SQUARE FUNCTIONS WITH GENERAL MEASURES

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ABSTRACT. We characterize the boundedness of square functions in the upper half-space with general measures. The short proof is based on an averaging identity over good Whitney regions.

1. INTRODUCTION

Let μ be a Borel measure on $\mathbb{R}^n$. We assume that $\mu(B(x, r)) \leq \lambda(x, r)$ for some $\lambda: \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty)$ satisfying the fact that $r \mapsto \lambda(x, r)$ is non-decreasing and $\lambda(x, 2r) \leq C_\lambda \lambda(x, r)$ for all $x \in \mathbb{R}^n$ and $r > 0$. Let

$$\theta_t f(x) = \int_{\mathbb{R}^n} s_t(x, y) f(y) d\mu(y), \quad x \in \mathbb{R}^n, t > 0,$$

where s_t is a kernel satisfying for some $\alpha > 0$ that

$$|s_t(x, y)| \lesssim \frac{t^\alpha}{t^\alpha \lambda(x, t) + |x - y|^\alpha \lambda(x, |x - y|)}$$

and

$$|s_t(x, y) - s_t(x, z)| \lesssim \frac{|y - z|^\alpha}{t^\alpha \lambda(x, t) + |x - y|^\alpha \lambda(x, |x - y|)}$$

whenever $|y - z| < t/2$. We use the ℓ^∞ metric on $\mathbb{R}^n$.

If $Q \subset \mathbb{R}^n$ is a cube with sidelength $\ell(Q)$, we define the associated Carleson box $\hat{Q} = Q \times (0, \ell(Q))$. In this note we will prove the following theorem:

1.1. Theorem. *Assume that there exists a function $b \in L^\infty(\mu)$ such that*

$$\left| \int_Q b(x) d\mu(x) \right| \gtrsim \mu(Q)$$

and

$$(1.2) \quad \iint_{\hat{Q}} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \mu(3Q)$$

for every cube $Q \subset \mathbb{R}^n$. Then it holds that

$$(1.3) \quad \iint_{\mathbb{R}_+^{n+1}} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \|f\|_{L^2(\mu)}^2, \quad f \in L^2(\mu).$$

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1.4. **Corollary.** *If*

$$\iint_{\widehat{Q}} |\theta_t \chi_Q(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \mu(Q)$$

for every cube $Q \subset \mathbb{R}^n$, then (1.3) holds.

To the best of our knowledge such a boundedness result was previously known only in the Lebesgue case (see [1], [8], [3], [2] for such results). Our framework covers, as is well-known, the doubling measures, the power bounded measures ($\mu(B(x, r)) \lesssim r^m$ for some m), and some other additional cases of interest (see Chapter 12 of [5] for an example in the context of Calderón–Zygmund operators).

The proof of our result follows by first establishing an averaging equality over good dyadic Whitney regions. Such an identity was inspired by Hytönen’s proof of the A_2 conjecture [4], which uses a very nice refinement of the Nazarov–Treil–Volberg method of random dyadic systems.

After this the probabilistic part of the proof ends, and we may study just one grid establishing a uniform (in the averaging parameter) bound for these good Whitney averages. Then we expand a function f in the same grid using the standard b -adapted martingale differences. It is not necessary to restrict this expansion into good cubes. The rest of the proof is a non-homogeneous Tb type summing argument (see e.g. [7] and [5]), which, in this setting, we manage to perform in a delightfully clear way. Indeed, it only takes a few pages. We find that the proof is of interest since it is, in particular, a very accessible application of the most recent non-homogeneous methods.

1.5. *Remark.* The property $\lambda(x, |x - y|) \sim \lambda(y, |x - y|)$ can be assumed without loss of generality. Indeed, in Proposition 1.1 of [6] it is shown that $\Lambda(x, r) := \inf_{z \in \mathbb{R}^n} \lambda(z, r + |x - z|)$ satisfies that $r \mapsto \Lambda(x, r)$ is non-decreasing, $\Lambda(x, 2r) \leq C_\lambda \Lambda(x, r)$, $\mu(B(x, r)) \leq \Lambda(x, r)$, $\Lambda(x, r) \leq \lambda(x, r)$ and $\Lambda(x, r) \leq C_\lambda \Lambda(y, r)$ if $|x - y| \leq r$. Therefore, we may (and do) assume that the dominating function λ satisfies the additional symmetry property $\lambda(x, r) \leq C\lambda(y, r)$ if $|x - y| \leq r$.

