into linear terms over $\mathbb{Z}[t]$, thanks to a theorem of Stanley [Sta72]; see also [OT92, Thm. 2.63]. Precisely, $\pi(\mathcal{A}, t)$ factors into linear terms as follows:

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (1 + b_i t),$$

where the coefficients b_i are positive integers.

Secondly, as mentioned above, a supersolvable arrangement $\mathcal{A}$ is always inductively free [JT84, Thm. 4.2]; see also [OT92, Thm. 4.85]. So we can talk about the exponents $\exp \mathcal{A}$ of $\mathcal{A}$ in this case; cf. §2.2. In the factorization of $\pi(\mathcal{A}, t)$ above, the occurring coefficients b_i are precisely the exponents of $\mathcal{A}$ [OT92, Thm. 4.137].

Thirdly, if $\mathcal{A}$ is supersolvable, then $\mathcal{A}$ is a $K(\pi, 1)$ -arrangement, for Falk and Randell [FR85] proved that *fiber-type* arrangements are always $K(\pi, 1)$ (cf. [OT92, Prop. 5.12]), and by work of Terao [Ter86], the classes of fiber-type arrangements and of supersolvable arrangements coincide (cf. [OT92, Thm. 5.112]). Therefore, every supersolvable arrangement is $K(\pi, 1)$.

Since a supersolvable arrangement is inductively free, we only need to consider the latter for our classification. We recall the main result from [HR12].

Theorem 1.1. *For W a finite complex reflection group, the reflection arrangement $\mathcal{A}(W)$ of W is inductively free if and only if W does not admit an irreducible factor isomorphic to a monomial group $G(r, r, \ell)$ for $r, \ell \geq 3$, $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$, or G_{34} .*

Jambu and Terao have already observed that $\mathcal{A}(D_4)$ is not supersolvable [JT84, Ex. 5.5]. Thus, supersolvable reflection arrangements form a proper subclass of the class of inductively free reflection arrangements.

It is straightforward to see that any rank 2 arrangement is supersolvable; cf. Remark 2.3. In 1962, Fadell and Neuwirth [FN62] showed that the braid arrangement is fiber-type, and in 1973, Brieskorn [Br73] proved this for the reflection arrangement of the hyperoctahedral group. In 1983, using Brieskorn's iterated fibration method, Orlik and Solomon [OS83, §4] observed that indeed every $\mathcal{A}(G(r, p, \ell))$ is fiber-type, for $r, \ell \geq 2$ and $p \neq r$. So $\mathcal{A}$ is supersolvable in each of these cases. Our classification asserts that this list is actually complete.

Theorem 1.2. *For W a finite complex reflection group, $\mathcal{A}(W)$ is supersolvable if and only if any irreducible factor of W is of rank at most 2 or is isomorphic either to a Coxeter group of type A_ℓ or B_ℓ for $\ell \geq 3$ or to a monomial group $G(r, p, \ell)$ for $r, \ell \geq 3$ and $p \neq r$.*

In 1962, Fadell and Neuwirth [FN62] proved that the braid arrangement $\mathcal{A}(A_\ell)$ is $K(\pi, 1)$. In 1973, Brieskorn [Br73] extended this result to a large class of Coxeter groups and conjectured that this is the case for every Coxeter group. This was proved subsequently by Deligne [Del72], who showed that every *simplicial* arrangement is $K(\pi, 1)$. All reflection arrangements $\mathcal{A}(W)$ have been known to be $K(\pi, 1)$ since the late 1980s, with the exception of the ones for the six exceptional groups listed in Theorem 1.1; see [OT92, §6.6]. Since these arrangements are not

inductively free, they are not supersolvable (or fiber-type), and so one cannot easily deduce that they are also $K(\pi, 1)$. These outstanding cases were settled only recently by Bessis [Be06].

The definition of supersolvability of $\mathcal{A}$ entails the existence of modular elements in $L(\mathcal{A})$ of any possible rank; see §2.4 for the notion of modular elements. Strikingly, our second main result shows that irreducible, supersolvable reflection arrangements are characterized merely by the presence of a modular element of rank 2.

Theorem 1.3. *For W a finite, irreducible complex reflection group of rank at least 2, $\mathcal{A}(W)$ is supersolvable if and only if there exists a modular element of rank 2 in its lattice $L(\mathcal{A}(W))$.*

The condition of irreducibility in Theorem 1.3 is necessary; see Remark 2.6.

The paper is organized as follows. In §2 we recall the required notation and facts about supersolvability of arrangements and reflection arrangements from [OT92, §4, §6]. Further, in Proposition 2.5, we show that supersolvable arrangements behave well with respect to the product construction for arrangements. Using this fact, it is easy to construct non-supersolvable arrangements admitting modular elements of every possible rank, Remark 2.6.

Theorems 1.2 and 1.3 are proved in §3 in a sequence of lemmas. Here we provide a particularly useful inductive tool for showing that the reflection arrangement of a given reflection group W is not supersolvable provided W admits a suitable parabolic subgroup whose reflection arrangement is not supersolvable; cf. Lemma 3.1.

For general information about arrangements and reflection groups we refer the reader to [Bou68] and [OT92].

2. RECOLLECTIONS AND PRELIMINARIES

2.1. Hyperplane arrangements. Let $V = \mathbb{C}^\ell$ be an ℓ -dimensional complex vector space. A *hyperplane arrangement* is a pair $(\mathcal{A}, V)$, where $\mathcal{A}$ is a finite collection of hyperplanes in V . Usually, we simply write $\mathcal{A}$ in place of $(\mathcal{A}, V)$. We only consider central arrangements. The empty arrangement in V is denoted by Φ_ℓ .

