APPROXIMATED BY FINITE-DIMENSIONAL HOMOMORPHISMS INTO SIMPLE C^* -ALGEBRAS WITH TRACIAL RANK ONE

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ABSTRACT. We discuss when a unital homomorphism $\phi : C(X) \rightarrow A$ can be approximated by finite-dimensional homomorphisms, where X is a compact metric space and A is a unital simple C^* -algebra with tracial rank one. In this paper, we will give a necessary and sufficient condition.

1. INTRODUCTION

In the theory of C^* -algebras, studying homomorphisms between two C^* -algebras is of fundamental importance. As a simple step, but also an important one, we study homomorphisms from some commutative C^* -algebra $C(X)$, where X is a compact metric space, into some simple C^* -algebra. Among these homomorphisms, the ones defined by evaluation at some finite points in X are the most simple cases, or equivalently, the ones with finite-dimensional range (we call them finite-dimensional homomorphisms). Now, it is natural to study their limits (in point-wise convergence topology).

In this paper, the target C^* -algebra of a homomorphism is in an important class of simple C^* -algebras in classification theory, the unital simple C^* -algebras with tracial rank no more than one. It was introduced by H. Lin to aid the program of classification of nuclear C^* -algebras ([Lin2]). H. Lin completely classified the unital nuclear separable simple C^* -algebras with tracial rank one which satisfy the UCT; see [Lin3] and [Lin5]. Let A be a unital simple C^* -algebra with tracial rank no more than one, and consider a unital monomorphism $\phi : C(X) \rightarrow A$. (For this problem, we only need to consider a monomorphism.) When X is path-connected and A is of tracial rank zero, it is proved that ϕ can be approximated by finite-dimensional homomorphisms if and only if ϕ induces a zero element in $KL(C_0(X), A)$ (see [HLX]). In the present paper, we shall extend the result to the tracial rank one case. It is worth mentioning that the latter C^* -algebras are not of real rank zero. It turns out that ϕ can be approximated by finite-dimensional homomorphisms

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if and only if $[\phi]$ vanishes on $\underline{K}(C_0(X))$, and, in addition, the induced maps $\widehat{\phi}$ maps $\text{Aff}T(C(X))$ into $\overline{\rho_A(K_0(A))}$ and $\phi^\dagger$ is trivial. For a general compact metric space X (not necessarily path-connected), we need some new generalized notation to describe $[\phi]$ (see equation (3.6)).

In the literature, this problem is related to the properties such as real rank zero, (FU) and (FN), corresponding to $X = [0, 1]$, $\mathbb{T}$, and a compact subset in the complex plane, respectively ([Lin1], [Lin9] and [Lin13]). In [Lin9], it is shown that a unitary in a unital simple C^* -algebra with tracial rank one can be approximated by unitaries with finite spectrum if and only if $u \in CU(A)$ and $u^n + \widehat{(u^n)^*}, i(\widehat{u^n - (u^n)^*}) \in \overline{\rho_A(K_0(A))}$ for all $n \geq 1$. As an application of the main result in this paper, we shall describe when a normal element in a unital simple C^* -algebra with tracial rank one can be approximated by normal elements with finite spectrum. In this paper, all C^* -algebras are assumed to be separable.

2. PRELIMINARIES

In this section, we gather some notation and well-known facts.

2.1. Let A, B be two C^* -algebras and let $\phi, \psi : A \rightarrow B$ be two maps. Suppose that $\mathcal{F} \subset A$ and $\epsilon > 0$. We write

$$\phi \approx_\epsilon \psi \quad \text{on } \mathcal{F}$$

if $\|\phi(x) - \psi(x)\| < \epsilon$ for all $a \in \mathcal{F}$. Similarly, we write

$$\phi \approx_\epsilon \text{adu} \circ \psi \quad \text{on } \mathcal{F}$$

if there is a unitary $u \in B$ such that $\|\phi(x) - u\psi(x)u^*\| < \epsilon$ for all $a \in \mathcal{F}$.

2.2. Suppose that $X = X_1 \sqcup \cdots \sqcup X_m$ is a disjoint union of path-connected compact metric spaces with each component X_i a base point $x_i \in X_i$ for $i = 1, \dots, m$. Let $C_0(X)$ be the set of continuous functions on X which vanish at all x_i . Put $rC(X) = C(X)/C_0(X) \cong \mathbb{C}^m$ ([DN] and [EG]).

