A CLASSIFICATION OF SOME UNIT LORENTZ BALLS AS NON-INTERSECTION BODIES

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(Communicated by Thomas Schlumprecht)

ABSTRACT. We show that the unit ball of the n -dimensional Lorentz space $\ell_{w,q}^n$ is not an intersection body for $q > 2$ and $n \geq 5$.

1. INTRODUCTION

A closed, bounded set K in $\mathbb{R}^n$ is called a star body if the origin is an interior point of K , if for every $x \in K$ each point on the line $[0, x)$ is an interior point of K , and if the boundary of K is continuous in the sense that the Minkowski functional

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is continuous. Following Lutwak [14] we say an origin-symmetric star body K is called an *intersection body of a star body* L if the radius of K in every direction $\xi \in S^n$ is equal to the $(n-1)$ -dimensional volume of the intersection of L and the $(n-1)$ -dimensional central hyperplane $\xi^\perp$ orthogonal to ξ . In other words,

$$\rho_K(\xi) = \text{Vol}_{n-1}(L \cap \xi^\perp),$$

for all $\xi \in S^{n-1}$, where $\rho_K(\xi) = \|\xi\|_K^{-1}$ is the radial function of K . Since we can express $\text{Vol}_{n-1}(L \cap \xi^\perp)$ as

$$\frac{1}{n-1} \int_{S^{n-1}} \|x\|_L^{-n+1} dx = R((n-1)^{-1} \|x\|_L^{-n+1}),$$

where

$$\xi \mapsto Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) dx, \quad \xi \in S^{n-1},$$

denotes the spherical Radon transform of the function f , then we can generalize our definition of an intersection body, as in [8], as follows: An origin-symmetric star body K is an *intersection body* if there exists a finite Borel measure μ on S^{n-1} such that $\|\cdot\|_K^{-1} = R\mu$ in the sense of linear functionals on the space $C(S^{n-1})$ of continuous function on the sphere, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x) dx = \int_{S^{n-1}} R(f)(x) d\mu$$

for all $f \in C(S^{n-1})$.

Received by the editors December 2, 2011 and, in revised form, April 5, 2012 and April 17, 2012.

2010 *Mathematics Subject Classification*. Primary 52A20; Secondary 42A38, 42A82, 46F12.

Key words and phrases. Lorentz space, intersection bodies.

The author would like to thank Alexander Koldobsky for all his guidance and support throughout this paper.

The concept of an intersection body is important in the dual Brunn-Minkowski theory, introduced by Lutwak in [14]. In particular, intersection bodies played an important role in the solution to the Busemann-Petty problem posed in 1956, which can be stated as follows: Suppose K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$. If $\text{Vol}_{n-1}(K \cap \xi^\perp) \leq \text{Vol}_{n-1}(L \cap \xi^\perp)$, for all $\xi \in S^{n-1}$, does it follow that $\text{Vol}_n(K) \leq \text{Vol}_n(L)$? The solution was completed in the late 1990's as a result of a series of papers [13], [1], [7], [2], [14], [15], [3], [5], [17], [19], [18], and [6]; see [4, pg. 343] or [12, pg. 3] for a history of the problem and its solution.

