DEGENERATE HYPERBOLIC EQUATIONS WITH LOWER DEGREE DEGENERACY

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ABSTRACT. We prove that the Cauchy problem of degenerate hyperbolic equations is well-posed if leading coefficients are degenerate at a low degree.

1. INTRODUCTION

The Cauchy problem is well-posed for (strictly) hyperbolic differential equations. However, we still do not know about the optimal assumption to ensure the well-posedness of the Cauchy problem of degenerate hyperbolic differential equations even if the space dimension is *one*.

In [3], Colombini and Spagnolo constructed a degenerate hyperbolic equation such that the Cauchy problem is not well-posed. The equation has a simple form

$$u_{tt} - K(t)u_{xx} = 0 \quad \text{in } \mathbb{R} \times (0, T),$$

where $K(t)$ is a nonnegative function in $(0, T)$. In fact, solutions do not exist for some smooth Cauchy data even in the distribution sense. In this example, the non-negative function $K(t)$ oscillates infinitely many times. Such an example clearly shows that certain conditions need to be imposed on degenerate leading coefficients to ensure the well-posedness of the Cauchy problem for degenerate hyperbolic equations. In [2], Colombini, Jannelli and Spagnolo conjectured that *the well-posedness in the C^∞ -category holds for the Cauchy problem of degenerate hyperbolic equations if the leading (degenerate) coefficients are analytic in time t* . Here, the analyticity in t ensures a *finite number of oscillations* in the t -direction, which excludes the example in [3].

It is proved in [11] that the Cauchy problem of degenerate hyperbolic equations is well-posed in C^∞ if leading (degenerate) coefficients are analytic in both time and space variables with appropriate assumptions on the lower order coefficients. For the smooth case, the Cauchy problem is well-posed for degenerate hyperbolic equations under additional conditions on oscillations of the degenerate coefficients with respect to time. Such conditions are in general imposed on *zero sets of time derivatives of degenerate coefficients*. In [4] and [7], those zero sets are allowed to

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be Lipschitz or even Hölder continuous (as space variables). However, conditions in [4] and [7] are hard to verify in general for many interesting cases.

Although appropriate assumptions on time derivatives of degenerate coefficients yield affirmative results, it seems that better candidates for natural assumptions are the degenerate coefficients themselves. The reason is clear since equations are degenerate if these coefficients vanish. It seems more natural to study *zero sets of degenerate coefficients* instead of those of time derivatives of degenerate coefficients. In [6], the first author derived classical energy estimates for solutions of degenerate hyperbolic equations when degenerate coefficients do not admit complex roots.

In this paper, we continue our study of the Cauchy problem of degenerate hyperbolic equations. Consider the Cauchy problem in a neighborhood of the origin in $\mathbb{R}^2$,

$$(1.1) \quad u_{tt} - a(x, t)u_{xx} - b_0(x, t)u_t - b(x, t)u_x - c(x, t)u = f(x, t),$$

$$(1.2) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x),$$

where a is a nonnegative function. This is a degenerate hyperbolic equation since a is allowed to vanish. We assume

$$(1.3) \quad |b| \leq C_b \sqrt{a},$$

for some positive constant C_b . This is the simplest form of the so-called *Levi condition*. It is necessary for the well-posedness of the Cauchy problem for degenerate hyperbolic equations.

If certain t -derivatives of a do not vanish, we can always write a locally in the form

$$(1.4) \quad a(x, t) = q(x, t)K(x, t),$$

where $q(x, t)$ is a positive smooth function and $K(x, t)$ is a nonnegative smooth function such that

$$(1.5) \quad \lambda \leq q \leq \Lambda,$$

for some positive constants λ and Λ , and

$$(1.6) \quad K(x, t) = t^m + c_1(x)t^{m-1} + \cdots + c_{m-1}(x)t + c_m(x) \geq 0,$$

for some even integer m and some smooth functions $c_1, \dots, c_m$ in a neighborhood of $0 \in \mathbb{R}$. We can achieve this by the Weierstrass preparation theorem.

Our primary goal is to discuss the well-posedness of degenerate hyperbolic differential equations. If a is bounded below away from zero, the corresponding equation is strictly hyperbolic. We can obtain energy estimates accordingly. In this paper, we are interested only in the case when a is *small*.

If a satisfies (1.4)-(1.6), the zero set of a has a relatively simple structure. For each fixed $x \in \mathbb{R}$, there are m zeros of K for t , and these zeros may be complex. In fact, it is precisely due to the presence of complex zeros that makes the discussion of the Cauchy problem difficult. If all zeros of K are real, these zeros decompose the plane into finitely many regions. We are able to derive energy estimates in each region and then patch them together to obtain global energy estimates. Refer to [6] for details. However, if complex zeros are present, we need to connect real zeros appropriately to form a decomposition. Among all possible choices, we need to choose one to obtain the desired a priori estimates. A natural choice seems to be the real parts of zeros. If $K(x, t)$ is assumed to be analytic in both t and x , the real parts of zeros indeed decompose the plane properly, as shown in [11].

However, when the analyticity in x is absent, zero sets are complicated to analyze in general. The real parts of zeros still work for smooth K in (1.6) with $m = 2$. In fact, it is the only known case. The real parts of zeros do *not* decompose the plane properly even for $m = 4$.

In this paper, we use the zero sets of *both* K and the t -derivative of K to decompose the plane. For $m = 4$, we find a nontrivial decomposition to derive energy estimates and therefore obtain the existence of solutions. Our main result is the following local existence and local energy estimates for the Cauchy problem (1.1)-(1.2). (Refer to Theorem 3.4 for the global version.)