Let A be a unital C^* -algebra and $\phi : C(X) \rightarrow A$ be a unital homomorphism. Then ϕ defines an element $[\phi]$ in $KK(C(X), A)$. We know that there is a split short exact sequence

$$0 \rightarrow C_0(X) \rightarrow C(X) \rightarrow rC(X) \rightarrow 0.$$

Then there is a natural decomposition,

$$KK(C(X), A) = KK(C_0(X), A) \oplus KK(rC(X), A).$$

If $\beta \in KK(C(X), A)$, we will write $\beta = (\beta_0, \beta_1)$ under this decomposition. In particular, suppose that $\phi : C(X) \rightarrow A$ is a unital homomorphism and denote by $e_i \in C(X)$ the identity of $C(X_i)$. Then $[\phi] = ([\phi]_0, [p_1], \dots, [p_m]) \in KK(C_0(X), A) \oplus KK(rC(X), A) = KK(C_0(X), A) \oplus K_0(A) \oplus \cdots \oplus K_0(A)$, where $p_i = \phi(e_i)$, $i = 1, \dots, m$.

From the universal coefficient theorem (see [RS] or [Bl]), for the C^* -algebras $C = C(X)$ and A as above, there is a split short exact sequence

$$0 \rightarrow \text{Ext}^1(K_*(C), K_*(A)) \rightarrow KK(C, A) \xrightarrow{\gamma} \text{Hom}^0(K_*(C), K_*(A)) \rightarrow 0.$$

Define

$$KL(C, A) = KK(C, A)/\text{Pext}^1(K_*(C), K_*(A)).$$

Then ϕ induces an element $[\phi]$ in $KL(C, A)$.

2.3. Let A be a C^* -algebra and let C_n be a commutative C^* -algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. We use the following notation:

$$K_*(A, \mathbb{Z}/n\mathbb{Z}) = K_*(A \otimes C_n)$$

and

$$\underline{K}(A) = K_*(A) \oplus \bigoplus_{n=2}^{\infty} K_*(A, \mathbb{Z}/n\mathbb{Z}).$$

Denote by $\text{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B))$ the set of systems of group homomorphisms which is compatible with all the Bockstein Operations (see [DL] for details). From [DL], we know that

$$KL(A, B) = \text{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B))$$

for A a separable amenable C^* -algebra which satisfies the UCT.

If $\alpha \in KL(C(X), B)$, we use the notation $\alpha = \{\alpha_n^i\}$, where

$$\alpha_n^i \in \text{Hom}(K_i(C(X)), \mathbb{Z}/n\mathbb{Z}) \rightarrow K_i(B, \mathbb{Z}/n\mathbb{Z}), \quad i = 0, 1.$$

2.4. Let X be a compact metric space and $x, y \in X$. Let $\delta > 0$. We write $x \sim_{\delta} y$ if there are points $x_0, x_1, \dots, x_m$ in X such that

$$x_0 = x, x_m = y, \text{ and } \text{dist}(x_i, x_{i+1}) < \delta,$$

for $i = 0, 1, \dots, m-1$. A subset $Y \subset X$ is said to be δ -connected if for any two points x, y in Y , one has $x \sim_{\delta} y$.

It is well-known that for any $\delta > 0$, X can be divided into finitely many disjoint δ -connected components. We will frequently use the following lemma.

Lemma 2.1 (Lemma 3.3 of [HLX]). *Let X be a compact metric space and $G \subset \underline{K}(C(X))$ be a finitely generated subgroup. Then there exists $\delta > 0$ satisfying the following:*

If $X = X_1 \sqcup \dots \sqcup X_m$, where each X_i is δ -connected, A is a unital C^ -algebra, and $\phi, \psi : C(X) \rightarrow A$ are two unital finite-dimensional homomorphisms such that $[\phi]([e_i]) = [\psi]([e_i])$ for $i = 1, \dots, m$, then*

$$[\phi] \mid_G = [\psi] \mid_G.$$

2.5. For a unital C^* -algebra A , let $T(A)$ denote the space of all tracial states of A . It is well-known that $T(A)$ is a Choquet simplex. Let $\text{Aff}T(A)$ be the space of all affine continuous real functions on $T(A)$. Then $\text{Aff}T(A)$ is an ordered Banach space with order unit.

If X is a compact Hausdorff space, then it is well-known that $\text{Aff}T(C(X)) = C_{\mathbb{R}}(X)$, the space of all real continuous functions on X .