By a result of Lutwak in [14], if K is an intersection body, the answer to the question of the Busemann-Petty problem is affirmative for any origin-symmetric star body L . On the other hand, if L is a non-intersection body it can be perturbed to construct a body K giving, together with L , a counterexample to the Busemann-Petty problem. As shown in [5], [18] and [6], every origin-symmetric, convex body in dimension $n \leq 4$ is an intersection body. On the other hand, in dimensions 5 and higher there exist symmetric, convex, non-intersection bodies, each of which provides a counterexample to the Busemann-Petty problem in these dimensions. The first example of a non-intersection body was given by Gardner [3] as a certain rounded cylinder in $\mathbb{R}^5$. Other examples of non-intersection bodies in dimensions $n \geq 5$ were given in [19], [6], [9], [11], and [16]. In particular, the unit balls of the spaces ℓ_q^n where $n \geq 5$, $q > 2$ are not intersection bodies, as was proved in [9]. [11] (or [12, sec. 4.4]) provides a characterization of intersection bodies in terms of the second derivative of the norm. For details and more results on intersection bodies see [4] or [12, ch. 4]. In this paper, we show that the unit ball of the n -dimensional Lorentz space $\ell_{w,q}^n$ is not an intersection body for $q > 2$ and $n \geq 5$. Our proof is based on a modification of the second derivative test for intersection bodies found in [11] or [12, pg. 83]. This second derivative test is applicable only when $q > 2$, so it should be noted that, as of the writing of this paper, it is unknown to the author whether the unit ball of $\ell_{w,q}^n$ is an intersection body for $0 < q \leq 2$ and $n \geq 5$.

2. MAIN THEOREM

Let $w = (a_1, \dots, a_n) \in \mathbb{R}^n$ be such that $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ and define a norm $\|\cdot\|_{w,q}$ on $\mathbb{R}^n$ as

$$(2.1) \quad \|x\|_{w,q} = \left(a_1(x_1^*)^q + \dots + a_n(x_n^*)^q \right)^{1/q},$$

where $x_1^*, \dots, x_n^*$ is the non-increasing rearrangement of $|x_1|, \dots, |x_n|$. We call

$$\ell_{w,q}^n = (\mathbb{R}^n, \|\cdot\|_{w,q})$$

the n -dimensional Lorentz space.

Now we appeal to the theory of distributions, which one can read more about in [12, sec. 2.5]. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and

$$\mathbb{N}_0^m = \{(a_1, \dots, a_m) : a_i \in \mathbb{N} \cup \{0\}, 1 \leq i \leq m\}.$$

Let $\mathcal{S}$ denote the set of all Schwartz functions on $\mathbb{R}^n$, i.e.

$$\mathcal{S}(\mathbb{R}^n) = \left\{ \phi \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)|_2 < \infty, \forall \alpha, \beta \in \mathbb{N}_0^n \right\}.$$

If f is a locally integrable function on $\mathbb{R}^n$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, then we let $\langle f, \phi \rangle$ denote the action of f on ϕ when f is viewed as a distribution, i.e.

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x) \phi(x) dx.$$

For $\phi \in \mathcal{S}$ we let $\widehat{\phi}(\xi)$ denote the Fourier transform

$$\int_{\mathbb{R}^n} \phi(x) e^{-ix \cdot \xi} dx$$

of ϕ at ξ . Here $x \cdot \xi$ denotes the scalar product of x and ξ .

Koldobsky [10, Thm. 1], [12, Thm. 4.1] showed that an origin-symmetric star body K is an intersection body if and only if $\|\cdot\|_K^{-1}$ is a positive definite distribution; i.e. $\langle \|\cdot\|_K^{-1}, \widehat{\phi} \rangle \geq 0$ for every non-negative $\phi \in \mathcal{S}$. This brings us to the main theorem:

Theorem 2.1. *Let $n \geq 5$ and $q > 2$. The function $\|\cdot\|_{w,q}^{-p}$ represents a positive definite distribution if and only if $p \in [n-3, n)$.*

Let $B_{w,q}^n$ denote the unit ball of $\ell_{w,q}^n$. We can use Theorem 2.1 along with the fact that $\|x\|_{w,q} = \|x\|_{B_{w,q}^n}$ to get the following:

Corollary 2.2. *The unit ball of $\ell_{w,q}^n$ is not an intersection body for $q > 2$ and $n \geq 5$.*

The proof of this theorem follows the second derivative test found in [11] or [12, Thm. 4.19], and as such requires Lemma 4.1 below, which is itself a modification of Lemma 1 of [11] (or [12, Lemma 4.20]). The original proof of this lemma relies on the fact that the function $x_1 \rightarrow \|x\|_{w,q}$ has a continuous first and second derivative on $\mathbb{R}$ for a fixed $(x_2, \dots, x_n) \in \mathbb{R}^{n-1} \setminus \{0\}$. The Lorentz norm doesn't have this property on all of $\mathbb{R}^n$, but fortunately the set on which this fails has measure zero.