The *lattice* $L(\mathcal{A})$ of $\mathcal{A}$ is the set of subspaces of V of the form $H_1 \cap \cdots \cap H_n$, where $\{H_1, \dots, H_n\}$ is a subset of $\mathcal{A}$. Note that V belongs to $L(\mathcal{A})$ as the intersection of the empty collection of hyperplanes. The lattice $L(\mathcal{A})$ is a partially ordered set by reverse inclusion: $X \leq Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathcal{A})$. We have a *rank* function on $L(\mathcal{A})$: $r(X) := \text{codim}_V(X)$. The *rank* $r(\mathcal{A})$ of $\mathcal{A}$ is the rank of a maximal element in $L(\mathcal{A})$ with respect to the partial order. With this definition $L(\mathcal{A})$ is a geometric lattice [OT92, p. 24]. The ℓ -arrangement $\mathcal{A}$ is called *essential* provided $r(\mathcal{A}) = \ell$.

The *product* $\mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ of two arrangements $(\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)$ is defined by

$$\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\};$$

see [OT92, Def. 2.13]. Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be the product of the two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$. We define a partial order on $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$ by $(X_1, X_2) \leq (Y_1, Y_2)$ provided $X_1 \leq Y_1$ and $X_2 \leq Y_2$. Then, by [OT92, Prop. 2.14], there is a lattice isomorphism

$$(2.1) \quad L(\mathcal{A}_1) \times L(\mathcal{A}_2) \cong L(\mathcal{A}) \quad \text{by} \quad (X_1, X_2) \mapsto X_1 \oplus X_2.$$

Note that $\mathcal{A} \times \Phi_0 = \mathcal{A}$ for any arrangement $\mathcal{A}$. If $\mathcal{A}$ is of the form $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_i \neq \Phi_0$ for $i = 1, 2$, then $\mathcal{A}$ is called *reducible*; otherwise $\mathcal{A}$ is *irreducible* [OT92, Def. 2.15]. For instance, the braid arrangement $\mathcal{A}(\ell)$ is the product of the empty 1-arrangement and an irreducible $(\ell - 1)$ -arrangement [OT92, Ex. 2.16].

2.2. Reflection groups and reflection arrangements. The irreducible finite complex reflection groups were classified by Shephard and Todd [ST54]. Let $W \subseteq \mathrm{GL}(V)$ be a finite complex reflection group. For $w \in W$, we write $\mathrm{Fix}(w) := \{v \in V \mid wv = v\}$ for the fixed point subspace of w . For $U \subseteq V$ a subspace, we define the *parabolic subgroup* W_U of W by $W_U := \{w \in W \mid U \subseteq \mathrm{Fix}(w)\}$.

The *reflection arrangement* $\mathcal{A} = \mathcal{A}(W)$ of W in V is the hyperplane arrangement consisting of the reflecting hyperplanes of the elements in W acting as reflections on V . By Steinberg's Theorem [Ste60, Thm. 1.5], for $U \subseteq V$ a subspace, the parabolic subgroup W_U is itself a complex reflection group, generated by the unitary reflections in W that are contained in W_U . This allows us to identify the reflection arrangement $\mathcal{A}(W_U)$ of W_U as a subarrangement of $\mathcal{A}$. This way, the lattice $L(\mathcal{A}(W_U))$ of $\mathcal{A}(W_U)$ is identified with a sublattice of $L(\mathcal{A})$. We make these identifications throughout.

Note that for $X \in L(\mathcal{A})$, we have $\mathcal{A}(W_X) = \mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$; cf. [OT92, Thm. 6.27, Cor. 6.28]. It follows that $L(\mathcal{A}(W_X)) = L(\mathcal{A}_X) = L(\mathcal{A})_X := \{Z \in L(\mathcal{A}) \mid X \subseteq Z\}$ by [OT92, Lem. 2.11].

Following [OT92, §6.4, App. C], we use the convention to label the W -orbit of $X \in L(\mathcal{A})$ by the type of the complex reflection group W_X . In that way we get a correspondence between the W -orbits in $L(\mathcal{A})$ and the conjugacy classes of parabolic subgroups of W .

2.3. Free and inductively free arrangements. Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V . If $x_1, \dots, x_\ell$ is a basis of V^* , then we identify S with the polynomial ring $\mathbb{C}[x_1, \dots, x_\ell]$. Letting S_p denote the $\mathbb{C}$ -subspace of S consisting of the homogeneous polynomials of degree p (along with 0), we see that S is naturally $\mathbb{Z}$ -graded: $S = \bigoplus_{p \in \mathbb{Z}} S_p$, where $S_p = 0$ for $p < 0$.

Let $\mathrm{Der}(S)$ be the S -module of $\mathbb{C}$ -derivations of S . For $i = 1, \dots, \ell$, let $D_i := \partial/\partial x_i$. Then $D_1, \dots, D_\ell$ is a $\mathbb{C}$ -basis of $\mathrm{Der}(S)$. We say that $\theta \in \mathrm{Der}(S)$ is *homogeneous of polynomial degree p* provided $\theta = \sum_{i=1}^{\ell} f_i D_i$, where $f_i \in S_p$ for each $1 \leq i \leq \ell$. In this case we write $\mathrm{pdeg} \theta = p$. Let $\mathrm{Der}(S)_p$ be the $\mathbb{C}$ -subspace of $\mathrm{Der}(S)$ consisting of all homogeneous derivations of polynomial degree p . Then $\mathrm{Der}(S)$ is a graded S -module: $\mathrm{Der}(S) = \bigoplus_{p \in \mathbb{Z}} \mathrm{Der}(S)_p$.

Following [OT92, Def. 4.4], for $f \in S$, we define the S -submodule $D(f)$ of $\mathrm{Der}(S)$ by $D(f) := \{\theta \in \mathrm{Der}(S) \mid \theta(f) \in fS\}$. Let $\mathcal{A}$ be an arrangement in V . Then for $H \in \mathcal{A}$ we fix $\alpha_H \in V^*$ with $H = \ker \alpha_H$. The *defining polynomial* $Q(\mathcal{A})$ of $\mathcal{A}$ is given by $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$.