Theorem 1.1. *Let q, b_0, b, c, K be smooth functions in $[-1, 1] \times [0, 1]$ with $a = qK$ satisfying (1.3) and (1.5). Suppose K is a nonnegative function given by (1.6) for $m = 4$. Then for any smooth function f in $[-1, 1] \times [0, 1]$ and smooth functions u_0, u_1 in $[-1, 1]$, the Cauchy problem for (1.1)-(1.2) admits a smooth solution in $[-X, X] \times [0, T]$, where $X, T \in (0, 1)$ are constants depending only λ, Λ and the C^N -norm of K , and N is a universal nonnegative integer. Moreover, for any $s \geq 0$,*

$$\begin{aligned} \|u\|_{H^s((-X, X) \times (0, T))} \leq C_s (&\|u_0\|_{H^{s+N}((-1, 1))} + \|u_1\|_{H^{s+N-1}((-1, 1))} \\ &+ \|f\|_{H^{s+N}((-1, 1) \times (0, 1))}, \end{aligned}$$

where C_s is a positive constant depending only on s, λ, Λ, C_b and the C^{s+N} -norms of q, K, b_0, b and c .

As pointed out earlier, the key point in deriving energy estimates is to decompose the plane appropriately. The decomposition should be related to zero sets of leading degenerate coefficients so that the degeneracy can be overcome by appropriately constructed weight functions. The decomposition of the plane and the construction of weight functions are intrinsically related. It is a delicate issue even for $m = 4$. The obvious choice of the real parts of complex zeros of K does not work. Although zeros of degree 4 polynomials can be expressed explicitly, our decomposition is not given by this expression. We need to substitute complex zeros by appropriate real functions since complex zeros do not show up in real planes. In fact, the decomposition has a complicated explicit expression in terms of coefficients. See (3.15) in Section 3.

We expect that Theorem 1.1 holds for arbitrary nonnegative polynomials K in (1.6) and that N depends only on m . However, it is unlikely that appropriate decompositions can be expressed explicitly in terms of coefficients. In fact, when complex zeros are absent, no explicit expressions are used in [6]. Instead, algebraic properties of zeros in terms of coefficients are employed.

Last, we point out that the discussions in Section 3 can be generalized to *some* polynomials of higher order.

The paper is organized as follows. In Section 2, we recall a criterion of appropriate decompositions such that energy estimates can be obtained and hence the Cauchy problem is well-posed. In Section 3, we verify that this criterion is satisfied for K in (1.6) for $m = 4$.

2. A CRITERION FOR ENERGY ESTIMATES

In this section, we recall a criterion in [6] leading to energy estimates of solutions of the Cauchy problem of degenerate hyperbolic equations. (See also [5].) In order to establish energy estimates for the Cauchy problem of degenerate hyperbolic

equations, we decompose the (upper half) plane into finitely many regions such that energy estimates can be obtained in each region. Then we patch these estimates in each region together to obtain global energy estimates.

First, we impose a basic assumption. Let a be a smooth nonnegative function in $\mathbb{R} \times \mathbb{R}$.

Assumption 2.1. *For some constant $\beta \in (0, 1]$ and some integer n , there exist n functions $\varphi_1, \dots, \varphi_n \in C^\beta(\mathbb{R}) \cap BV(\mathbb{R})$, with $\varphi_n \leq \dots \leq \varphi_1$ in $\mathbb{R}$, which separate $\mathbb{R} \times \mathbb{R}$ into $n + 1$ regions*

$$\begin{aligned}\Omega_1 &= \{(x, t); t > \varphi_1(x)\}, \\ \Omega_k &= \{(x, t); \varphi_k(x) < t < \varphi_{k-1}(x)\} \quad \text{for } k = 2, \dots, n, \\ \Omega_{n+1} &= \{(x, t); t < \varphi_n(x)\},\end{aligned}$$

such that for some constant $\eta \in (0, 1)$,

$$(2.1) \quad |\partial_x \varphi_k| \sqrt{a}|_{t=\varphi_k} \leq \eta \quad \text{for } k = 1, \dots, n,$$

and for some nonnegative constant C_* ,

$$(2.2) \quad \partial_t a \geq -C_* a \quad \text{in } \Omega_1 \quad \text{and} \quad \partial_t a \leq C_* a \quad \text{in } \Omega_{n+1},$$

and for $k = 2, \dots, n$, there exist a positive function ω_k in Ω_k and a nonnegative integer d_k such that in Ω_k ,

$$(2.3) \quad \frac{\partial_t \omega_k}{\omega_k} \leq c_1, \quad \frac{\partial_t \omega_k}{\omega_k} + \frac{|\partial_x \omega_k|}{\omega_k} \sqrt{a} \leq c_2, \quad \frac{\partial_t(\omega_k a)}{\omega_k a} + \frac{|\partial_x \omega_k|}{\omega_k} \sqrt{a} \leq c_3,$$

and

$$(2.4) \quad c_0 \leq \omega_k \leq (t - \varphi_k(x))^{-d_k} \quad \text{for any } (x, t) \in \Omega_k,$$

for some positive constants c_0, c_1, c_2 and c_3 .

We remark that the first of the three inequalities in (2.3) is superfluous since it follows from the second one. In practice, we need to verify this first and then move on to the second and third inequalities. This is the reason we include it here.