3. PROPERTIES OF THE LORENTZ NORM

Fix $q > 2$ and w as above. If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then y will denote $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. Let $e_1, e_2, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. Throughout the paper $\|x\|$ denotes $\|x_1 e_1 + \dots + x_n e_n\|_{w,q}$ and $\|y\|$ denotes $\|e_2 x_2 + \dots + e_n x_n\|$. We will sometimes revert back to $\|x\|_{w,q}$ for emphasis. We let $|\cdot|_2$ and B_2^n denote the norm and unit ball of ℓ_2^n respectively.

Let S_n be the group of permutations on n symbols and let

$$A_\sigma = \{x \in \mathbb{R}^n : |x_{\sigma(1)}| > |x_{\sigma(2)}| > \dots > |x_{\sigma(n)}|\}$$

for all $\sigma \in S_n$. For each such σ , we note that A_σ is open and that $\mathbb{R}^n \setminus (\bigcup_\sigma A_\sigma)$ is of Lebesgue measure zero.

For a given $\sigma \in S_n$ and $y = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$, let $\|x\|'_{x_1}(\cdot, y)$ and $\|x\|''_{x_1}(\cdot, y)$ denote the first and second derivative of the function

$$x_1 \rightarrow \|x_1 e_1 + x_2 e_2 + \dots + x_n e_n\|$$

respectively and let $A_\sigma^1(y) = \{\xi \in \mathbb{R} : (\xi, y) \in A_\sigma\}$.

Proposition 3.1. *For $q > 2$, $\sigma \in S_n$ and a fixed $y = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$ with $y \neq 0$, the function $x_1 \rightarrow \|x_1 e_1 + x_2 e_2 + \dots + x_n e_n\|$ has a continuous first and second derivative on the set $A_\sigma^1(y)$.*

Moreover, for every $(x_1, y) \in A_\sigma$ with $\|y\| = 1$ we have

$$(3.1) \quad \|x\|'_{x_1}(0, y) = \|x\|''_{x_1}(0, y) = 0,$$

and there exists a $K > 0$ such that $\|x\|''_{x_1}(x_1, y) < K$ for every $y \in \mathbb{R}^{n-1}$ with $\|y\| = 1$ and every $x_1 \in A_\sigma^1(y)$.

Proof. Fix $\sigma \in S_n$ and $y \in \mathbb{R}^{n-1} \setminus \{0\}$. We will now consider the function $x_1 \rightarrow \|x_1 e_1 + x_2 e_2 + \dots + x_n e_n\|$ defined on the set $A_\sigma^1(y)$. This function is even, so it suffices to assume $x_1 \geq 0$, which implies $(|x_1|^q)' = qx_1^{q-1}$. Let $i \in \{1, \dots, n\}$ be such that $x_i^* = |x_1|$. Since we are only considering $(x_1, y) \in A_\sigma$, the index i is constant. Let $C_{\sigma, y} = \sum_{j \neq i} a_j (x_j^*)^q$, which is constant for a fixed σ and y . A direct calculation shows

$$(3.2) \quad \|x\|'_{x_1}(x_1, y) = \frac{a_i x_1^{q-1}}{\left(a_i x_1^q + C_{\sigma, y}\right)^{1-\frac{1}{q}}}$$

and

$$(3.3) \quad \|x\|''_{x_1}(x_1, y) = \frac{(q-1)a_i x_1^{q-2} C_{\sigma, y}}{\left(a_i x_1^q + C_{\sigma, y}\right)^{2-\frac{1}{q}}}.$$