The *module of $\mathcal{A}$ -derivations* of $\mathcal{A}$ is defined by $D(\mathcal{A}) := D(Q(\mathcal{A}))$. We say that $\mathcal{A}$ is *free* if the module of $\mathcal{A}$ -derivations $D(\mathcal{A})$ is a free S -module. The notion of freeness was introduced by Saito in his seminal work [S80].

With the $\mathbb{Z}$ -grading of $\mathrm{Der}(S)$, the module of $\mathcal{A}$ -derivations becomes a graded S -module $D(\mathcal{A}) = \bigoplus_{p \in \mathbb{Z}} D(\mathcal{A})_p$, where $D(\mathcal{A})_p = D(\mathcal{A}) \cap \mathrm{Der}(S)_p$ [OT92, Prop. 4.10]. If $\mathcal{A}$ is a free arrangement, then the S -module $D(\mathcal{A})$ admits a basis of ℓ homogeneous derivations, say $\theta_1, \dots, \theta_\ell$ [OT92, Prop. 4.18]. While the θ_i 's are not unique, their

polynomial degrees $\text{pdeg } \theta_i$ are unique (up to ordering). This multiset is the set of *exponents* of the free arrangement $\mathcal{A}$ and is denoted by $\exp \mathcal{A}$.

There is a stronger notion of freeness, motivated by the so-called *Addition-Deletion Theorem*; see [OT92, Thm. 4.51].

Definition 2.1. The class $\mathcal{IF}$ of *inductively free* arrangements is the smallest class of arrangements subject to

- (i) $\Phi_\ell \in \mathcal{IF}$ for each $\ell \geq 0$;
- (ii) if there exists an $H \in \mathcal{A}$ such that both the subarrangement $\mathcal{A} \setminus \{H\}$ of $\mathcal{A}$ and the restriction of $\mathcal{A}$ to H , $\mathcal{A}^H := \{H' \cap H \mid H' \in \mathcal{A} \setminus \{H\}\}$, belong to $\mathcal{IF}$, and $\exp \mathcal{A}^H \subseteq \exp(\mathcal{A} \setminus \{H\})$, then $\mathcal{A}$ also belongs to $\mathcal{IF}$.

Terao [Ter80] proved that each reflection arrangement is free. In [HR12], we classified all inductively free reflection arrangements; cf. Theorem 1.1.

2.4. Supersolvable arrangements. Let $\mathcal{A}$ be an arrangement. Following [OT92, §2], we say that $X \in L(\mathcal{A})$ is *modular* provided $X + Y \in L(\mathcal{A})$ for every $Y \in L(\mathcal{A})$. (This is not the actual definition of a modular element, but it is equivalent to the definition in our case [OT92, Cor. 2.26].) Let $\mathcal{A}$ be a central (and essential) ℓ -arrangement. The following notion is due to Stanley [Sta72].

Definition 2.2. We say that $\mathcal{A}$ is *supersolvable* provided there is a maximal chain

$$V = X_0 < X_1 < \dots < X_{\ell-1} < X_\ell = \{0\}$$

of modular elements X_i in $L(\mathcal{A})$; cf. [OT92, Def. 2.32].

This terminology owes to the fact that the lattice of subgroups of a finite supersolvable group satisfies the condition in Definition 2.2.

Remark 2.3. By [OT92, Ex. 2.28], V , $\{0\}$ and the members in $\mathcal{A}$ are always modular in $L(\mathcal{A})$. It follows that all 0-, 1-, and 2-arrangements are supersolvable.

As mentioned in the Introduction, supersolvable arrangements are always inductively free; see also [OT92, Thm. 4.58]. In general, a free 3-arrangement need not be supersolvable as such an arrangement need not be inductively free; see [OT92, Ex. 4.59].

The following observation is immediate from Definition 2.2 and Remark 2.3.

Lemma 2.4. A 3-arrangement $\mathcal{A}$ is supersolvable if and only if there exists a modular rank 2 element in $L(\mathcal{A})$.

Thanks to [OT92, Prop. 4.28], free arrangements behave well with respect to the product construction for arrangements. This is also the case for supersolvable arrangements.

Proposition 2.5. Let $\mathcal{A}_1, \mathcal{A}_2$ be two arrangements. Then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is supersolvable if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are supersolvable, and in that case the multiset of exponents of $\mathcal{A}$ is given by $\exp \mathcal{A} = \{\exp \mathcal{A}_1, \exp \mathcal{A}_2\}$.

Proof. Let $\mathcal{A}_i = (\mathcal{A}_i, V_i)$, $\ell_i = \dim V_i$ for $i = 1, 2$, let $(\mathcal{A}, V) = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$, and let $\ell = \ell_1 + \ell_2 = \dim V$.

First suppose that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are supersolvable with

$$V_i = X_i^0 < X_i^1 < \dots < X_i^{\ell_i} = \{0\}$$

being a maximal chain of modular elements in $L(\mathcal{A}_i)$ for $i = 1, 2$. For $0 \leq j \leq \ell_1$ set $Z^j := X_1^j \oplus V_2$ and for $\ell_1 + 1 \leq j \leq \ell$ set $Z^j := \{0\} \oplus X_2^{j-\ell_1} = X_2^{j-\ell_1}$. By (2.1), $Z^j \in L(\mathcal{A})$ for each $0 \leq j \leq \ell$. We claim that

$$V = Z^0 < Z^1 < \dots < Z^\ell = \{0\}$$

is a maximal chain of modular elements in $L(\mathcal{A})$. Clearly, this is a proper chain of elements in $L(\mathcal{A})$ of length ℓ by construction; cf. (2.1). Let $Y = Y_1 \oplus Y_2 \in L(\mathcal{A})$. Then, for $0 \leq j \leq \ell_1$,

$$Z^j + Y = (X_1^j \oplus V_2) + (Y_1 \oplus Y_2) = (X_1^j + Y_1) \oplus V_2$$

belongs to $L(\mathcal{A})$ by (2.1), since X_1^j is modular in $L(\mathcal{A}_1)$, so that $X_1^j + Y_1 \in L(\mathcal{A}_1)$. Likewise,

$$Z^j + Y = (\{0\} \oplus X_2^{j-\ell_1}) + (Y_1 \oplus Y_2) = Y_1 \oplus (X_2^{j-\ell_1} + Y_2)$$

lies in $L(\mathcal{A})$, for $\ell_1 + 1 \leq j \leq \ell$ by (2.1). As $Y \in L(\mathcal{A})$ is arbitrary, $\mathcal{A}$ is supersolvable.