1.6. *Remark.* Condition (1.2) is necessary for (1.3) to hold. Indeed, one writes $b = b\chi_{3Q} + b\chi_{(3Q)^c}$ and notices that in (1.2) the part with $b\chi_{3Q}$ is dominated by $\|b\chi_{3Q}\|_{L^2(\mu)}^2 \lesssim \mu(3Q)$, if one assumes (1.3). For the other part, we note that for every $x \in Q$ it holds that

$$\begin{aligned} |\theta_t(b\chi_{(3Q)^c})(x)| &\lesssim \int_{(3Q)^c} \frac{t^\alpha}{|x - y|^\alpha \lambda(x, |x - y|)} d\mu(y) \\ &\lesssim t^\alpha \int_{Q^c} \frac{|y - c_Q|^{-\alpha}}{\lambda(c_Q, |y - c_Q|)} d\mu(y) \lesssim t^\alpha \ell(Q)^{-\alpha}. \end{aligned}$$

This implies that

$$\iint_{\widehat{Q}} |\theta_t(b\chi_{(3Q)^c})(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \mu(Q) \cdot \ell(Q)^{-2\alpha} \int_0^{\ell(Q)} t^{2\alpha-1} dt \lesssim \mu(Q).$$

The assumption of Corollary 1.4 is also necessary. However, even there one may weaken the assumption by replacing on the right-hand side $\mu(Q)$ with, say, $\mu(3Q)$ (note that Theorem 1.1 is true with $\mu(3Q)$ replaced by $\mu(\kappa Q)$, $\kappa > 1$).

2. PROOF OF THE MAIN THEOREM

2.1. A random dyadic grid. Let us be given a random dyadic grid $\mathcal{D} = \mathcal{D}(w)$, $w = (w_i)_{i \in \mathbb{Z}} \in (\{0, 1\}^{\mathbb{Z}})^{\mathbb{Z}}$. This means that $\mathcal{D} = \{Q + \sum_{i: 2^{-i} < \ell(Q)} 2^{-i} w_i : Q \in \mathcal{D}_0\} = \{Q + w : Q \in \mathcal{D}_0\}$, where we simply have defined $Q + w := Q + \sum_{i: 2^{-i} < \ell(Q)} 2^{-i} w_i$. Here $\mathcal{D}_0$ is the standard dyadic grid of $\mathbb{R}^n$.

We set $\gamma = \alpha/(2d + 2\alpha)$, where $\alpha > 0$ appears in the kernel estimates and $d := \log_2 C_\lambda$. A cube $Q \in \mathcal{D}$ is called bad if there exists another cube $\tilde{Q} \in \mathcal{D}$ so that $\ell(\tilde{Q}) \geq 2^r \ell(Q)$ and $d(Q, \partial \tilde{Q}) \leq \ell(Q)^\gamma \ell(\tilde{Q})^{1-\gamma}$. Otherwise it is good. One notes that $\pi_{\text{good}} := \mathbb{P}_w(Q + w \text{ is good})$ is independent of $Q \in \mathcal{D}_0$. The parameter r is a fixed constant so large that $\pi_{\text{good}} > 0$ and $2^{r(1-\gamma)} \geq 3$.

Furthermore, it is important to note that for a fixed $Q \in \mathcal{D}_0$ the set $Q + w$ depends on w_i with $2^{-i} < \ell(Q)$, while the goodness (or badness) of $Q + w$ depends on w_i with $2^{-i} \geq \ell(Q)$. In particular, these notions are independent.

2.2. Averaging over good Whitney regions. Let $f \in L^2(\mu)$. For $R \in \mathcal{D}$, let $W_R = R \times (\ell(R)/2, \ell(R))$ be the associated Whitney region. We can assume that w is such that $\mu(\partial R) = 0$ for every $R \in \mathcal{D} = \mathcal{D}(w)$ (this is the case for a.e. w). Using that $\pi_{\text{good}} := \mathbb{P}_w(R + w \text{ is good}) = E_w \chi_{\text{good}}(R + w)$ for any $R \in \mathcal{D}_0$ we may now write