Since $y \neq 0$, $\|x\|'_{x_1}$ and $\|x\|''_{x_1}$ are both well defined, continuous functions on $A_\sigma^1(y)$. Notice also that

$$\|x\|'_{x_1}(0, y) = \|x\|''_{x_1}(0, y) = 0.$$

Now we would like to find the constant K . If $x_1 \leq C_{\sigma, y}$, then $x_1^{q-2} \leq C_{\sigma, y} \leq a_i x_1^q + C_{\sigma, y} \leq \left(a_i x_1^q + C_{\sigma, y}\right)^{2-\frac{1}{q}}$, and hence

$$\|x\|''_{x_1}(x_1, y) \leq (q-1)a_i C_{\sigma, y}.$$

If $C_{\sigma, y} < x_1$, then $x_1^{-2} < C_{\sigma, y}$, and hence $x_1^{q-2} \leq x_1^q C_{\sigma, y}$. This implies $a_1 x_1^{q-2} \leq a_1 x_1^q C_{\sigma, y} \leq a_1 x_1^q C_{\sigma, y} + C_{\sigma, y} \leq C_{\sigma, y}^{2-\frac{1}{q}} \left(a_1 x_1^q + 1\right)^{2-\frac{1}{q}}$. Therefore

$$\|x\|''_{x_1}(x_1, y) \leq (q-1)C_{\sigma, y}^{2-\frac{1}{q}}.$$

We can assume that $C_{\sigma, y} > 1$, so in either case we have bounded $\|x\|''_{x_1}$ by a constant which depends on $C_{\sigma, y}^2$. We want this constant to be the same for any $y \in \mathbb{R}^{n-1}$ with $\|y\| = 1$. In order to do this first notice $B_{w, q}^n$ is compact so there exist constants $k_1, k_2 > 0$ such that $k_1 B_2^n \subset B_{w, q}^n \subset k_2 B_2^n$, which implies $k_1 |y|_2 \leq \|y\| \leq k_2 |y|_2$ for all $y \in \mathbb{R}^{n-1}$. If $M = \max(x_j : j \neq i)$ and $\|y\| = 1$, then, for some $c > 0$,

$$\|x\|''_{x_1}(x_1, y) \leq (c \cdot C_{\sigma, y})^2 \leq (cMk_2)^2.$$

This finally shows $\|x\|''_{x_1}(x_1, y)$ is bounded by a constant independent of x_1 and y , with $\|y\| = 1$, and we call this constant K . $\square$

Proposition 3.1 shows that the functions $\|x\|'_{x_1}(\cdot, y)$ and $\|x\|''_{x_1}(\cdot, y)$ exist on $A_\sigma^1(y)$ for every fixed $\sigma \in S_n$ and every $y \in \mathbb{R}^{n-1} \setminus \{0\}$, which is why we defined the sets A_σ as we did.

Proposition 3.2. *Let $0 < p < n - 3$ and fix $\sigma \in S_n$.*

- (i) *The function $y \mapsto \|y\|^{-p-2}$ is locally integrable on $\mathbb{R}^{n-1}$.*
- (ii) *For every fixed $y \in \mathbb{R}^{n-1}$, the function $x_1 \mapsto \|x_1 e_1 + \dots + x_n e_n\|$ is a convex differentiable function whose derivative at $x_1 = 0$ is zero, and therefore $\|x\| \geq \|y\|$ for every $x = (x_1, y) \in A_\sigma$.*
- (iii) *The function $\|x\|_{x_1}''$ is non-negative, homogeneous of degree -1 . If K is the upper bound found in Proposition 3.1, then for every $(x_1, y) \in A_\sigma$ with $y \neq 0$ we have*

$$(3.4) \quad \|x\|_{x_1}''(x) = \|x\|_{x_1}'' \left(\|y\| \frac{x}{\|y\|} \right) = \frac{1}{\|y\|} \|x\|_{x_1}'' \left(\frac{x}{\|y\|} \right) \leq \frac{K}{\|y\|}.$$

- (iv) *Convergence in the limit $\lim_{x_1 \rightarrow 0} \|x\|_{x_1}''(x_1, y) = 0$ is uniform with respect to $y \in \mathbb{R}^{n-1}$ when $\|y\| = 1$.*

Proof.