Let $a = qK$, where q is a positive function with a uniform positive lower bound. In this case, (2.2) and (2.3) can be derived from

$$(2.5) \quad \partial_t K \geq 0 \quad \text{in } \Omega_1 \quad \text{and} \quad \partial_t K \leq 0 \quad \text{in } \Omega_{n+1},$$

and

$$(2.6) \quad \frac{\partial_t \omega_k}{\omega_k} \leq c_1, \quad \frac{\partial_t \omega_k}{\omega_k} + \frac{|\partial_x \omega_k|}{\omega_k} \sqrt{K} \leq c_2, \quad \frac{\partial_t(\omega_k K)}{\omega_k K} + \frac{|\partial_x \omega_k|}{\omega_k} \sqrt{K} \leq c_3.$$

Assumption 2.1 is sufficient for energy estimates.

Theorem 2.2. *Let a, b_0, b, c, q, K be smooth functions in $\mathbb{R} \times [0, T]$ with $a = qK$ satisfying (1.3) and (1.5). Suppose a satisfies Assumption 2.1. Then for any smooth solution u of (1.1)-(1.2), there holds for any $s \geq 0$,*

$$\|u\|_{s, \mathbb{R} \times [0, T]} \leq C(\|u_0\|_{s+N, \mathbb{R}} + \|u_1\|_{s+N-1, \mathbb{R}} + \|f\|_{s+N, \mathbb{R} \times [0, T]}),$$

where C is a positive constant depending only on s, Λ, C_b , the C^{s+N} -norms of a, b_0, b and c , and the C^β -norms and the BV -norms of $\varphi_1, \dots, \varphi_n$ in Assumption 2.1, and N is a positive constant depending only on m .

Here and hereafter, $\|\cdot\|_{s,\Omega}$ is the H^s -norm in Ω .

To prove Theorem 2.2, we first derive an estimate with weight ω_k in Ω_k . Then we eliminate weights and patch these estimates together. Refer to [6] for a detailed proof. See also [5]. We point out that integration by parts can be performed in Ω_k since its boundary is given by functions of bounded variation. Moreover, Sobolev embedding holds in Ω_k since its boundary is Hölder continuous. In fact, we have

$$H^k(\Omega_k) \subset L^\infty(\Omega_k) \quad \text{for } k > \frac{1}{2}\left(1 + \frac{1}{\beta}\right).$$

Refer to [9] for details.

We now consider the nonnegative functions $K = K(x, t)$ given by (1.6). In the following, a zero, or a root, of K is referred to as a function $t = t(x)$ satisfying $K(x, t(x)) = 0$. For each fixed $x \in \mathbb{R}$, $K(x, t)$ is a polynomial of t of degree m and hence has m complex roots $t_1(x), \dots, t_m(x)$, which can be arranged to be continuous.

It is proved in [6] that Assumption 2.1 is satisfied if all t_i are real. The proof is based on a careful analysis of root functions of polynomials. The functions φ_k are simply the root functions t_i . When complex roots are present, the first natural choice is to choose real parts of zeros. However, such a choice does not always work! The subtlety of verifying Assumption 2.1 to a large extent is to find an appropriate function other than real parts of zeros when complex zeros are present.

Next, we briefly discuss the regularity of zeros of K . The following result asserts that they are Hölder continuous if coefficients are Hölder continuous. Refer to Lemma 2.3 and Lemma 2.5 of [8] for a proof.

Lemma 2.3. *Let $I \subset \mathbb{R}$ be an interval, K be a function in (1.6) and $t = t(x)$ be a (complex-valued) continuous function on I satisfying $K(x, t) = 0$ on I . If $c_1, \dots, c_m$ are $C^\alpha(I)$ for some $\alpha \in (0, 1]$, then t is $C^{\alpha/m}$. Moreover,*

$$\|t\|_{C^{\frac{\alpha}{m}}(I)} \leq 4m \sup_{1 \leq i \leq m} \|c_i\|_{C^\alpha(I)}^{1/i}.$$

In general, the root function in Lemma 2.3 is not Lipschitz. Concerning whether it is of bounded variation, we have the following result in the simplest case. Refer to Lemma 1 in [2] for nonnegative f and the main result in [12] for the general case.

Lemma 2.4. *Suppose f is a C^m -function on $I \subset \mathbb{R}$. Then $\sqrt[m]{|f|}$ is of bounded variation on I and*

$$(2.7) \quad \|(\sqrt[m]{|f|})'\|_{L^1(I)} \leq C(m, I) \|f\|_{C^m(I)}^{\frac{1}{m}},$$

where $C(m, I)$ is a positive constant depending only on m and I .

Lemma 2.4 implies that the curve $y = \sqrt[m]{|f|}$ has a finite arc length over I with the length bounded by $C(m, I) \|f\|_{C^m(I)}^{1/m}$.

3. NONNEGATIVE POLYNOMIALS OF DEGREE FOUR

The main goal in this section is to verify that Assumption 2.1 is satisfied by nonnegative degree four polynomials. To simplify our presentation, we assume that (1.4)-(1.6) hold globally in $\mathbb{R} \times [0, T]$, for some $T > 0$.

We start with an arbitrary nonnegative polynomial of degree m in (1.6). Obviously, m must be even since $K \geq 0$. We begin with the simplest case when $m = 2$. Suppose K is a quadratic polynomial in t given by

$$K(x, t) = t^2 + c_1(x)t + c_2(x) \quad \text{in } \mathbb{R} \times \mathbb{R},$$

where c_1 and c_2 are smooth functions on $\mathbb{R}$. Then $\partial_t K = 2t + c_1$, and hence

$$\partial_t K > 0 \text{ if } t > -\frac{c_1}{2} \quad \text{and} \quad \partial_t K < 0 \text{ if } t < -\frac{c_1}{2}.$$

Note that the curve $t = -c_1/2$ is smooth. Then Assumption 2.1 holds for such a K with

$$\Omega_1 = \{(x, t); t > -\frac{1}{2}c_1(x)\}, \quad \Omega_2 = \{(x, t); t < -\frac{1}{2}c_1(x)\}.$$

Since $K \geq 0$, then $4c_2 \geq c_1^2$ and $K(x, t) = 0$ is satisfied by

$$t(x) = \frac{1}{2}(-c_1(x) \pm i\sqrt{4c_2(x) - c_1^2(x)}).$$

Hence the real part of zeros of K is exactly $-c_1/2$.