$$\begin{aligned} \iint_{\mathbb{R}_+^{n+1}} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} &= E_w \sum_{R \in \mathcal{D}} \iint_{W_R} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} \\ &= E_w \sum_{R \in \mathcal{D}_0} \iint_{W_{R+w}} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} \\ &= \frac{1}{\pi_{\text{good}}} \sum_{R \in \mathcal{D}_0} \pi_{\text{good}} E_w \iint_{W_{R+w}} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} \\ &= \frac{1}{\pi_{\text{good}}} \sum_{R \in \mathcal{D}_0} E_w [\chi_{\text{good}}(R+w)] E_w \iint_{W_{R+w}} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} \\ &= \frac{1}{\pi_{\text{good}}} \sum_{R \in \mathcal{D}_0} E_w \left[\chi_{\text{good}}(R+w) \iint_{W_{R+w}} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} \right] \\ &= \frac{1}{\pi_{\text{good}}} E_w \sum_{R \in \mathcal{D}_{\text{good}}} \iint_{W_R} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t}. \end{aligned}$$

Notice that we used the independence of $\chi_{\text{good}}(R+w)$ and $\iint_{W_{R+w}} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t}$ for a fixed $R \in \mathcal{D}_0$. In [4], Hytönen used averaging equalities to represent a general Calderón–Zygmund operator as an average of dyadic shifts. These techniques are similar in spirit.

We now fix one w . It is enough to show that

$$(2.1) \quad \sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) \leq 2^s}} \iint_{W_R} |\theta_t f(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \|f\|_{L^2(\mu)}^2$$

with every large $s \in \mathbb{Z}$. Let us now fix the s as well.

2.3. Adapted decomposition of f . We now perform the standard b -adapted martingale difference decomposition of f . We define $\langle f \rangle_Q = \mu(Q)^{-1} \int_Q f \, d\mu$,

$$E_Q = \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \chi_Q b$$

and

$$\Delta_Q f = \sum_{Q' \in \text{ch}(Q)} \left[\frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'}} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \right] \chi_{Q'} b.$$

We can write in $L^2(\mu)$ that

$$f = \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 2^s}} \Delta_Q f + \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) = 2^s}} E_Q f.$$

It also holds that

$$\|f\|_{L^2(\mu)}^2 \sim \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 2^s}} \|\Delta_Q f\|_{L^2(\mu)}^2 + \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) = 2^s}} \|E_Q f\|_{L^2(\mu)}^2.$$

We plug this decomposition into (2.1) noting that we need to prove that

$$\sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) \leq 2^s}} \iint_{W_R} \left| \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 2^s}} \theta_t \Delta_Q f(x) \right|^2 d\mu(x) \frac{dt}{t} \lesssim \|f\|_{L^2(\mu)}^2,$$

where we abuse notation by redefining the operator Δ_Q to be $\Delta_Q + E_Q$ if $\ell(Q) = 2^s$.

2.4. The case $\ell(Q) < \ell(R)$. Here we show that

$$\sum_{R: \ell(R) \leq 2^s} \iint_{W_R} \left| \sum_{Q: \ell(Q) < \ell(R)} \theta_t \Delta_Q f(x) \right|^2 d\mu(x) \frac{dt}{t} \lesssim \|f\|_{L^2(\mu)}^2.$$

Let us set

$$A_{QR} = \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{D(Q, R)^\alpha \sup_{z \in Q \cup R} \lambda(z, D(Q, R))} \mu(Q)^{1/2} \mu(R)^{1/2},$$

where $D(Q, R) = \ell(Q) + \ell(R) + d(Q, R)$. Notice that $A_{QR} = A_{RQ}$. We record the following proposition (this is Proposition 6.3 of [5]):

2.2. Proposition. *It holds that*

$$\sum_{Q, R} A_{QR} x_Q y_R \lesssim \left(\sum_Q x_Q^2 \right)^{1/2} \left(\sum_R y_R^2 \right)^{1/2}$$

for $x_Q, y_R \geq 0$. In particular, it holds that

$$\left(\sum_R \left[\sum_Q A_{QR} x_Q \right]^2 \right)^{1/2} \lesssim \left(\sum_Q x_Q^2 \right)^{1/2}.$$