Now suppose that $\mathcal{A}$ is supersolvable. Let

$$V = X^0 < X^1 < \dots < X^{\ell-1} < X^\ell = \{0\}$$

be a maximal chain of modular elements in $L(\mathcal{A})$. Then $X^j = X_1^j \oplus X_2^j$ for each $1 \leq j \leq \ell$ by (2.1). As each X^j is modular in $L(\mathcal{A})$, for every $Y = Y_1 \oplus Y_2 \in L(\mathcal{A})$, the sum

$$X^j + Y = (X_1^j \oplus X_2^j) + (Y_1 \oplus Y_2) = (X_1^j + Y_1) \oplus (X_2^j + Y_2)$$

belongs to $L(\mathcal{A})$. By (2.1), it follows that $X_i^j + Y_i$ belongs to $L(\mathcal{A}_i)$ for every $1 \leq j \leq \ell$ and $i = 1, 2$. Since $Y \in L(\mathcal{A})$ is arbitrary, each X_i^j is modular in $L(\mathcal{A}_i)$ for every $1 \leq j \leq \ell$ and $i = 1, 2$. Since $\ell = \ell_1 + \ell_2$, it follows from our construction that there are subsequences $1 \leq j_1 < \dots < j_{\ell_1}$ and $1 \leq k_1 < \dots < k_{\ell_2}$ of the integers from 1 to ℓ such that

$$V_1 = X_1^0 < X_1^{j_1} < \dots < X_1^{j_{\ell_1}} = \{0\}$$

and

$$V_2 = X_2^0 < X_2^{k_1} < \dots < X_2^{k_{\ell_2}} = \{0\}$$

are maximal chains of modular elements in $L(\mathcal{A}_1)$ and $L(\mathcal{A}_2)$, respectively. Thus both $\mathcal{A}_1$ and $\mathcal{A}_2$ are supersolvable, as desired.

Finally, the statement on the exponents follows from [OT92, Prop. 4.28] and the fact that supersolvable arrangements are free [JT84, Thm. 4.2]. $\square$

Our final observation in this section shows that it is easy to construct an arrangement $\mathcal{A}$ which is not supersolvable but still admits modular elements of every possible rank.

Remark 2.6. Suppose that $\mathcal{A}_i = (\mathcal{A}_i, V_i)$ (for $i = 1, 2$) are arrangements so that $\mathcal{A}_1$ is supersolvable but $\mathcal{A}_2$ is not and that $\ell_1 = \dim V_1 \geq \ell_2 = \dim V_2$. Consider the product $(\mathcal{A}, V) = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$. While $\mathcal{A}$ is again not supersolvable, thanks to Proposition 2.5, it is easy to see that $L(\mathcal{A})$ admits modular elements of every possible rank r for any $0 \leq r \leq \ell_1 + \ell_2 = r(\mathcal{A})$, for let

$$V_1 = X_0 < X_1 < \dots < X_{\ell_1} = \{0\}$$

be a maximal chain of modular elements in $L(\mathcal{A}_1)$. Then, by (2.1), $Z_s := X_s \oplus V_2$ and $Z'_t := X_t \oplus \{0\}$ belong to $L(\mathcal{A})$ for each $0 \leq s \leq \ell_2 \leq \ell_1$ and $1 \leq t \leq \ell_1$. Since $\text{codim}_V Z_s = \text{codim}_{V_1} X_s = s$ and $\text{codim}_V Z'_t = \text{codim}_{V_1} X_t + \dim V_2 = t + \ell_2$, the

rank of Z_s is s for each $0 \leq s \leq \ell_2$, and that of Z'_t is $\ell_2 + t$ for each $1 \leq t \leq \ell_1$. Now let $Y = Y_1 \oplus Y_2 \in L(\mathcal{A})$ be arbitrary. Then, since each X_s is modular in $L(\mathcal{A}_1)$, it follows from (2.1) that

$$Z_s + Y = (X_s \oplus V_2) + (Y_1 \oplus Y_2) = (X_s + Y_1) \oplus V_2 \in L(\mathcal{A})$$

and

$$Z'_t + Y = (X_t \oplus \{0\}) + (Y_1 \oplus Y_2) = (X_t + Y_1) \oplus Y_2 \in L(\mathcal{A}).$$

Hence Z_s is modular of rank s for $0 \leq s \leq \ell_2$ and Z'_t is modular of rank $\ell_2 + t$ for $1 \leq t \leq \ell_1$. In particular, $L(\mathcal{A})$ admits modular elements of every possible rank.

3. PROOFS OF THEOREMS 1.2 AND 1.3

Our first result allows us to conclude that $\mathcal{A}(W)$ is not supersolvable given that the reflection arrangement of a suitable parabolic subgroup of W is not supersolvable. While this is an elementary observation, it is nevertheless a rather effective inductive tool.