2.6. Let $\phi : C \rightarrow A$ be a unital homomorphism. Denote by $\phi^T : T(A) \rightarrow T(C)$ the affine continuous map induced by ϕ , i.e. $\phi^T(\tau) = \tau \circ \phi$ for all τ in $T(A)$. It then also induces a unital positive linear map $\widehat{\phi} : \text{Aff}T(C) \rightarrow \text{Aff}T(A)$ defined by $\widehat{\phi}(f) = f \circ \phi^T$ for all f in $\text{Aff}T(C)$.

Let A be a unital C^* -algebra, and denote by $(K_0(A), K_0(A)^+, [1_A])$ the associated scaled ordered group and $SK_0(A)$ the state space of $K_0(A)$ (see [R]). There is an affine continuous map $r_A : T(A) \rightarrow SK_0(A)$ defined by $r_A(\tau)([p]) = \sum_{i=1}^n \tau(p_{ii})$, where $\tau \in T(A)$ and $[p] \in K_0(A)$ is the element presented by the projection $p \in M_n(A)$. Then r_A defines a canonical map $\rho_A : K_0(A) \rightarrow \text{Aff}T(A)$ by $\rho_A(g)(\tau) = r_A(\tau)(g)$ for τ in $T(A)$ and g in $K_0(A)$.

Let $\pi_A : \text{Aff}T(A) \rightarrow \text{Aff}T(A)/\overline{\rho_A(K_0(A))}$ denote the canonical quotient map.

2.7. Let $U(A)$ be the unitary group of A and $CU(A)$ be the closure of the commutator subgroup of $U(A)$. Using the de la Harpe-Skandalis determinant, by Theorem 3.2 of [Th], one has the following split short exact sequence:

$$(2.1) \quad 0 \rightarrow \text{Aff}T(A)/\overline{\rho_A(K_0(A))} \rightarrow U_\infty(A)/CU_\infty(A) \rightarrow K_1(A) \rightarrow 0.$$

We then have

$$(2.2) \quad U_\infty(A)/CU_\infty(A) \cong \text{Aff}T(A)/\overline{\rho_A(K_0(A))} \oplus K_1(A).$$

For a unital homomorphism $\phi : C \rightarrow A$, it induces a group homomorphism $\phi^\dagger : U_\infty(C)/CU_\infty(C) \rightarrow U_\infty(A)/CU_\infty(A)$. With respect to the decomposition (2.2), we can view $\phi^\dagger$ as a matrix

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

Here α_{21} is automatically zero by exactness, α_{11} is induced by $\widehat{\phi}$, and α_{22} is the map ϕ_{*1} . Denote the remaining homomorphism, α_{12} , by

$$\phi^\dagger : K_1(C) \rightarrow \text{Aff}T(A)/\overline{\rho_A(K_0(A))}.$$

2.8. We shall recall the definition and some basic properties of unital simple C^* -algebras of tracial ranks (see [Lin2] and [Lin5]).

Let A be a unital simple C^* -algebra and $k \in \mathbb{N}$. We say that A has **tracial rank no more than k** if for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$ and any non-zero positive element $a \in A_+$, there exist a non-zero projection $p \in A$ and a C^* -subalgebra B of A with $1_B = p$ such that

(1) B has form $B = \bigoplus_{i=1}^q M_{n_i}(C(X_i))$, where X_i is a finite CW complex with $\dim(X_i) \leq k$ for each i ,

(2) $\|px - xp\| < \epsilon$ for all $x \in \mathcal{F}$,

(3) $pxp \in_\epsilon B$ for all $x \in \mathcal{F}$,

(4) $[1 - p] \leq [a]$.

We will write $TR(A) \leq k$ if A has tracial rank no more than k . Especially, $TR(A) = 0$ means that A has tracial rank zero; i.e. the above B can be chosen to be finite-dimensional.

Recently, H. Lin proved that if a unital simple C^* -algebra A with $TR(A) \leq k$ satisfies the UCT, then A actually has tracial rank no more than one (see [Lin12]). Hence we focus on A with $TR(A) \leq 1$.

Suppose that A is a unital simple C^* -algebra with $TR(A) \leq 1$. Then it is well-known that A has stable rank one, real rank no more than one, and a weakly unperforated K_0 -group with Riesz interpolation property and fundamental comparison property (see [Lin5]). For the $TR(A) = 0$ case, we know that A has real rank zero and the canonical map $\rho_A : K_0(A) \rightarrow \text{Aff}T(A)$ in 2.6 has dense range.