Notice that $\ell(Q) < \ell(R) \leq 2^s$ implies that $\int \Delta_Q f \, d\mu = 0$. Let $(x, t) \in W_R$. We write

$$\theta_t \Delta_Q f(x) = \int_Q [s_t(x, y) - s_t(x, c_Q)] \Delta_Q f(y) \, d\mu(y),$$

where c_Q is the center of the cube Q . Noting that $|y - c_Q| \leq \ell(Q)/2 \leq \ell(R)/4 < t/2$ for every $y \in Q$, we may estimate

$$\begin{aligned} |s_t(x, y) - s_t(x, c_Q)| &\lesssim \frac{\ell(Q)^\alpha}{\ell(R)^\alpha \lambda(x, \ell(R)) + d(Q, R)^\alpha \lambda(x, d(Q, R))} \\ &\lesssim \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{D(Q, R)^\alpha \sup_{z \in Q \cup R} \lambda(z, D(Q, R))}. \end{aligned}$$

The last estimate is seen as follows: In the numerator, simply estimate $\ell(Q)^\alpha \leq \ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}$. In the denominator we split into two cases. If $d(Q, R) \leq \ell(R)$ one has $D(Q, R) \lesssim \ell(R)$, while in the case $d(Q, R) > \ell(R)$ one has $D(Q, R) \lesssim d(Q, R)$. It remains to note that if $z \in Q \cup R$, then $|x - z| \lesssim D(Q, R)$.

We conclude that

$$|\theta_t \Delta_Q f(x)| \lesssim \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{D(Q, R)^\alpha \sup_{z \in Q \cup R} \lambda(z, D(Q, R))} \mu(Q)^{1/2} \|\Delta_Q f\|_{L^2(\mu)},$$

$$(x, t) \in W_R.$$

This yields that

$$\begin{aligned} \sum_{R: \ell(R) \leq 2^s} \iint_{W_R} \left| \sum_{Q: \ell(Q) < \ell(R)} \theta_t \Delta_Q f(x) \right|^2 d\mu(x) \frac{dt}{t} &\lesssim \sum_R \left[\sum_Q A_{QR} \|\Delta_Q f\|_{L^2(\mu)} \right]^2 \\ &\lesssim \sum_Q \|\Delta_Q f\|_{L^2(\mu)}^2 \lesssim \|f\|_{L^2(\mu)}^2. \end{aligned}$$

2.5. The case $\ell(Q) \geq \ell(R)$ and $d(Q, R) > \ell(R)^\gamma \ell(Q)^{1-\gamma}$. In this subsection we deal with

$$\sum_{R: \ell(R) \leq 2^s} \iint_{W_R} \left| \sum_{\substack{Q: \ell(R) \leq \ell(Q) \leq 2^s \\ d(Q, R) > \ell(R)^\gamma \ell(Q)^{1-\gamma}}} \theta_t \Delta_Q f(x) \right|^2 d\mu(x) \frac{dt}{t}.$$

Let $(x, t) \in W_R$. The size estimate gives that

$$|\theta_t \Delta_Q f(x)| \lesssim \int_Q \frac{\ell(R)^\alpha}{d(Q, R)^\alpha \lambda(x, d(Q, R))} |\Delta_Q f(y)| d\mu(y),$$

where we claim that

$$\frac{\ell(R)^\alpha}{d(Q, R)^\alpha \lambda(x, d(Q, R))} \lesssim \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{D(Q, R)^\alpha \lambda(x, D(Q, R))}.$$

This estimate is trivial if $d(Q, R) \geq \ell(Q)$, because in that case $D(Q, R) \lesssim d(Q, R)$. So assume that $d(Q, R) < \ell(Q)$, in which case $D(Q, R) \lesssim \ell(Q)$. This case is more tricky. Note that

$$\begin{aligned} \lambda(x, \ell(Q)) &= \lambda(x, (\ell(Q)/\ell(R))^\gamma \ell(R)^\gamma \ell(Q)^{1-\gamma}) \\ &\lesssim C_\lambda^{\log_2(\frac{\ell(Q)}{\ell(R)})^\gamma} \lambda(x, \ell(R)^\gamma \ell(Q)^{1-\gamma}) = \left(\frac{\ell(Q)}{\ell(R)} \right)^{\gamma d} \lambda(x, \ell(R)^\gamma \ell(Q)^{1-\gamma}). \end{aligned}$$

Using the assumption $d(Q, R) > \ell(R)^\gamma \ell(Q)^{1-\gamma}$ and the identity $\gamma d + \gamma \alpha = \alpha/2$, we conclude that

$$\frac{\ell(R)^\alpha}{d(Q, R)^\alpha \lambda(x, d(Q, R))} \lesssim \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{\ell(Q)^\alpha \lambda(x, \ell(Q))} \lesssim \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{D(Q, R)^\alpha \lambda(x, D(Q, R))}.$$

Noting again that if $z \in Q \cup R$, then $|x - z| \lesssim D(Q, R)$, we have shown that

$$|\theta_t \Delta_Q f(x)| \lesssim \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{D(Q, R)^\alpha \sup_{z \in Q \cup R} \lambda(z, D(Q, R))} \mu(Q)^{1/2} \|\Delta_Q f\|_{L^2(\mu)},$$

$$(x, t) \in W_R.$$

This is enough by Proposition 2.2, as in the previous subsection.