Next we consider a degree four polynomial K in t given by

$$(3.1) \quad K(x, t) = t^4 + c_1(x)t^3 + c_2(x)t^2 + c_3(x)t + c_4(x) \quad \text{in } \mathbb{R} \times \mathbb{R}.$$

We prove that Assumption 2.1 holds for such a K .

Lemma 3.1. *Let K be given in (3.1). Then there exist two functions $t = \varphi_1(x)$, $t = \varphi_2(x)$ in $C^{1/4}(\mathbb{R}) \cap BV(\mathbb{R})$, with $\varphi_2 \leq \varphi_1$ on $\mathbb{R}$, such that*

- (i) $\partial_t K \geq 0$ for any $t > \varphi_1(x)$ and $\partial_t K \leq 0$ for any $t < \varphi_2(x)$,
- (ii) $\frac{\partial_t K}{K} \leq \frac{4}{t - \varphi_2(x)}$ for $\varphi_2(x) < t < \varphi_1(x)$,
- (iii) $|\partial_x \varphi_2| \sqrt{K}|_{\varphi_2 \leq t \leq \varphi_1} \leq K_0$ and $|\partial_x \varphi_1| \sqrt{K}|_{t=\varphi_1} \leq \kappa$,

for some constants $K_0 > 0$ and $\kappa \in (0, 1)$, if $K|_{t=\varphi_1}$ is sufficiently small.

The smallness of K on $t = \varphi_1$ will be quantified in the proof.

With Lemma 3.1, we decompose $\mathbb{R} \times \mathbb{R}$ by

$$\Omega_1 = \{t > \varphi_1\}, \quad \Omega_2 = \{\varphi_2 < t < \varphi_1\}, \quad \Omega_3 = \{t < \varphi_2\}.$$

In Ω_2 , we take

$$\omega_2 = (t - \varphi_2)^{-N},$$

for N sufficiently large. Then Assumption 2.1 is satisfied.

Before proving Lemma 3.1, we first explain through some straightforward calculation why real parts of zeros do not decompose the plane properly. In other words, we cannot take φ_i as real parts of zeros of K .

First, by changing $t \rightarrow t - c_1/4$, we may assume $c_1 = 0$. Hence K is given by

$$(3.2) \quad K(x, t) = t^4 + c_2(x)t^2 + c_3(x)t + c_4(x) \quad \text{in } \mathbb{R} \times \mathbb{R}.$$

Then the zeros of K are given by $a \pm b_1 i$ and $-a \pm b_2 i$, with $a \geq 0$, and hence

$$(3.3) \quad K = [(t - a)^2 + b_1^2][(t + a)^2 + b_2^2].$$

A simple comparison with (3.2) yields

$$(3.4) \quad \begin{aligned} -2a^2 + b_1^2 + b_2^2 &= c_2, \\ 2a(b_1^2 - b_2^2) &= c_3, \\ (a^2 + b_1^2)(a^2 + b_2^2) &= c_4. \end{aligned}$$

With $K \geq 0$, we have

$$(3.5) \quad c_4 \geq 0, \quad 2\sqrt{c_4} + c_2 \geq 0.$$

In fact, by taking $t = 0$ in (3.2), we have $c_4 \geq 0$. Next, by taking $t = -\operatorname{sgn}(c_3) \cdot \sqrt[4]{c_4}$ in (3.2), we have $2c_4 + c_2\sqrt{c_4} \geq |c_3|\sqrt[4]{c_4}$, and hence $2\sqrt{c_4} + c_2 \geq 0$.

We first consider a special case where K has two real zeros a and $-a$. In this case, with (3.4), we have $c_3 = 0$, $c_2 = -2\sqrt{c_4}$ and

$$a = \sqrt[4]{c_4}.$$

This implies that for $-a \leq t \leq a$,

$$\sqrt{K} = (a - t)(t + a) \leq 4a^2 = 4\sqrt{c_4},$$

and hence

$$|\partial_x a| \sqrt{K} \Big|_{|t| \leq a} \leq c_4^{-\frac{1}{4}} |\partial_x c_4| \leq 2\sqrt[4]{c_4} |\partial_x \sqrt{c_4}|,$$

which is bounded. Hence, Lemma 3.1(iii) is satisfied for $\varphi_1 = \sqrt[4]{c_4}$ and $\varphi_2 = -\sqrt[4]{c_4}$.

When complex zeros are present, we need to extend real zeros appropriately to decompose $\mathbb{R} \times \mathbb{R}$. It is natural to choose the real part of complex zeros or the module of complex zeros. However, this does not always work. We consider the case $c_3 = 0$ for an illustration. A straightforward calculation shows that (3.5) is sufficient for $K \geq 0$ in the case when $c_3 = 0$. In fact, if $c_3 = 0$, we have, for $c_4 \geq 0$, $|c_2| \leq 2\sqrt{c_4}$,

$$K = \left((t + \frac{1}{2}\sqrt{2\sqrt{c_4} - c_2})^2 + \frac{1}{4}(2\sqrt{c_4} + c_2) \right) \left((t - \frac{1}{2}\sqrt{2\sqrt{c_4} - c_2})^2 + \frac{1}{4}(2\sqrt{c_4} + c_2) \right),$$

and, for $c_4 \geq 0$, $c_2 \geq 2\sqrt{c_4}$,

$$K = (t^2 + \frac{c_2}{2} + \sqrt{\frac{c_2^2}{4} - c_4})(t^2 + \frac{c_2}{2} - \sqrt{\frac{c_2^2}{4} - c_4}).$$