3. MAIN RESULTS

First of all, we discuss the case when X is path-connected. The following result is the main theorem in [HLX].

Lemma 3.1 (Theorem 3.9 of [HLX]). *Let X be a compact path-connected metric space and A be a unital simple C^* -algebra with $TR(A) = 0$. Suppose that $\phi : C(X) \rightarrow A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$[\phi]_0 = 0 \text{ in } KL(C_0(X), A). \quad \square$$

By using the uniqueness theorem recently proved by H. Lin ([Lin11]) and the method used in [Lin9], we obtain the following result.

Theorem 3.2. *Let X be a compact path-connected metric space with a base point x_0 and A be a unital simple infinite dimensional C^* -algebra with $TR(A) \leq 1$. Suppose that $\phi : C(X) \rightarrow A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$\begin{aligned} [\phi]_0 &= 0 \text{ in } KL(C_0(X), A), \\ \pi_A \circ \hat{\phi} &= 0, \text{ and} \\ \phi^\dagger &= 0. \end{aligned}$$

Proof. Firstly, suppose that $\psi : C(X) \rightarrow A$ is a unital finite-dimensional homomorphism. Then we can write ψ as

$$\psi(f) = \sum_{k=1}^m f(x_k) p_k$$

for all $f \in C(X)$, where $x_k \in X$ and $p_1, \dots, p_m$ are mutually orthogonal projections in A with $\sum_{k=1}^m p_k = 1$. Define $\psi_0 : C(X) \rightarrow A$ by

$$\psi_0(f) = f(x_0) \cdot 1_A$$

for all $f \in C(X)$. Since X is path-connected, ψ is homotopic to ψ_0 , and hence

$$[\psi]_0 = [\psi_0]_0 = 0, \text{ in } KL(C_0(X), A).$$

Also, since $\overline{\rho(K_0(A))}$ is an $\mathbb{R}$ -linear subspace of $\text{Aff}T(A)$ (see Proposition 3.6 of [Lin9]), we see that $\hat{\psi}$ maps $C_{\mathbb{R}}(X)$ into $\overline{\rho(K_0(A))}$. Next, if $u \in U(C(X))$, then $|u(x_k)| = 1$; we write $u(x_k) = \exp(i\theta_k)$, where $\theta_k \in \mathbb{R}$ for $k = 1, \dots, m$. Put $h = \sum_{k=1}^m \theta_k p_k \in A_{sa}$; then

$$\psi(u) = \sum_{k=1}^m u(x_k) p_k = \exp(ih).$$

Note that $\hat{h} \in \overline{\rho(K_0(A))}$, $\psi(u) \in CU(A)$ by Theorem 2.9 of [Lin9]. Hence $\psi^\dagger(U(C(X))) \subset CU(A)$. Similarly, $\psi^\dagger(U_n(C(X))) \subset CU_n(A)$. Then $\psi^\dagger = 0$, and hence $\psi^\dagger = 0$.

Now if ϕ can be approximated by finite-dimensional homomorphisms, ϕ must satisfy the above-mentioned three conditions.

Conversely, suppose that $\phi : C(X) \rightarrow A$ is a unital homomorphism such that $[\phi]_0 = 0$, $\pi_A \circ \widehat{\phi} = 0$, and $\phi^\dagger = 0$. As the proof of Lemma 4.1 of [Lin9], we can choose a unital simple C^* -subalgebra $B \subset A$ with tracial rank zero such that the inclusion $\iota : B \rightarrow A$ induces an isomorphism:

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) \cong (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Then $[\iota]$ is a KK -equivalence in $KK(B, A)$ (see 7.3 of [RS]), and hence there is a $\beta \in KK(A, B)$ such that

$$[\iota] \times \beta = [id_B] \text{ and } \beta \times [\iota] = [id_A].$$

Define $\kappa = [\phi] \times \beta \in KL_e(C(X), B)^{++}$. Since $TR(B) = 0$, we know that $\text{Aff}T(B) = \overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$. By assumption, $\widehat{\phi}$ maps $C_{\mathbb{R}}(X)$ into $\overline{\rho_A(K_0(A))}$; then $\widehat{\phi}$ defines a unital strictly positive linear map $\gamma : C_{\mathbb{R}}(X) \rightarrow \text{Aff}T(B)$ such that $\widehat{\iota} \circ \gamma = \widehat{\phi}$. We now check that the defined pair (κ, γ) is compatible, that is, $\rho_B \circ \kappa_0^0 = \gamma \circ \rho_{C(X)}$. It suffices to show that $\widehat{\iota} \circ \rho_B \circ \kappa_0^0 = \widehat{\iota} \circ \gamma \circ \rho_{C(X)}$, since $\widehat{\iota}$ is injective. This is equivalent to

$$(3.1) \quad \rho_A \circ \iota_{*0} \circ \kappa_0^0 = \widehat{\phi} \circ \rho_{C(X)}.$$