2.6. The case $\ell(R) \leq \ell(Q) \leq 2^r \ell(R)$ and $d(Q, R) \leq \ell(R)^\gamma \ell(Q)^{1-\gamma}$. Here we bound

$$\begin{aligned} & \sum_{R: \ell(R) \leq 2^s} \iint_{W_R} \left| \sum_{\substack{Q: \ell(R) \leq \ell(Q) \leq \min(2^s, 2^r \ell(R)) \\ d(Q, R) \leq \ell(R)^\gamma \ell(Q)^{1-\gamma}}} \theta_t \Delta_Q f(x) \right|^2 d\mu(x) \frac{dt}{t} \\ & \lesssim \sum_Q \sum_{R: R \sim Q} \iint_{W_R} |\theta_t \Delta_Q f(x)|^2 d\mu(x) \frac{dt}{t}, \end{aligned}$$

where we have written $Q \sim R$ to mean $\ell(Q) \sim \ell(R)$ and $d(Q, R) \lesssim \min(\ell(Q), \ell(R))$. We also used the fact that given R there are $\lesssim 1$ cubes Q for which $Q \sim R$.

Let $(x, t) \in W_R$. The size estimate gives that

$$\begin{aligned} |\theta_t \Delta_Q f(x)| & \lesssim \frac{\mu(Q)^{1/2}}{\lambda(x, t)^{1/2}} \frac{1}{\lambda(x, t)^{1/2}} \|\Delta_Q f\|_{L^2(\mu)} \\ & \lesssim \frac{\mu(Q)^{1/2}}{\lambda(c_Q, \ell(Q))^{1/2}} \frac{1}{\lambda(c_R, \ell(R))^{1/2}} \|\Delta_Q f\|_{L^2(\mu)} \leq \mu(R)^{-1/2} \|\Delta_Q f\|_{L^2(\mu)}. \end{aligned}$$

Therefore, we have that

$$\iint_{W_R} |\theta_t \Delta_Q f(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \|\Delta_Q f\|_{L^2(\mu)}^2,$$

and so

$$\sum_Q \sum_{R: R \sim Q} \iint_{W_R} |\theta_t \Delta_Q f(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \sum_Q \|\Delta_Q f\|_{L^2(\mu)}^2 \sum_{R: R \sim Q} 1 \lesssim \|f\|_{L^2(\mu)}^2.$$

2.7. The case $\ell(Q) > 2^r \ell(R)$ and $d(Q, R) \leq \ell(R)^\gamma \ell(Q)^{1-\gamma}$. We finally utilize the goodness of R to conclude that in this case we must actually have that $R \subset Q$. This means that

$$\begin{aligned} & \sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) \leq 2^s}} \iint_{W_R} \left| \sum_{\substack{Q: 2^r \ell(R) < \ell(Q) \leq 2^s \\ d(Q, R) \leq \ell(R)^\gamma \ell(Q)^{1-\gamma}}} \theta_t \Delta_Q f(x) \right|^2 d\mu(x) \frac{dt}{t} \\ & = \sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) < 2^{s-r}}} \iint_{W_R} \left| \sum_{k=r+1}^{s+\text{gen}(R)} \theta_t \Delta_{R^{(k)}} f(x) \right|^2 d\mu(x) \frac{dt}{t}, \end{aligned}$$

where $\text{gen}(R)$ is determined by $\ell(R) = 2^{-\text{gen}(R)}$, and $R^{(k)} \in \mathcal{D}$ is the unique cube for which $\ell(R^{(k)}) = 2^k \ell(R)$ and $R \subset R^{(k)}$. We decompose