Then the zero of K is given by

$$t = \begin{cases} \pm \frac{1}{2}\sqrt{2\sqrt{c_4} - c_2} \pm \frac{1}{2}i\sqrt{2\sqrt{c_4} + c_2} & \text{if } |c_2| \leq 2\sqrt{c_4}, \\ \pm i\sqrt{\frac{c_2}{2} \pm \sqrt{\frac{c_2^2}{4} - c_4}} & \text{if } c_2 \geq 2\sqrt{c_4}. \end{cases}$$

Then, we have

$$\operatorname{Ret} = \begin{cases} \pm \frac{1}{2}\sqrt{2\sqrt{c_4} - c_2} & \text{if } |c_2| \leq 2\sqrt{c_4}, \\ 0 & \text{if } c_2 \geq 2\sqrt{c_4} \end{cases}$$

and

$$|t| = \begin{cases} \sqrt[4]{c_4} & \text{if } |c_2| \leq 2\sqrt{c_4}, \\ \sqrt{\frac{c_2}{2} \pm \sqrt{\frac{c_2^2}{4} - c_4}} & \text{if } c_2 \geq 2\sqrt{c_4}. \end{cases}$$

If $a(t, x)$ is real analytic in (t, x) , then the real parts of complex zeros satisfy Lemma 3.1 (iii). This was proved in Lemma 2.2 (3) in [11] (for general degree). Before proving Lemma 3.1, we demonstrate by choosing c_2 and c_4 appropriately

that Lemma 3.1(iii) is *not* satisfied for the real parts of complex zeros or the modules of complex zeros!

We first consider the real parts of zeros:

$$\varphi(x) = \frac{1}{2}\sqrt{(2\sqrt{c_4} - c_2)^+}.$$

For $|c_2| \leq 2\sqrt{c_4}$, we have

$$K|_{t=\varphi} = \frac{1}{4}(2\sqrt{c_4} + c_2)(2\sqrt{c_4} - c_2 + \frac{1}{4}(2\sqrt{c_4} + c_2)),$$

and hence

$$\sqrt{K}|_{t=\varphi} \geq \frac{1}{4}\sqrt{2\sqrt{c_4} + c_2} \cdot \sqrt{2\sqrt{c_4} - c_2} + \frac{1}{8}(2\sqrt{c_4} + c_2).$$

This implies that

$$|\varphi'|\sqrt{K}|_{t=\varphi} \geq \frac{1}{32} \left(\sqrt{2\sqrt{c_4} + c_2} \cdot |(2\sqrt{c_4} - c_2)'| + \frac{2\sqrt{c_4} + c_2}{\sqrt{2\sqrt{c_4} - c_2}} |(2\sqrt{c_4} - c_2)'| \right).$$

Obviously, the first term is bounded as $\sqrt{c_4}$ is Lipschitz. However, the second term may not be bounded. There arises a problem at points where $c_2 = 2\sqrt{c_4}$ and $c_2c_4 \neq 0$. This example illustrates that *the real parts of complex zeros are not a good candidate for the decomposition*.

Now, we examine the module of complex zeros:

$$\psi = \begin{cases} \sqrt[4]{c_4} & \text{if } |c_2| \leq 2\sqrt{c_4}, \\ \frac{1}{\sqrt{2}}\sqrt{c_2 + \sqrt{c_2^2 - 4c_4}} & \text{if } c_2 \geq 2\sqrt{c_4}. \end{cases}$$

For $|c_2| \leq 2\sqrt{c_4}$, we have

$$K|_{t=\psi} = 2c_4 + c_2\sqrt{c_4} \leq 4c_4,$$

and hence

$$|\psi'|\sqrt{K}|_{t=\psi} \leq \frac{1}{2}c_4^{-1/4}|c_4'| = \sqrt[4]{c_4}|(\sqrt{c_4})'|.$$

Obviously, this is bounded, as $\sqrt{c_4}$ is Lipschitz. In fact, we also have

$$|\psi'|\sqrt{K}|_{0 \leq t \leq \psi} \leq \sqrt[4]{c_4}|(\sqrt{c_4})'|.$$

Next, for $c_2 \geq 2\sqrt{c_4}$, we have

$$K|_{t=\psi} = c_2(c_2 + \sqrt{c_2^2 - 4c_4}),$$

and hence

$$\begin{aligned} |\psi'|\sqrt{K}|_{t=\psi} &= \frac{1}{\sqrt{2}}\sqrt{c_2} \cdot \sqrt{c_2 + \sqrt{c_2^2 - 4c_4}} \cdot |(\sqrt{c_2 + \sqrt{c_2^2 - 4c_4}})'| \\ &= \frac{1}{2\sqrt{2}}\sqrt{c_2} \cdot |(c_2 + \sqrt{c_2^2 - 4c_4})'|. \end{aligned}$$

Note that we only consider this for $c_2 \geq 2\sqrt{c_4}$. Hence $(\sqrt{c_2^2 - 4c_4})'$ is unbounded at $x_0 \in \partial\{c_2^2 > 4c_4\}$ with $(c_2^2 - 4c_4)'(x_0) \neq 0$. This example illustrates that *the modules of complex zeros are not a good candidate for the decomposition*.

Now we start to prove Lemma 3.1. As we have explained before, we need to include all real zeros in the decomposition and to extend these functions appropriately. As we will see, such an extension is in fact between the real parts and the modules of complex zeros. (See (3.9) in Lemma 3.3.)