Since $\iota_{*0} \circ \kappa_0^0 = \phi_{*0}$, equation (3.1) becomes

$$\rho_A \circ \phi_{*0} = \widehat{\phi} \circ \rho_{C(X)},$$

and this is well-known to be true. The following diagram shows the above calculation:

$$\begin{array}{ccccc} K_0(C(X)) & \xrightarrow{\kappa_0^0} & K_0(B) & \xleftarrow[\iota_{*0}]{\rho_0^0} & K_0(A) \\ \downarrow \rho_{C(X)} & & \downarrow \rho_B & & \downarrow \rho_A \\ C_{\mathbb{R}}(X) & \xrightarrow{\gamma} & \text{Aff}T(B) & \xrightarrow[\widehat{\iota}]{} & \text{Aff}T(A) \end{array}$$

Now apply Theorem 5.2 of [Lin7]; there exists a unital monomorphism $h : C(X) \rightarrow B$ such that

$$(3.2) \quad [h] = \kappa \text{ in } KL(C(X), B) \quad \text{and} \quad \widehat{h} = \gamma.$$

Then

$$(3.3) \quad [\iota \circ h] = [h] \times [\iota] = \kappa \times [\iota] = [\phi] \times \beta \times [\iota] = [\phi]$$

and

$$(3.4) \quad \widehat{(\iota \circ h)} = \widehat{\iota} \circ \widehat{h} = \widehat{\iota} \circ \gamma = \widehat{\phi}.$$

Note that $TR(B) = 0$, $h^\dagger$ is automatically zero and then

$$(3.5) \quad (\iota \circ h)^\dagger = \phi^\dagger = 0.$$

Combine (3.3) – (3.5) and use Theorem 5.10 of [Lin11]; then we conclude that the two homomorphisms ϕ and $\iota \circ h$ are approximately unitarily equivalent. Finally, since $[\phi]_0 = 0$ in $KL(C_0(X), A)$, it is obvious that $[h]_0 = 0$ in $KL(C_0(X), B)$. From Lemma 3.1, h can be approximated by finite-dimensional homomorphisms. Hence ϕ can also be approximated by finite-dimensional homomorphisms. $\square$

Corollary 3.3. *Let X be a disjoint union of finitely many compact path-connected metric spaces and A be a unital simple infinite dimensional C^* -algebra with $TR(A) \leq 1$. Suppose that $\phi : C(X) \rightarrow A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$[\phi]_0 = 0 \text{ in } KL(C_0(X), A),$$

$$\pi_A \circ \hat{\phi} = 0, \text{ and}$$

$$\phi^\dagger = 0.$$

Proof. Write ϕ as a finite direct sum and apply Theorem 3.2. $\square$

Now we turn to consider a general (not necessarily path-connected) compact metric space X . We need a condition which generalizes the first condition of Corollary 3.3. For any $x \in X$, the evaluation map $\pi_x : C(X) \rightarrow \mathbb{C}$ defines an element $[\pi_x] : \underline{K}(C(X)) \rightarrow \underline{K}(\mathbb{C})$.

Let A be a unital C^* -algebra and $\phi : C(X) \rightarrow A$ be a unital homomorphism. We shall use the equation

$$(3.6) \quad [\phi] \mid_{\bigcap_{x \in X} \ker[\pi_x]} = 0$$

to describe ϕ . It is easy to see that (3.6) is equivalent to saying that $[\phi]_0 = 0$ when X is a disjoint union of finitely many path-connected compact spaces.