$$\Delta_{R^{(k)}} f = -B_{R^{(k-1)}} \chi_{\mathbb{R}^n \setminus R^{(k-1)}} b + \sum_{\substack{S \in \text{ch}(R^{(k)}) \\ S \neq R^{(k-1)}}} \chi_S \Delta_{R^{(k)}} f + B_{R^{(k-1)}} b,$$

where

$$B_{R^{(k-1)}} = \langle \Delta_{R^{(k)}} f / b \rangle_{R^{(k-1)}} = \begin{cases} \frac{\langle f \rangle_{R^{(k-1)}}}{\langle b \rangle_{R^{(k-1)}}} - \frac{\langle f \rangle_{R^{(k)}}}{\langle b \rangle_{R^{(k)}}}, & \text{if } r+1 \leq k < s + \text{gen}(R), \\ \frac{\langle f \rangle_{R^{(k-1)}}}{\langle b \rangle_{R^{(k-1)}}}, & k = s + \text{gen}(R). \end{cases}$$

Noticing that

$$\sum_{k=r+1}^{s+\text{gen}(R)} B_{R^{(k-1)}} = \frac{\langle f \rangle_{R^{(r)}}}{\langle b \rangle_{R^{(r)}}},$$

we have that $\sum_{k=r+1}^{s+\text{gen}(R)} \theta_t \Delta_{R^{(k)}} f$ equals

$$- \sum_{k=r+1}^{s+\text{gen}(R)} B_{R^{(k-1)}} \theta_t (\chi_{\mathbb{R}^n \setminus R^{(k-1)}} b) + \sum_{k=r+1}^{s+\text{gen}(R)} \sum_{\substack{S \in \text{ch}(R^{(k)}) \\ S \neq R^{(k-1)}}} \theta_t (\chi_S \Delta_{R^{(k)}} f) + \frac{\langle f \rangle_{R^{(r)}}}{\langle b \rangle_{R^{(r)}}} \theta_t b.$$

Let us first deal with the last term. We bound

$$\begin{aligned} & \sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) < 2^{s-r}}} \frac{|\langle f \rangle_{R^{(r)}}|^2}{|\langle b \rangle_{R^{(r)}}|^2} \iint_{W_R} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} \\ & \lesssim \sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) < 2^{s-r}}} |\langle f \rangle_{R^{(r)}}|^2 \iint_{W_R} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} \\ & \leq \sum_{S \in \mathcal{D}} |\langle f \rangle_S|^2 \sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ S=R^{(r)}}} \iint_{W_R} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} =: \sum_{S \in \mathcal{D}} |\langle f \rangle_S|^2 a_S \lesssim \|f\|_{L^2(\mu)}^2, \end{aligned}$$

where the last estimate follows from the Carleson embedding theorem and the next lemma.

2.3. Lemma. *The sequence $(a_S)_{S \in \mathcal{D}}$ satisfies the Carleson condition; i.e. it holds that*

$$\sum_{S \subset R} a_S \lesssim \mu(R)$$

for every $R \in \mathcal{D}$.

Proof. Fix $R \in \mathcal{D}$, and let $\mathcal{F}(R)$ denote the maximal $Q \in \mathcal{D}$ such that $\ell(Q) \leq 2^{-r}\ell(R)$ and $d(Q, R^c) \geq 3\ell(Q)$. Notice that

$$\begin{aligned} \sum_{S \subset R} a_S &= \sum_{S \subset R} \sum_{\substack{Q \in \mathcal{D}_{\text{good}} \\ S=Q^{(r)}}} \iint_{W_Q} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} \\ &\leq \sum_{\substack{Q \in \mathcal{D} \\ \ell(Q) \leq 2^{-r}\ell(R) \\ d(Q, R^c) \geq 3\ell(Q)}} \iint_{W_Q} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} \\ &= \sum_{Q \in \mathcal{F}(R)} \sum_{\tilde{Q} \subset Q} \iint_{W_{\tilde{Q}}} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} \\ &\leq \sum_{Q \in \mathcal{F}(R)} \iint_{\tilde{Q}} |\theta_t b(x)|^2 d\mu(x) \frac{dt}{t} \lesssim \sum_{Q \in \mathcal{F}(R)} \mu(3Q) \lesssim \mu(R), \end{aligned}$$

where we used goodness, the fact that $2^{r(1-\gamma)} \geq 3$, our assumption about b and the fact that $\sum_{Q \in \mathcal{F}(R)} \chi_{3Q} \lesssim \chi_R$. $\square$