Lemma 3.1. *Let W be a complex reflection group and $\mathcal{A} = \mathcal{A}(W)$ its reflection arrangement. Suppose that there are $X \in L(\mathcal{A})$ and $r \in \mathbb{N}$ such that $L(\mathcal{A}(W_X))$ does not admit modular elements of rank r and that every W -orbit of elements in $L(\mathcal{A})$ of rank r meets $L(\mathcal{A}(W_X))$. Then $L(\mathcal{A})$ does not admit modular elements of rank r . In particular, $\mathcal{A}$ is not supersolvable.*

Proof. Let $Y \in L(\mathcal{A})$ be of rank r . Recall from Section 2.2 that $L(\mathcal{A}(W_X)) = L(\mathcal{A})_X$. Therefore, by our hypotheses there exists a $w \in W$ so that $wY \in L(\mathcal{A})_X$, and there is a $Z \in L(\mathcal{A})_X$ so that $U := wY + Z \notin L(\mathcal{A})_X$. It suffices to show that $U \notin L(\mathcal{A})$, as then $Y + w^{-1}Z$ does not belong to $L(\mathcal{A})$, and so Y is not modular. So, for a contradiction, suppose that $U \in L(\mathcal{A})$. Since both wY and Z belong to $L(\mathcal{A})_X$, we have $X \subseteq wY + Z = U$, and so $U \in L(\mathcal{A})_X$ which is absurd, since $U \notin L(\mathcal{A})_X$ by construction. Consequently, $U \notin L(\mathcal{A})$, as desired. $\square$

Note that the first condition on $L(\mathcal{A}(W_X))$ in Lemma 3.1 says that $\mathcal{A}(W_X)$ is not supersolvable.

Lemma 3.2. *If W is of type D_4 , F_4 , H_3 , or $W = G_{25}$ or G_{26} , then there are no modular elements in $L(\mathcal{A}(W))$ of rank 2. In particular, $\mathcal{A}(W)$ is not supersolvable.*

Proof. Let $\mathcal{A} = \mathcal{A}(W)$. Using the explicit data on the W -orbits in $L(\mathcal{A})$ from [OT92, §6, App. C], we choose an orbit representative X for each W -orbit in $L(\mathcal{A})$ of elements of rank 2, and give an explicit $Y \in L(\mathcal{A})$ in each case so that $X + Y \notin L(\mathcal{A})$. We denote the coordinate functions in S simply by a, b, c , etc., and for $f \in V^*$, we write H_f for $\ker f$.

(i) Let W be of type D_4 . Jambu and Terao have already observed that $\mathcal{A} = \mathcal{A}(D_4)$ is not supersolvable [JT84, Ex. 5.5]. We give a different, elementary argument showing that $L(\mathcal{A})$ does not admit modular elements of rank 2. The defining polynomial of $\mathcal{A} = \mathcal{A}(D_4)$ is

$$Q(D_4) := (a-b)(a+b)(a-c)(a+c)(a-d)(a+d) \\ (b-c)(b+c)(b-d)(b+d)(c-d)(c+d).$$

There are four W -orbits of elements of rank 2 in $L(\mathcal{A})$, corresponding to three conjugacy classes of parabolic subgroups of W of type A_1^2 and one of type A_2 . The conjugacy classes of type A_1^2 are fused by the action of the group of graph

automorphisms of W . It thus suffices to only consider one orbit in $L(\mathcal{A})$ with a parabolic subgroup of type A_1^2 and another one with point stabilizer of type A_2 , where representatives of these orbits are given by

$$X_1 = H_{a+b} \cap H_{a-b} \quad \text{and} \quad X_2 = H_{a-b} \cap H_{b-c},$$

respectively. One readily checks that $X_1 + (H_{b+d} \cap H_{b-d}) = H_b \notin \mathcal{A}$, as well as that $X_2 + (H_{a+b} \cap H_{c-d} \cap H_{c+d}) = H_{a+b-2c} \notin \mathcal{A}$. Thus both X_1 and X_2 are not modular.

(ii) Let W be of type F_4 . The defining polynomial of $\mathcal{A} = \mathcal{A}(F_4)$ is given by

$$\begin{aligned} Q(F_4) := & abcd(a+b)(b+c)(c+d)(b+2c)(a+b+c)(b+c+d)(a+b+2c) \\ & (a+b+c+d)(b+2c+d)(a+2b+2c)(a+b+2c+d)(b+2c+2d) \\ & (a+2b+2c+d)(a+b+2c+2d)(a+2b+3c+d)(a+2b+2c+2d) \\ & (a+2b+3c+2d)(a+2b+4c+2d)(a+3b+4c+2d)(2a+3b+4c+2d). \end{aligned}$$

There are four W -orbits of elements of rank 2 in $L(\mathcal{A})$ (cf. [OT92, Table C.9]) corresponding to the four conjugacy classes of parabolic subgroups of W of type A_2 , $\tilde{A}_2$, $A_1 \times \tilde{A}_1$, and B_2 with representatives

$$X_1 = H_a \cap H_b, \quad X_2 = H_c \cap H_d, \quad X_3 = H_{c+d} \cap H_{a+2b+2c+2d}, \quad \text{and} \quad X_4 = H_a \cap H_{b+c},$$

respectively. We claim that none of these elements is modular. First note that we have $X_1 + X_3 = H_{a+2b} \notin \mathcal{A}$, so both X_1 and X_3 are not modular. Moreover, one checks that $X_2 + (H_{a+2b+3c+d} \cap H_{a+2b+2c+2d}) = H_{c-d} \notin \mathcal{A}$ and also that $X_4 + (H_b \cap H_{a+b+c+d} \cap H_{a+2b+4c+2d}) = H_{a-2b-2c} \notin \mathcal{A}$, and so X_2 and X_4 are not modular either.