Let X be a compact metric space. It is well-known that there is an inductive system $\{C(X_n), h_n\}$ such that $C(X) = \varinjlim C(X_n, h_n)$, where each X_n is a finite CW complex. Suppose that $h_{n,\infty} : C(X_n) \rightarrow C(X)$ is the induced homomorphism. Also there are continuous maps $f_n : X \rightarrow X_n$ that induce $h_{n,\infty}$. Write $X_n = X_n^1 \sqcup \cdots \sqcup X_n^{r(n)}$, where X_n^i is the path-connected component of X_n for $i = 1, \dots, r(n)$. Fix a point x_n^i in each X_n^i .

Lemma 3.4. *With the above notation, let A be a unital C^* -algebra and $\phi : C(X) \rightarrow A$ be a unital homomorphism. If (3.6) holds, then we have*

$$(3.7) \quad [\phi \circ h_{n,\infty}] \mid_{\bigcap_{i=1}^{r(n)} \ker[\pi_{x_n^i}]} = 0,$$

for all n, i .

Proof. Fix a number n . For any $g \in \bigcap_{i=1}^{r(n)} \ker[\pi_{x_n^i}]$, let G be the subgroup of $\underline{K}(C(X))$ generated by $[h_{n,\infty}](g)$. Let $\delta > 0$ be as required by Lemma 2.1 for G . Write $X = Y_1 \sqcup \cdots \sqcup Y_N$, where each Y_k is a δ -connected component of X . For each $1 \leq k \leq N$, there is $1 \leq i(n, k) \leq r(n)$ such that $f_n(Y_k) \cap X_n^{i(n, k)} \neq \emptyset$. Choose points $y_k \in Y_k$ and $z_n^{i(n, k)} \in X_n^{i(n, k)}$ such that

$$f_n(y_k) = z_n^{i(n, k)},$$

for each k . Then

$$(3.8) \quad [\pi_{y_k}](h_{n,\infty}(g)) = [\pi_{z_n^{i(n, k)}}](g) = [\pi_{x_n^{i(n, k)}}](g) = 0.$$

By Lemma 2.1, we obtain that $[\pi_y](h_{n,\infty}(g)) = 0$ for all $y \in Y_k$. This is true for each k and then for all $y \in X$. Hence, (3.7) now follows from (3.6). $\square$

The key point of using equation (3.6) is shown in the next lemma.

Lemma 3.5. *Let $\phi : C(X) \rightarrow A$ be a unital homomorphism as above. Then the following two conditions are equivalent:*

(i) $[\phi] \mid_{\bigcap_{x \in X} \ker[\pi_x]} = 0$;

(ii) *for any finitely generated subgroup $G \subset \underline{K}(C(X))$, there is a unital finite-dimensional homomorphism $\psi : C(X) \rightarrow A$ such that*

$$[\phi] \mid_G = [\psi] \mid_G.$$

Proof. Firstly, we prove “(i) $\Rightarrow$ (ii)”. Suppose that $C(X) = \varinjlim (C(X_n), h_n)$, where each X_n is a finite CW complex. Write $X_n = X_n^1 \sqcup \cdots \sqcup X_n^{r(n)}$, where X_n^i is the path-connected component of X_n for $i = 1, \dots, r(n)$. Also, denote by e_n^i the unit of the summand $C(X_n^i)$. Then, we can further assume that h_n is unital and $h_n(e_n^i) \neq 0$ for all n, i , since otherwise we can delete the summand $C(X_n^i)$ from the inductive system without changing the inductive limit. Suppose that $h_{n,\infty} : C(X_n) \rightarrow C(X)$ is the induced map. It follows that

$$(3.9) \quad h_{n,\infty}(e_n^i) \neq 0$$

for all n, i . On the other hand, there are continuous maps $f_n : X \rightarrow X_n$ that induce $h_{n,\infty}$. Then (3.9) is equivalent to

$$(3.10) \quad f_n(X) \cap X_n^i \neq \emptyset$$

for $i = 1, \dots, r(n)$.

Given a finitely generated subgroup $G \subset \underline{K}(C(X))$, we can choose a large n such that $G \subset [h_{n,\infty}](\underline{K}(C(X_n)))$. From (3.10), we can choose $z_n^i \in X_n^i$ and $y^i \in X$ such that

$$(3.11) \quad f_n(y^i) = z_n^i$$

for each $i = 1, \dots, r(n)$. Now, consider $\phi \circ h_{n,\infty} : C(X_n) \rightarrow A$. By Lemma 3.4 and noting that $[\pi_{x_n^i}] = [\pi_{z_n^i}]$, we have

$$(3.12) \quad [\phi \circ h_{n,\infty}] \mid_{\bigcap_{i=1}^{r(n)} \ker[\pi_{z_n^i}]} = 0.$$

Define $\psi : C(X) \rightarrow A$ by

$$\psi(f) = \sum_{i=1}^{r(n)} f(y^i) p_i,$$

where $p_i = \phi \circ h_{n,\infty}(e_n^i)$. From (3.12) and the fact that the homomorphisms are unital, we obtain that $[\phi \circ h_{n,\infty}] = [\psi \circ h_{n,\infty}]$, and hence $[\phi] \mid_G = [\psi] \mid_G$.