To complete the proof of our main theorem, it remains to control

$$\sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) < 2^{s-r}}} \iint_{W_R} \left| \sum_{k=r+1}^{s+\text{gen}(R)} B_{R^{(k-1)}} \theta_t(\chi_{\mathbb{R}^n \setminus R^{(k-1)}} b)(x) \right|^2 d\mu(x) \frac{dt}{t}$$

and

$$\sum_{\substack{R \in \mathcal{D}_{\text{good}} \\ \ell(R) < 2^{s-r}}} \iint_{W_R} \left| \sum_{k=r+1}^{s+\text{gen}(R)} \sum_{\substack{S \in \text{ch}(R^{(k)}) \\ S \neq R^{(k-1)}}} \theta_t(\chi_S \Delta_{R^{(k)}} f)(x) \right|^2 d\mu(x) \frac{dt}{t}.$$

By the accretivity condition for b , it holds that

$$|B_{R^{(k-1)}}| \lesssim \mu(R^{(k-1)})^{-1/2} \|\Delta_{R^{(k)}} f\|_{L^2(\mu)}.$$

Let $(x, t) \in W_R$. The size estimate gives that

$$\begin{aligned} |\theta_t(\chi_{\mathbb{R}^n \setminus R^{(k-1)}} b)(x)| &\lesssim \ell(R)^\alpha \int_{\mathbb{R}^n \setminus B(x, d(R, \mathbb{R}^n \setminus R^{(k-1)}))} \frac{|x-y|^{-\alpha}}{\lambda(x, |x-y|)} d\mu(y) \\ &\lesssim \ell(R)^\alpha d(R, \mathbb{R}^n \setminus R^{(k-1)})^{-\alpha} \lesssim 2^{-\alpha k/2}, \end{aligned}$$

where goodness was used to conclude that $d(R, \mathbb{R}^n \setminus R^{(k-1)}) \geq \ell(R)^{1/2} \ell(R^{(k-1)})^{1/2}$.

Then let $S \in \text{ch}(R^{(k)})$, $S \subset R^{(k)} \setminus R^{(k-1)}$. Notice that $d(R, S) \geq \ell(R)^\gamma \ell(S)^{1-\gamma}$. Therefore, an estimate as in subsection 2.5 gives that

$$|\theta_t(\chi_S \Delta_{R^{(k)}} f)(x)| \lesssim 2^{-\alpha k/2} \mu(R^{(k-1)})^{-1/2} \|\Delta_{R^{(k)}} f\|_{L^2(\mu)}.$$

What we need then readily follows from the following estimate:

$$\begin{aligned}
& \sum_{R: \ell(R) < 2^{s-r}} \mu(R) \left[\sum_{k=r+1}^{s+\text{gen}(R)} 2^{-\alpha k/2} \mu(R^{(k-1)})^{-1/2} \|\Delta_{R^{(k)}} f\|_{L^2(\mu)} \right]^2 \\
& \lesssim \sum_{R: \ell(R) < 2^{s-r}} \mu(R) \sum_{k=r+1}^{s+\text{gen}(R)} 2^{-\alpha k/2} \mu(R^{(k-1)})^{-1} \|\Delta_{R^{(k)}} f\|_{L^2(\mu)}^2 \\
& = \sum_{k=r+1}^{\infty} 2^{-\alpha k/2} \sum_{m=k-s}^{\infty} \sum_{S: \ell(S)=2^{k-m-1}} \|\Delta_{S^{(1)}} f\|_{L^2(\mu)}^2 \mu(S)^{-1} \sum_{\substack{R: \ell(R)=2^{-m} \\ R \subset S}} \mu(R) \\
& = \sum_{k=r+1}^{\infty} 2^{-\alpha k/2} \sum_{m=k-s}^{\infty} \sum_{S: \ell(S)=2^{k-m-1}} \|\Delta_{S^{(1)}} f\|_{L^2(\mu)}^2 \\
& \lesssim \sum_{k=r+1}^{\infty} 2^{-\alpha k/2} \sum_{m=k-s}^{\infty} \sum_{S: \ell(S)=2^{k-m}} \|\Delta_S f\|_{L^2(\mu)}^2 \lesssim \sum_{S: \ell(S) \leq 2^s} \|\Delta_S f\|_{L^2(\mu)}^2 \lesssim \|f\|_{L^2(\mu)}^2.
\end{aligned}$$

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