Suppose $t = a$ is a real zero of K . Since $K \geq 0$, then $t = a$ is also a zero of $\partial_t K$. In other words, a real zero of K is a common zero of K and $\partial_t K$. Hence we may consider a linear combination of K and $\partial_t K$.

For $K \geq 0$ as in (3.2), we consider

$$(3.6) \quad H(t) = t\partial_t K - K = 3t^4 + c_2 t^2 - c_4.$$

Then H has two real zeros:

$$(3.7) \quad \alpha = \sqrt{\frac{\sqrt{c_2^2 + 12c_4} - c_2}{6}}, \quad -\alpha = -\sqrt{\frac{\sqrt{c_2^2 + 12c_4} - c_2}{6}}.$$

In the following, we set $\varphi_1 = \alpha$ and $\varphi_2 = -\alpha$, and verify that (i)-(iii) in Lemma 3.1 are satisfied.

We emphasize again that a zero of K is referred to as an expression $t = t(x)$ such that $K(x, t(x)) = 0$. All c_i , α and zeros of K are functions of $x \in \mathbb{R}$.

We first discuss the regularity of α .

Lemma 3.2. *Let α be as in (3.7). Then $\alpha \in C^{1/4}(\mathbb{R}) \cap BV(\mathbb{R})$.*

Proof. First, we note that α is a continuous zero of H in (3.6). Then, $\alpha \in C^{1/4}(\mathbb{R})$ by Lemma 2.3. Next, we calculate α' . Here, the derivative is taken with respect to x . By

$$3\alpha^4 + c_2\alpha^2 - c_4 = 0,$$

we get

$$2\alpha' = -\frac{\alpha}{6\alpha^2 + c_2}c_2' + \frac{1}{\alpha(6\alpha^2 + c_2)}c_4'.$$

By the expression for α in (3.7), we obtain $6\alpha^2 + c_2 = \sqrt{c_2^2 + 12c_4}$. With $\pm c_2 \leq |c_2| \leq \sqrt{c_2^2 + 12c_4}$ and the explicit expression of α in (3.7), it is easy to see that

$$\frac{\alpha}{6\alpha^2 + c_2} = \frac{1}{\sqrt{6}} \frac{\sqrt{\sqrt{c_2^2 + 12c_4} - c_2}}{\sqrt{c_2^2 + 12c_4}} \leq \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt[4]{c_2^2 + 12c_4}}$$

and

$$\frac{1}{\alpha(6\alpha^2 + c_2)} = \frac{\sqrt{6}}{\sqrt{\sqrt{c_2^2 + 12c_4} - c_2} \cdot \sqrt{c_2^2 + 12c_4}} \leq \frac{1}{\sqrt{c_4} \sqrt[4]{c_2^2 + 12c_4}}.$$

Hence, we obtain

$$(3.8) \quad |\alpha'| \leq \frac{1}{2\sqrt{3}} \cdot \frac{1}{\sqrt[4]{c_2^2 + 12c_4}} |c_2'| + \frac{1}{2} \cdot \frac{1}{\sqrt{c_4} \sqrt[4]{c_2^2 + 12c_4}} |c_4'|.$$

Then

$$|\alpha'| \leq \frac{1}{2\sqrt{3}} \cdot \frac{1}{\sqrt{|c_2|}} |c_2'| + \frac{1}{2\sqrt[4]{12}} \cdot \frac{1}{\sqrt[4]{c_4^3}} |c_4'| \leq |(\sqrt{|c_2|})'| + |(\sqrt[4]{c_4})'|.$$

Therefore by Lemma 2.4, we get for any interval $I \subset \mathbb{R}$,

$$\int_I |\alpha'| \leq C(I) (\|c_2\|_{C^2(I)}^{\frac{1}{2}} + \|c_4\|_{C^4(I)}^{\frac{1}{4}}).$$

Hence, α is of bounded variation on I . □

We note that α is in fact a $C^{1/2}$ -function by the explicit expression in (3.7).

Lemma 3.3. *Let $K \geq 0$ be as in (3.2) with roots $a \pm b_1 i$ and $-a \pm b_2 i$, and α as in (3.7). Then*

$$(3.9) \quad |a| \leq \alpha \leq \min\{\sqrt[4]{c_4}, \sqrt{a^2 + b_i^2}\} \quad \text{for } i = 1, 2,$$

$$(3.10) \quad K_t > 0 \text{ for } t > \alpha \text{ and } K_t < 0 \text{ for } t < -\alpha,$$

$$(3.11) \quad \frac{\partial_t K}{K} \leq \frac{4}{t + \alpha} \quad \text{for any } |t| \leq \alpha,$$

$$(3.12) \quad |\alpha'| \sqrt{K} \leq \sqrt[4]{K} (|c'_2| + |(\sqrt{c_4})'|) \quad \text{for any } |t| \leq \alpha.$$

Note that the expression in the parentheses on the right hand side of (3.12) is bounded since $c_4 \geq 0$.