Conversely, suppose that condition (ii) is true. Let g be an element in $\bigcap_{x \in X} \ker[\pi_x]$. Let G be the subgroup generated by g . Now we can choose a unital finite-dimensional homomorphism $\psi : C(X) \rightarrow A$ such that

$$(3.13) \quad [\phi] \mid_G = [\psi] \mid_G.$$

Choose a large n satisfying the conditions as above. Then there is $\tilde{g} \in \underline{K}(C(X_n))$ such that $g = [h_{n,\infty}](\tilde{g})$. Since $g \in \bigcap_{x \in X} \ker[\pi_x]$, by (3.11), $\tilde{g} \in \bigcap_{i=1}^{r(n)} \ker[\pi_{x_n^i}]$. Therefore by (3.13),

$$[\phi](g) = [\phi] \circ [h_{n,\infty}](\tilde{g}) = [\psi] \circ [h_{n,\infty}](\tilde{g}) = 0. \quad \square$$

Lemma 3.6. *Let A be a unital C^* -algebra satisfying the fundamental comparison property, and let $\phi : C(X) \rightarrow A$ be a unital homomorphism. If $\widehat{\phi}(C_{\mathbb{R}}(X)) \subset \overline{\rho_A(K_0(A))}$, then for any finite subset $\mathcal{F} \subset C_{\mathbb{R}}(X)$ and $\epsilon > 0$, there exists a unital finite-dimensional homomorphism $\psi : C(X) \rightarrow A$ such that*

$$\widehat{\phi} \approx_{\epsilon} \widehat{\psi} \quad \text{on } \mathcal{F}.$$

Proof. The proof is contained in the proof of Theorem 3.6 of [HLX]. $\square$

For our purpose, we need the following approximated version of the uniqueness theorem which is deduced from Theorem 5.3 of [Lin11], combined with Proposition 11.1 of [Lin6] and Lemma 3.4 of [Lin11].

Lemma 3.7. *Let X be a compact metric space and A be a unital simple C^* -algebra with $TR(A) \leq 1$. Suppose that $\phi : C(X) \rightarrow A$ is a unital monomorphism. For any finite subset $\mathcal{F} \subset C(X)$ and $\epsilon > 0$, there exist a finitely generated subgroup $G \subset \underline{K}(C(X))$, a finite subset $\mathcal{G} \subset C_{\mathbb{R}}(X)$ and $\gamma > 0$ such that for any unital homomorphism $\psi : C(X) \rightarrow A$, if*

$$\begin{aligned} [\phi] |_G &= [\psi] |_G, \\ \widehat{\phi} &\approx_{\gamma} \widehat{\psi} \quad \text{on } \mathcal{G}, \end{aligned}$$

and

$$\phi^{\dagger} = \psi^{\dagger},$$

then there is a unitary $u \in A$ such that

$$\phi \approx_{\epsilon} \text{adu} \circ \psi \quad \text{on } \mathcal{F}. \quad \square$$

With the above preparations, we can now prove the main lemma.

Lemma 3.8. *Let X be a compact metric space and A be a unital simple infinite dimensional C^* -algebra with $TR(A) \leq 1$. Suppose that $\phi : C(X) \rightarrow A$ is a unital monomorphism satisfying the following conditions:*

(i) *for any finitely generated subgroup $G \subset \underline{K}(C(X))$, there is a unital finite-dimensional homomorphism $\psi : C(X) \rightarrow A$ such that*

$$[\phi] |_G = [\psi] |_G,$$

(ii)

$$\phi^{\dagger} = 0.$$

Then ϕ can be approximated by finite-dimensional homomorphisms.