(iii) Let W be of type H_3 . The defining polynomial of $\mathcal{A} = \mathcal{A}(H_3)$ is

$$\begin{aligned} Q(H_3) := & abc(a - \omega b)(a - (\omega + 1)b)(b + c)(a + b)(a - \omega b - \omega c) \\ & (a - (\omega + 1)b - (\omega + 1)c)(a + b + c)(a - \omega b - (\omega + 1)c)(a - \omega b + c) \\ & (a + b + (\omega + 2)c)(a + b - (\omega + 1)c)(a - 2(\omega + 1)b - (\omega + 1)c), \end{aligned}$$

where $\omega = \eta^2 + \eta^3$ and η is a primitive 5-th root of unity. There are three W -orbits of elements of rank 2 in $L(\mathcal{A})$; cf. [OT92, Table C.4]. They correspond to the three conjugacy classes of parabolic subgroups of W of types $I_2(5)$, A_2 and A_1^2 with respective representatives

$$X_1 = H_a \cap H_b, \quad X_2 = H_a \cap H_{a-\omega b-(\omega+1)c}, \quad \text{and} \quad X_3 = H_c \cap H_{a-2(\omega+1)b-(\omega+1)c}.$$

One calculates that $X_1 + X_3 = H_{a-2(\omega+1)b}$ is not in $\mathcal{A}$ and moreover that $X_2 + (H_{a+b} \cap H_{a-\omega b+c}) = H_{2a-(\omega-1)b+c} \notin \mathcal{A}$; hence none of these elements is modular.

(iv) Let $W = G_{25}$. The defining polynomial of $\mathcal{A} = \mathcal{A}(G_{25})$ is

$$\begin{aligned} Q(G_{25}) := & abc(a + b + c)(a + b + \zeta c)(a + b + \zeta^2 c)(a + \zeta b + c) \\ & (a + \zeta b + \zeta c)(a + \zeta b + \zeta^2 c)(a + \zeta^2 b + c)(a + \zeta^2 b + \zeta c)(a + \zeta^2 b + \zeta^2 c), \end{aligned}$$

where ζ is a primitive 3-rd root of unity. There are two W -orbits of elements of rank 2 in $L(\mathcal{A})$ [OT92, Table C.6]. They correspond to the two conjugacy classes of parabolic subgroups of W isomorphic to $C(3)^2$ and G_4 with respective representatives

$$X_1 = H_a \cap H_b \quad \text{and} \quad X_2 = H_c \cap H_{a+b+c}.$$

Since $X_1 + X_2 = H_{a+b} \notin \mathcal{A}$, both X_1 and X_2 are not modular.

(v) Let $W = G_{26}$. Falk and Randell [FR87, (3.3)] already asserted that $\mathcal{A} = \mathcal{A}(G_{26})$ is not supersolvable. We give a different, elementary argument showing that $L(\mathcal{A})$ does not admit modular elements of rank 2. The defining polynomial of $\mathcal{A}$ is

$$\begin{aligned} Q(G_{26}) := & abc(a-b)(a-c)(b-c)(a-\zeta b)(a-\zeta^2 b)(a-\zeta c)(a-\zeta^2 c)(b-\zeta c) \\ & (b-\zeta^2 c)(a+b+c)(a+b+\zeta c)(a+b+\zeta^2 c)(a+\zeta b+c)(a+\zeta b+\zeta c) \\ & (a+\zeta b+\zeta^2 c)(a+\zeta^2 b+c)(a+\zeta^2 b+\zeta c)(a+\zeta^2 b+\zeta^2 c), \end{aligned}$$

where ζ is a primitive 3-rd root of unity. There are three W -orbits of elements of rank 2 in $L(\mathcal{A})$ [OT92, Table C.7]. They correspond to the three conjugacy classes of parabolic subgroups of W isomorphic to $A_1 \times C(3)$, G_4 , and $G(3, 1, 2)$ with representatives

$$X_1 = H_b \cap H_{a-\zeta c}, \quad X_2 = H_c \cap H_{a+\zeta b+c}, \quad \text{and} \quad X_3 = H_a \cap H_b,$$

respectively. One checks that $X_2 + X_3 = H_{a+\zeta b} \notin \mathcal{A}$ and moreover that $X_1 + (H_{a-b} \cap H_{b-\zeta^2 c}) = H_{a-(\zeta+2)b-\zeta c} \notin \mathcal{A}$, and so none of these elements is modular.

This completes the proof of Lemma 3.2. $\square$

Armed with Lemmas 3.1 and 3.2, we can now readily derive that a large class of unitary reflection groups does not admit a supersolvable reflection arrangement. For W of types E_6 , E_7 , E_8 , and H_4 , Falk and Randell had raised this question in [FR87, (2.1)].

Lemma 3.3. *If W is of type D_ℓ , for $\ell \geq 5$, E_6 , E_7 , E_8 , H_4 , or $W = G_{32}$, then there are no modular elements in $L(\mathcal{A}(W))$ of rank 2. In particular, $\mathcal{A}(W)$ is not supersolvable.*

Proof. Let $\mathcal{A} = \mathcal{A}(W)$. If W is one of the Weyl groups as in the statement, let W_X be a parabolic subgroup of W of type D_4 , if $W = H_4$, let $W_X = H_3$, and if $W = G_{32}$, then let $W_X = G_{25}$. By Lemma 3.2, $L(\mathcal{A}(W_X))$ does not admit modular elements of rank 2. From the information on the W -orbits in $L(\mathcal{A})$ given in [OT92, §6, App. C], we infer that with these choices, every W -orbit of elements in $L(\mathcal{A})$ of rank 2 meets the sublattice $L(\mathcal{A}(W_X))$. The desired result follows from Lemma 3.1. $\square$

Proof of Theorem 1.2. Thanks to Proposition 2.5, the question of supersolvability reduces to the case when $\mathcal{A} = \mathcal{A}(W)$ is irreducible. Therefore, we may assume that W is irreducible.

(i) Let W be of rank at most 2. Then $\mathcal{A}(W)$ is supersolvable by Remark 2.3.