- (i) Let $B_{w,q}^n$, k_1 , and k_2 be as in the proof of Proposition 3.1 so that $k_1|y|_2 \leq \|y\| \leq k_2|y|_2$ for all $y \in \mathbb{R}^{n-1}$. For $\lambda > 0$, we let λB_2^n be the origin centered ball of ℓ_2^n with radius λ . When we switch to polar coordinates we get

$$(3.5) \quad \int_{\lambda B_2^n} \|y\|^{-p-2} dy \leq 2\pi k_1^{-p-2} \frac{\lambda^{n-3-p}}{n-3-p}.$$

This shows (i) holds because the last expression in (3.5) is finite.

- (ii) A direct computation using the definition of convexity shows that the function $x_1 \mapsto \|x_1 e_1 + \dots + x_n e_n\|$ is convex on the set $A_\sigma^1(y)$. By Proposition 3.1 we know that $\|x\|_{x_1}'(0, y) = 0$, so, by convexity, $\|x\| \geq \|y\|$.
- (iii) The homogeneity of $\|x\|_{x_1}''$ follows from the fact that $\|x\|_{x_1}'$ is homogeneous of degree 0.
- (iv) For a fixed $y \in \mathbb{R}^{n-1}$ with $\|y\| = 1$ let i and $C_{\sigma,y}$ be as in Proposition 3.1. If $C_{\sigma,y} = 0$, then $\|x\|_{x_1}'' = 0$ by (3.3). Otherwise, if $m = \min\{a_j : j \neq i\}$ and $M = \max\{a_j : j \neq i\}$, then m, M , and $|y|_2$ are all positive, so, with k_1 and k_2 as in part (i), we have

$$(3.6) \quad \frac{(q-1)a_i x_1^{q-2} m}{(a_i x_1^{q-2} + M k_1^{-1})^{2-\frac{1}{q}}} \leq \frac{(q-1)a_i x_1^{q-2} C_{\sigma,y}}{(a_i x_1^q + C_{\sigma,y})^{2-\frac{1}{q}}} \leq \frac{(q-1)a_i x_1^{q-2}}{(m k_2^{-1})^{1-\frac{1}{q}}}.$$

The left and right hand sides of (3.6) are independent of y , so (iv) holds. $\square$

4. LEMMA

Define a family of even Schwartz functions $(\phi_m)_{m \in \mathbb{N}}$ as

$$(4.1) \quad \phi_m(x) = h_m(x_1)u(x_2, \dots, x_n), \quad x \in \mathbb{R}^n,$$

where for each $m \in \mathbb{N}$,

$$h_m(x_1) = \frac{m}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2 m^2}{2}\right)$$

and

$$u(x_2, \dots, x_n) = \frac{1}{(2\pi)^{(n-1)/2}} \exp\left(-\frac{x_2^2 + \dots + x_n^2}{2}\right).$$

Lemma 4.1. *Let $p \in (0, n-3)$. For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $-\epsilon \leq \langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \rangle$ whenever $m \geq N$.*

Proof. We will first show this theorem holds when $\langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \rangle$ is replaced with

$$\int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x) dx,$$

where $\sigma \in S_n$ is arbitrary.