Proof. (1) First by $H(\alpha) = 0$, we have

$$0 \leq K(\alpha) + K(-\alpha) = 2(\alpha^4 + c_2\alpha^2 + c_4) = -4\alpha^4 + 4c_4.$$

This implies $|\alpha| \leq \sqrt[4]{c_4}$. In fact, it can be proved directly with the explicit expression of α in (3.7). We simply note by a straightforward calculation that

$$\sqrt{\frac{-c_2 + \sqrt{c_2^2 + 12c_4}}{6}} \leq \sqrt[4]{c_4} \Leftrightarrow 0 \leq c_2 + 2\sqrt{c_4}.$$

Next, by (3.3) and (3.4), we have

$$a^4 + (b_1^2 + b_2^2)a^2 + b_1^2b_2^2 = c_4$$

or

$$3a^4 + c_2a^2 + b_1^2b_2^2 = c_4.$$

Hence, we get

$$3a^4 + c_2a^2 \leq c_4.$$

Note, by $H(\alpha) = 0$, that

$$3\alpha^4 + c_2\alpha^2 = c_4.$$

By taking a difference, we obtain

$$(a^2 - \alpha^2)(3a^2 + 3\alpha^2 + c_2) \leq 0.$$

With $c_4 = \alpha^2(3\alpha^2 + c_2) \geq 0$ or $3a^2 + c_2 = a^2 + b_1^2 + b_2^2 \geq 0$, we obtain $a^2 - \alpha^2 \leq 0$.

To prove another upper bound of α , we simply note for $i = 1, 2$,

$$\begin{aligned} \alpha^2 \leq a^2 + b_i^2 &\Leftrightarrow \frac{-c_2 + \sqrt{c_2^2 + 12c_4}}{6} \leq a^2 + b_i^2 \\ &\Leftrightarrow \sqrt{c_2^2 + 12c_4} \leq c_2 + 6(a^2 + b_i^2) \\ &\Leftrightarrow c_4 \leq c_2(a^2 + b_i^2) + 3(a^2 + b_i^2)^2 \\ &\Leftrightarrow a^2 + b_j^2 \leq c_2 + 3(a^2 + b_i^2) \quad \text{for } j \neq i \\ &\Leftrightarrow 0 \leq 4b_i^2, \end{aligned}$$

where we used the expressions of c_2 and c_4 in (3.4).

(2) By (3.3), we have

$$\frac{\partial_t K}{K} = \frac{2(t-a)}{(t-a)^2 + b_1^2} + \frac{2(t+a)}{(t+a)^2 + b_2^2}.$$

Then (3.10) follows easily. For (3.11), we only need to consider $\partial_t K > 0$. By (3.9), we have $\alpha^2 \leq a^2 + b_i^2$ for $i = 1, 2$. Then an elementary calculation shows that

$$\begin{aligned} \frac{t-a}{(t-a)^2 + b_1^2} &\leq \frac{1}{t+\alpha} \quad \text{for any } t \in (a, \alpha), \\ \frac{t+a}{(t+a)^2 + b_2^2} &\leq \frac{1}{t+\alpha} \quad \text{for any } t \in (-a, \alpha). \end{aligned}$$

This implies (3.11).

In the above proof, we showed that each term in $\partial_t K/K$ satisfies the required estimate. In the following, we provide another proof to control $\partial_t K/K$ itself. We consider

$$\frac{\partial_t K}{K} - \frac{4}{t+\alpha} = \frac{(t+\alpha)\partial_t K - 4K}{(t+\alpha)K}.$$

Set

$$g(t) = (t+\alpha)\partial_t K - 4K.$$

We need to prove $g(t) \leq 0$ for $|t| \leq \alpha$. Note

$$g(-\alpha) = -4K(-\alpha) < 0,$$

$$g(\alpha) = 2\alpha\partial_t K(\alpha) - 4K(\alpha) = 2H(\alpha) - 2K(\alpha) = -2K(\alpha) < 0.$$

To proceed, we write

$$g(t) = 4\alpha t^3 - 2c_2 t^2 + 2c_2 \alpha t - 3c_3 t + c_3 \alpha - 4c_4.$$

By setting $t = s\alpha$ and $h(s) = g(s\alpha)$, with $|s| \leq 1$, we have

$$h(s) = 4s^3\alpha^4 + (2s - 2s^2)c_2\alpha^2 + (1 - 3s)c_3\alpha - 4c_4.$$

With $K(\alpha) = \alpha^4 + c_2\alpha^2 + c_3\alpha + c_4$, we get

$$h(s) = (4s^3 + 3s - 1)\alpha^4 + (-2s^2 + 5s - 1)c_2\alpha^2 + (3s - 5)c_4 + (1 - 3s)K(\alpha).$$

By $H(\alpha) = 3\alpha^4 + c_2\alpha^2 - c_4 = 0$, we have $c_2\alpha^2 = -3\alpha^4 + c_4$, and hence

$$\begin{aligned} h(s) &= (4s^3 + 6s^2 - 12s + 2)\alpha^4 + (-2s^2 + 8s - 6)c_4 + (1 - 3s)K(\alpha) \\ &= 2(s - 1)[(2s^2 + 5s - 1)\alpha^4 - 2(s - 3)c_4] + (1 - 3s)K(\alpha). \end{aligned}$$

We first consider $1/3 < s < 1$. Note that $2s^2 + 5s - 1 > 0$ for $1/3 < s < 1$. Hence, we obtain $h(s) \leq 0$ for $1/3 < s < 1$. Next, we consider $-1 < s < 1/3$. Note that

$$K(\alpha) + K(-\alpha) = -4\alpha^4 + 4c_4.$$

We therefore obtain

$$\begin{aligned} h(s) &= (4s^3 + 6s^2 - 2)\alpha^4 + (-2s^2 - 4s - 2)c_4 - (1 - 3s)K(-\alpha) \\ &= 2(s + 1)^2[(2s - 1)\alpha^4 - 2c_4] - (1 - 3s)K(-\alpha). \end{aligned}$$

Hence $h(s) \leq 0$ for $-1 < s < 1/3$. This finishes the proof.