(ii) Let W be an irreducible Coxeter group of rank at least 3. Fadell and Neuwirth [FN62] proved that the braid arrangement $\mathcal{A}(A_\ell)$ is fiber-type, and Brieskorn [Br73] showed that the reflection arrangement of the hyperoctahedral group $\mathcal{A}(B_\ell)$ is also fiber-type. Thus $\mathcal{A}(A_\ell)$ and $\mathcal{A}(B_\ell)$ are supersolvable by [Ter86]. It follows from Lemmas 3.2 and 3.3 that for all other types $\mathcal{A}(W)$ is not supersolvable.

(iii) Let W be a monomial group $G(r, p, \ell)$. Note that the reflection arrangements for $G(r, 1, \ell)$ and $G(r, p, \ell)$ for $p \neq r$ coincide. It thus suffices to consider the case $G(r, 1, \ell)$. Let $\mathcal{A} = \mathcal{A}(G(r, 1, \ell))$ for $r, \ell \geq 3$. Orlik and Solomon [OS83, §4] showed that $\mathcal{A}$ is fiber-type, and so $\mathcal{A}$ is supersolvable. We give an elementary,

direct argument for the supersolvability of $\mathcal{A}$. Let $H_i = \ker x_i$ and define $X_i = H_1 \cap H_2 \cap \dots \cap H_i$ for $1 \leq i \leq \ell$. Set $X_0 = V$. We claim that

$$V = X_0 < X_1 < \dots < X_\ell = \{0\}$$

is a maximal chain of modular elements in $L(\mathcal{A})$. Let $\zeta = e^{2\pi i/r}$ be a primitive r -th root of unity. The hyperplanes of $\mathcal{A}$ are of the form $H_i = \ker x_i$ for $1 \leq i \leq \ell$ and $H_{i,j}(m) = \ker(x_i - \zeta^m x_j)$ for $1 \leq i < j \leq \ell$ and $0 \leq m \leq r-1$. So for an intersection of such hyperplanes $Y \in L(\mathcal{A})$, the coordinates of the subspace Y are either 0 or some coincide, while others differ by a power of ζ . Now X_k consists of all vectors in V whose first k coordinates are zero and all others are arbitrary, for $1 \leq k \leq \ell$. Thus the relations among the coordinates of a vector in the sum $X_k + Y$ of the two subspaces are the same as for the ones in Y for the first k coordinates, and for higher indices there are no restrictions. So $X_k + Y$ coincides with the intersection of those hyperplanes H_i and $H_{i,j}(m)$ that define Y subject to the requirement that all occurring indices i, j are at most k . In particular, $X_k + Y$ belongs again to $L(\mathcal{A})$, and so X_k is modular for $1 \leq k \leq \ell$, as claimed.

Note that this argument also applies to the braid arrangement $\mathcal{A}(G(1, 1, \ell))$ (cf. [OT92, Ex. 2.33]), as well as to the reflection arrangement of the hyperoctahedral group $\mathcal{A}(G(2, 1, \ell))$.

By Theorem 1.1 and [JT84, Thm. 4.2], the reflection arrangements of the monomial groups $G(r, r, \ell)$ for $r, \ell \geq 3$ are not supersolvable.

(iv) Finally, let W be an irreducible, non-real, unitary reflection group of rank at least 3. By Theorem 1.1 and [JT84], the reflection arrangements of G_{24} , G_{27} , G_{29} , G_{31} , G_{33} , and G_{34} are not supersolvable; neither are the ones of G_{25} , G_{26} and G_{32} by Lemmas 3.2 and 3.3.

This completes the proof of Theorem 1.2. $\square$

We now attend to Theorem 1.3. Our first observation is immediate from Theorem 1.2 and Lemma 2.4.

Lemma 3.4. *If W is a monomial group $G(r, r, 3)$ for $r \geq 3$, G_{24} or G_{27} , then there are no modular elements in $L(\mathcal{A}(W))$ of rank 2.*

Lemma 3.5. *If W is a monomial group $G(r, r, 4)$ for $r \geq 3$, G_{29} or G_{31} , then there are no modular elements in $L(\mathcal{A}(W))$ of rank 2.*

Proof. Let $\mathcal{A} = \mathcal{A}(W)$. As before, we denote the coordinate functions in S simply by a, b, c , and d , and for $f \in V^*$, we write H_f for $\ker f$.

(i) Let $W = G(r, r, 4)$ for $r \geq 3$. There are three W -orbits in $L(\mathcal{A})$ of elements of rank 2 corresponding to the conjugacy classes of parabolic subgroups of types A_1^2 , A_2 and $I_2(r)$; cf. [OT92, §6.4]. Let $W_X = G(r, r, 3)$ be a parabolic subgroup, where $X \in L(\mathcal{A})$ is of rank 3. There are two W_X -orbits in $L(\mathcal{A}(W_X))$ of elements of rank 2 corresponding to the conjugacy classes of parabolic subgroups of types A_2 and $I_2(r)$. It follows from Lemma 3.4 and the argument in the proof of Lemma 3.1 that for each $Y \in L(\mathcal{A})$ of rank 2 whose W -orbit meets $L(\mathcal{A}(W_X))$, Y is not modular. This leaves the W -orbit in $L(\mathcal{A})$ whose elements have a point stabilizer of type A_1^2 to be considered. One such element is given by $Y = H_{a-b} \cap H_{c-d}$. But $Y + (H_{b-c} \cap H_{a-d}) = H_{a-b+c-d} \notin \mathcal{A}$, so Y is not modular.