Fix $\sigma \in S_n$. By Fubini's theorem we have

$$(4.2) \quad \int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x) dx = \int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A_\sigma^1(y)} \|x\|^{-p} \frac{d^2}{dx_1^2} h_m(x_1) dx_1 \right) dy.$$

Since $(\|x\|^{-p})''_{x_1}(\cdot, y)$ exists on each $A_\sigma^1(y)$ and since $h_m(x_1) \rightarrow 0$ as $x_1 \rightarrow \pm\infty$ for all $m \in \mathbb{N}$, we can integrate by parts twice to see that (4.2) is equal to

$$(4.3) \quad \int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A_\sigma^1(y)} (\|x\|^{-p})''_{x_1^2}(x_1, y) h_m(x_1) dx_1 \right) dy.$$

Now notice that

$$(\|x\|^{-p})''_{x_1^2}(x) = p(p+1)\|x\|^{-p-2}(\|x\|'_{x_1}(x))^2 - p\|x\|^{-p-1}\|x\|''_{x_1^2}(x),$$

so that (4.3) breaks into the sum of two integrals, the first of which is positive and the second is negative. Therefore it suffices to show that the second integral

$$(4.4) \quad \int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A_\sigma^1(y)} (\|x\|^{-p-1}\|x\|''_{x_1^2}(x_1, y) h_m(x_1) dx_1 \right) dy$$

converges to 0 as m grows large.

Let $\epsilon > 0$. By part (i) of Lemma 3.2, $\|y\|^{-p-2}$ is locally integrable on $\mathbb{R}^{n-1}$; hence

$$L = \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy < \infty.$$

By (3.5) we can choose $d > 0$ small enough so that

$$(4.5) \quad \int_{\|y\| < d} \|y\|^{-p-2} u(y) dy < \frac{\epsilon}{3K},$$

where K is the constant from Proposition 3.1. By (ii) and (iii) of Lemma 3.2 we have

$$\begin{aligned} & \int_{\|y\| < d} u(y) \left(\int_{A_\sigma^1(y)} \|x\|^{-p-1}\|x\|''_{x_1^2}(x_1, y) h_m(x_1) dx_1 \right) dy \\ & \leq \int_{\|y\| < d} u(y) \left(\int_{A_\sigma^1(y)} \|y\|^{-p-2} h_m(x_1) dx_1 \right) dy \\ (4.6) \quad & \leq K \int_{\mathbb{R}} h_m(x_1) dx_1 \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy \leq \frac{\epsilon}{3}. \end{aligned}$$

By (iv) there exists a $\delta > 0$ such that

$$(4.7) \quad \|x\|''_{x_1^2}(x_1, y) < \frac{\epsilon}{3L}$$

whenever $|x_1| < \delta$ and $\|y\| = 1$. By (iii) the second derivative of $\|x\|''_{x_1}$ is homogeneous of degree -1 , so this, along with uniform convergence, implies that

$$(4.8) \quad \|x\|''_{x_1}(x_1, y) = \frac{1}{\|y\|} \|x\|''_{x_1} \left(\frac{x_1}{\|y\|}, \frac{y}{\|y\|} \right) \leq \frac{\epsilon}{3L\|y\|}$$

whenever $|x_1|/\|y\| < \delta$.

Now consider the sets

$$E_1(y) := \{x_1 \in A_\sigma^1(y) : |x_1| < \delta d\},$$

$$E_2(y) := \{x_1 \in A_\sigma^1(y) : |x_1| > \delta d\}.$$

If $\|y\| \geq d$ and $x_1 \in E_2(y)$, then $|x_1|/\|y\| < \delta$, so we can use (4.8) to get

$$(4.9) \quad \begin{aligned} & \int_{\|y\| \geq d} u(y) \left(\int_{E_1(y)} \|x\|^{-p-1} \|x\|''_{x_1}(x_1, y) h_m(x_1) dx \right) dy \\ & \leq \frac{\epsilon}{3L} \int_{\mathbb{R}} h_m(x_1) dx_1 \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy \leq \frac{\epsilon}{3}. \end{aligned}$$

Finally, notice for a given $\eta > 0$ that

$$\lim_{m \rightarrow \infty} \int_{\{x_1 : |x_1| > \eta\}} h_m(x_1) dx_1 = 0;$$

hence we can choose $N \in \mathbb{N}$ so that for all $m > N$, the following holds:

$$(4.10) \quad \int_{\{x_1 : |x_1| > \delta d\}} h_m(x_1) dx_1 < \frac{\epsilon}{3KL}.$$