Proof. It suffices to prove the following: for any finite subset $\mathcal{F} \subset C(X)$ and $\epsilon > 0$, there is a unital finite-dimensional homomorphism $\psi : C(X) \rightarrow A$ and a unitary $u \in A$ such that

$$\phi \approx_{\epsilon} \text{adu} \circ \psi \quad \text{on } \mathcal{F}.$$

Let $G \subset \underline{K}(C(X))$ be a finitely generated subgroup, $\mathcal{G} \subset C_{\mathbb{R}}(X)$ a finite subset and $\gamma > 0$ with the properties described in Lemma 3.7 corresponding to $\mathcal{F}$ and ϵ . For this finitely generated subgroup G , we can choose $\delta > 0$ as in Lemma 2.1. Write $X = X_1 \sqcup \cdots \sqcup X_m$, where each X_k is a δ -connected component of X .

Now let G_0 be the subgroup generated by G and $\{[e_0], \dots, [e_m]\}$. From condition (i) we can choose a unital finite-dimensional homomorphism $\psi_0 : C(X) \rightarrow A$ such that

$$(3.14) \quad [\phi] |_{G_0} = [\psi_0] |_{G_0}.$$

Under the identification $C(X) = C(X_1) \oplus \cdots \oplus C(X_m)$ we consider the homomorphism $\phi_i : C(X_i) \rightarrow p_i A p_i$ induced by ϕ , where $p_i = \phi(e_i)$ for $i = 1, \dots, m$. Also view $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_m$. Since $\phi^\dagger = 0$, we have

$$(3.15) \quad \widehat{\phi}(C_{\mathbb{R}}(X)) \subset \overline{\rho_A(K_0(A))}.$$

Then we also have $\widehat{\phi}_i(C_{\mathbb{R}}(X_i)) \subset \overline{\rho_{p_i A p_i}(K_0(p_i A p_i))}$. For $\mathcal{G}_i$ and $\gamma > 0$, by applying Lemma 3.6, there is a unital finite-dimensional homomorphism $\phi'_i : C(X_i) \rightarrow p_i A p_i$ such that

$$(3.16) \quad \widehat{\phi}_i \approx_{\gamma} \widehat{\phi}'_i \quad \text{on } \mathcal{G}_i.$$

Define $\psi : C(X) \rightarrow A$ by

$$\psi = \phi'_1 \oplus \cdots \oplus \phi'_m.$$

It follows from Lemma 2.1, (3.14), and the definition of ψ that

$$(3.17) \quad [\psi] \mid_G = [\psi_0] \mid_G = [\phi] \mid_G.$$

From (3.16), we have

$$(3.18) \quad \widehat{\phi} \approx_{\gamma} \widehat{\psi} \quad \text{on } \mathcal{G}.$$

Note that $\phi^\dagger = \psi^\dagger = 0$. We complete the proof by applying Lemma 3.7. $\square$

Theorem 3.9. *Let X be a compact metric space and A be a unital simple infinite dimensional C^* -algebra with $TR(A) \leq 1$. Suppose that $\phi : C(X) \rightarrow A$ is a unital monomorphism. Then ϕ can be approximated by finite-dimensional homomorphisms if and only if*

$$\begin{aligned} [\phi] \mid_{\bigcap_{x \in X} \text{Ker}[\pi_x]} &= 0, \\ \pi_A \circ \widehat{\phi} &= 0, \text{ and} \\ \phi^\dagger &= 0. \end{aligned}$$

Proof. For necessity, we need only to check the first condition. This is easy by Lemma 3.5. Conversely, combine Lemmas 3.5 and 3.8, and also note that the three conditions imply that $\phi^\dagger = 0$; see Section 2.7. $\square$

As an application, we now obtain a necessary and sufficient condition for a normal element to be approximated by normal elements with finite spectrum.

Corollary 3.10. *Let A be a unital simple infinite-dimensional C^* -algebra with $TR(A) \leq 1$ and $x \in A$ be a normal element. Suppose that $\phi : C(sp(x)) \rightarrow A$ is the map induced by the continuous functional calculus. Then x can be approximated by normal elements with finite spectrum if and only if*

$$\phi^\dagger = 0.$$

Proof. Put $X = sp(x)$. We know that $K_*(C(X))$ is torsion free; then

$$KL(C(X), A) = \bigoplus_{i=0,1} \text{Hom}(K_i(C(X)), K_i(A)).$$

Note that in this case, $[\phi]_0 = 0$ is equivalent to saying that $\phi_{*1} = 0$. Then the three conditions in Theorem 3.9 follow by $\phi^\dagger = 0$. $\square$

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