(ii) Let $W = G_{29}$. There are three W -orbits in $L(\mathcal{A})$ of elements of rank 2 corresponding to the conjugacy classes of parabolic subgroups of types A_1^2 , A_2 and B_2 ; cf. [OT92, Table C.10]. Let $W_X = G(4, 4, 3)$ be a parabolic subgroup of W ,

where X has rank 3. There are two W_X -orbits in $L(\mathcal{A}(W_X))$ of elements of rank 2 corresponding to the conjugacy classes of parabolic subgroups of types A_2 and B_2 . It follows from Lemma 3.4 and the argument in the proof of Lemma 3.1 that for each $Y \in L(\mathcal{A})$ of rank 2 whose W -orbit meets $L(\mathcal{A}(W_X))$, Y is not modular. This leaves the W -orbit whose point stabilizer is of type A_1^2 to be considered. Let $i = \sqrt{-1}$. The defining polynomial of $\mathcal{A}$ is given by

$$\begin{aligned} Q(G_{29}) := & abcd(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \\ & (a+c)(a+b)(a+d)(b+c)(b+d)(c+d) \\ & (a-b+ic+id)(a-b+ic-id)(a-b-ic-id)(a-b-ic+id) \\ & (a+b+ic+id)(a+b-ic-id)(a+b-ic+id)(a+b+ic-id) \\ & (a-ib+ic+d)(a-ib-c-id)(a-ib-c+id)(a-ib+ic-d) \\ & (a-ib-ic+d)(a-ib+c-id)(a-ib-ic-d)(a+ib-c+id) \\ & (a+ib-c-id)(a+ib-ic+d)(a+ib-ic-d)(a+ib+c+id) \\ & (a+ib+ic+d)(a+ib+ic-d)(a-ib+c+id)(a+ib+c-id). \end{aligned}$$

A member in $L(\mathcal{A})$ with point stabilizer A_1^2 is given by $Y = H_{a-b+ic+id} \cap H_{a+ib-c-id}$. One checks that $Y + (H_{a+ib-ic+d} \cap H_{b-d}) = H_{a+(2i-1)b-ic-(i-2)d} \notin \mathcal{A}$, so Y is not modular.

(iii) Let $W = G_{31}$. Let $i = \sqrt{-1}$. The defining polynomial of $\mathcal{A} = \mathcal{A}(G_{31})$ is given by

$$\begin{aligned} Q(G_{31}) := & abcd(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \\ & (a+b)(a+c)(a+d)(b+c)(b+d)(c+d) \\ & (a-ib)(a-ic)(a-id)(b-ic)(b-id)(c-id) \\ & (a+ib)(a+ic)(a+id)(b+ic)(b+id)(c+id) \\ & (a-b-c-d)(a-b+c+d)(a-b+c-d)(a-b-c+d) \\ & (a+b+c+d)(a+b-c+d)(a+b-c-d)(a+b+c-d) \\ & (a-b-ic-id)(a-b+ic+id)(a-b-ic+id)(a-b+ic-id) \\ & (a+b-ic-id)(a+b+ic+id)(a+b+ic-id)(a+b-ic+id) \\ & (a-ib-c-id)(a-ib+c-id)(a-ib-c+id)(a-ib+c+id) \\ & (a-ib+ic+d)(a-ib+ic-d)(a-ib-ic+d)(a-ib-ic-d) \\ & (a+ib+c-id)(a+ib-c-id)(a+ib-c+id)(a+ib+c+id) \\ & (a+ib-ic+d)(a+ib-ic-d)(a+ib+ic-d)(a+ib+ic+d). \end{aligned}$$

There are three W -orbits of elements of rank 2 in $L(\mathcal{A})$; cf. [OT92, Table C.12]. They correspond to the three conjugacy classes of parabolic subgroups of W of types A_1^2 , A_2 , and $G(4, 2, 2)$ with respective representatives

$$X_1 = H_a \cap H_{b-c}, \quad X_2 = H_{a+id} \cap H_{a+b-c-d}, \quad \text{and} \quad X_3 = H_a \cap H_{a+ib}.$$

One checks that $X_1 + X_2 = H_{2a+(1+i)b-(1+i)c} \notin \mathcal{A}$, so that X_1 and X_2 are not modular. Further, one calculates that $X_3 + (H_{a-b-c-d} \cap H_{a-ic} \cap H_{a-b-c+d}) = H_{2a+(i-1)b} \notin \mathcal{A}$, so X_3 is not modular either.

This completes the proof of Lemma 3.5. $\square$

Lemma 3.6. *If W is a monomial group $G(r, r, \ell)$ for $r \geq 3$, $\ell \geq 5$, G_{33} , or G_{34} , then there are no modular elements in $L(\mathcal{A}(W))$ of rank 2.*

Proof. Let $\mathcal{A} = \mathcal{A}(W)$. If $W = G(r, r, \ell)$ for $r \geq 3$, $\ell \geq 5$, G_{33} , and G_{34} , let W_X be the parabolic subgroup $W_X = G(r, r, 4)$, $W_X = W(D_4)$, and $W_X = W(D_4)$, respectively. By Lemmas 3.2 and 3.5, $L(\mathcal{A}(W_X))$ does not admit modular elements of rank 2. From the information on the W -orbits in $L(\mathcal{A})$ given in [OT92, §6, Table C.14, C.17], we infer that with these choices, every W -orbit of elements in $L(\mathcal{A})$ of rank 2 meets the sublattice $L(\mathcal{A}(W_X))$. The desired result follows from Lemma 3.1. $\square$

Proof of Theorem 1.3. The forward implication is clear from Definition 2.2.

The reverse implication follows for general central arrangements of rank up to 3 by Remark 2.3 and Lemma 2.4. So assume that $\mathcal{A}(W)$ is irreducible of rank at least 4. It follows readily from Theorem 1.2 and Lemmas 3.2 through 3.6 that if $\mathcal{A}$ is not supersolvable, then there are no modular elements of rank 2 in $L(\mathcal{A})$. $\square$

While Theorem 1.3 asserts that an irreducible reflection arrangement is supersolvable precisely when its lattice admits a modular element of rank 2, this assertion is false for reducible reflection arrangements; see Remark 2.6.

The defining polynomials of the reflection arrangements for F_4 , H_3 , G_{25} , G_{26} and G_{31} used in the proofs of Lemmas 3.2 and 3.5 were obtained using the functionality for complex reflection groups provided by the CHEVIE package in GAP (and some GAP code by J. Michel) (see [S⁺97] and [GHL⁺96]).

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