(3) We first claim for $|t| \leq \alpha$ that

$$(3.13) \quad K \leq 4c_4.$$

By replacing t by $\text{sgn}(c_3)|t|$ in $K \geq 0$, we get

$$|c_3 t| \leq t^4 + c_2 t^2 + c_4.$$

Hence,

$$K \leq t^4 + c_2 t^2 + |c_3 t| + c_4 \leq 2(t^4 + c_2 t^2 + c_4).$$

If $c_2 > 0$, we have for $|t| \leq \alpha$ that

$$K \leq 2(\alpha^4 + c_2 \alpha^2 + c_4) = 4(c_4 - \alpha^4) \leq 4c_4.$$

If $c_2 \leq 0$, we have for $|t| \leq \alpha$ that

$$K \leq 2(\alpha^4 + c_4) \leq 4c_4,$$

where we used $\alpha \leq \sqrt[4]{c_4}$. This proves (3.13). By (3.8), we have for $|t| \leq \alpha$ that

$$|\alpha'| \leq \frac{1}{\sqrt[4]{c_2^2 + 12c_4}} (|c_2'| + |(\sqrt{c_4})'|),$$

and hence with (3.13),

$$|\alpha'| \sqrt[4]{K} \leq \frac{\sqrt[4]{4c_4}}{\sqrt[4]{c_2^2 + 12c_4}} (|c_2'| + |(\sqrt{c_4})'|) \leq |c_2'| + |(\sqrt{c_4})'|.$$

This proves (3.12). $\square$

Note that Lemma 3.1 follows easily from Lemma 3.3. We should point out that there may be other ways to extend real zeros of K . The point is whether we can construct functions other than H in (3.6) so that their zeros satisfy Lemma 3.1.

We now prove the following global energy estimates and the existence of solutions under an extra assumption that K is small.

Theorem 3.4. *Let q, b_0, b, c, K be smooth functions in $\mathbb{R} \times [0, T]$ with $a = qK$ satisfying (1.3) and (1.5). Suppose K is a nonnegative function given by (3.2) and satisfies*

$$(3.14) \quad \sqrt[4]{K} (|c_2'| + |(\sqrt{c_4})'|) \leq \frac{1}{2\sqrt{\Lambda}} \quad \text{in } \mathbb{R} \times [0, T].$$

Then the Cauchy problem for (1.1)-(1.2) admits a smooth solution in $\mathbb{R} \times [0, T]$ for any smooth function f in $\mathbb{R} \times [0, T]$ and any smooth functions u_0, u_1 in $\mathbb{R}$. Moreover, for any $s \geq 0$,

$$\|u\|_{s, \mathbb{R} \times [0, T]} \leq C_s (\|u_0\|_{s+N, \mathbb{R}} + \|u_1\|_{s+N-1, \mathbb{R}} + \|f\|_{s+N, \mathbb{R} \times [0, T]}),$$

where C_s is a positive constant depending only on s, λ, Λ, C_b and the C^{s+N} -norms of q, K, b_0, b and c , and N is a nonnegative integer which is universal.

Proof. The proof is based on a standard regularization process. Instead of (1.1), we consider a modified equation:

$$u_{tt} - q(K + \varepsilon)u_{xx} - b_0 u_t - b u_x - c u = f.$$

This is a (strictly) hyperbolic differential equation for $\varepsilon > 0$. Hence there exists a smooth solution u_ε with the given initial values. We note that (3.14) is the smallness assumption mentioned in Lemma 3.1 and that it also holds for $K + \varepsilon$. When we apply Lemma 3.1 to $K + \varepsilon$, the resulting φ_1 and φ_2 have $C^{1/4}$ -norms and BV -norms independent of ε , although φ_1 and φ_2 themselves depend on ε . Hence, with Theorem 2.2, we get energy estimates for u_ε which are uniform in ε . With $\varepsilon \rightarrow 0+$, we obtain the desired result. $\square$

By comparing Theorem 1.1 and Theorem 3.4, we note that the former concerns the local existence and the latter the global existence. Now we prove Theorem 1.1.

Proof of Theorem 1.1. We first note that the C^N -norm of K in $[-1, 1] \times [0, 1]$ is bounded, where N is as in Theorem 3.4. We always take $N \geq 2$. For a constant $\delta > 0$ to be determined, either $K \geq \delta$ or $K \leq 10\delta$ near $(0, 0)$. In the case $K \geq \delta$, the equation is strictly hyperbolic. Hence, the local existence and energy estimates are standard. In the case $K \leq 10\delta$, we have (3.14) by choosing δ small. Here, c'_2 and $(\sqrt{c_4})'$ are bounded since the C^2 -norm of K is bounded. It is straightforward to extend all involved functions to $\mathbb{R} \times [0, T]$ such that the extended functions satisfy assumptions in Theorem 3.4. Then, Theorem 1.1 follows from Theorem 3.4. $\square$

To end this section, we give an explicit expression of the decomposition used in this paper. Let $K(x, t)$ be a nonnegative function given by (3.1). Then with the help of (3.7), φ_1 and φ_2 in Lemma 3.1 are given by

$$(3.15) \quad \begin{aligned} \varphi_1(x) &= -\frac{1}{4}c_1 + \frac{1}{\sqrt{6}}((c_2^2 - 3c_1c_3 + 12c_4)^{\frac{1}{2}} - c_2 + \frac{3}{8}c_1^2)^{\frac{1}{2}}, \\ \varphi_2(x) &= -\frac{1}{4}c_1 - \frac{1}{\sqrt{6}}((c_2^2 - 3c_1c_3 + 12c_4)^{\frac{1}{2}} - c_2 + \frac{3}{8}c_1^2)^{\frac{1}{2}}. \end{aligned}$$

This is by no means a trivial decomposition.

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