This along with (ii) and (iii) implies for all $m > N$,

$$(4.11) \quad \begin{aligned} & \int_{\|y\| \geq d} u(y) \left(\int_{E_2(y)} \|x\|^{-p-1} \|x\|''_{x_1}(x_1, y) h_m(x_1) dx \right) dy \\ & \leq K \frac{\epsilon}{3L} \int_{\mathbb{R}} h_m(x_1) dx_1 \int_{\mathbb{R}^{n-1}} \|y\|^{-p-2} u(y) dy \leq \frac{\epsilon}{3}. \end{aligned}$$

Combining (4.6), (4.9), and (4.11) gives us

$$(4.12) \quad \lim_{m \rightarrow \infty} \int_{\mathbb{R}^{n-1}} u(y) \left(\int_{A_\sigma^1(y)} \|x\|^{-p-1} \|x\|''_{x_1}(x_1, y) h_m(x_1) dx_1 \right) dy = 0.$$

Since ϵ and σ were arbitrary we must have that this theorem holds when $\langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \rangle$ is replaced with $\int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x) dx$, where $\sigma \in S_n$. Notice that

$$(4.13) \quad \begin{aligned} \langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \rangle &= \int_{\mathbb{R}^n} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x) dx \\ &= \sum_{\sigma \in S_n} \int_{A_\sigma} \|x\|^{-p} \frac{d^2}{dx_1^2} \phi_m(x) dx. \end{aligned}$$

S_n is finite, so (4.13) along with our previous work completes the lemma. $\square$

5. PROOF OF THE MAIN THEOREM

Proof. Let $n \geq 5$. With $B_{w,q}^n$ still denoting the unit ball of $\ell_{w,q}^n$ we have that $B_{w,q}^n$ is an origin-symmetric convex body in $\mathbb{R}^n$, so Corollary 4.9 of [12] shows that if $p \in [n-3, n)$, then $\|x\|_{B_{w,q}^n}^{-p} = \|\cdot\|_{w,q}^{-p}$ represents a positive definite distribution.

Now suppose $p \in (0, n-3)$ and that, by way of contradiction, $\|\cdot\|^{-p} = \|\cdot\|_{w,q}^{-p}$ is a positive definite distribution. By Corollary 2.26 of [12] there exists a finite Borel measure μ on S^{n-1} such that for every even test function ϕ ,

$$(5.1) \quad \langle \|\cdot\|^{-p}, \phi \rangle = \int_{\mathbb{R}^n} \|x\|^{-p} \phi(x) dx = \int_{S^{n-1}} \left(\int_0^\infty t^{p-1} \widehat{\phi}(t\xi) dt \right) d\mu(\xi).$$

Since $(\partial_{x_1}^2 \phi)^\wedge(\xi) = -\xi_1^2 \widehat{\phi}(\xi)$ we have

$$(5.2) \quad \begin{aligned} \left\langle (\|\cdot\|^{-p})''_{x_1}, \phi \right\rangle &= \langle \|\cdot\|^{-p}, (\partial_{x_1}^2 \phi)(\xi) \rangle \\ &= - \int_{S^{n-1}} \left(\int_{\mathbb{R}} |t|^{p+1} \widehat{\phi}(t\xi) dt \right) \xi_1^2 d\mu(\xi). \end{aligned}$$

This holds for all even test functions ϕ , so let us consider the functions ϕ_m as defined in (4.1). Taking the Fourier transform we have

$$(5.3) \quad \widehat{\phi}_m(\xi) = \exp\left(-\frac{\xi_1^2}{2m^2}\right) \exp\left(-\frac{\xi_2^2 + \dots + \xi_n^2}{2}\right).$$

Using (5.1), (5.2), and (5.3) along with the change of variable

$$x = \frac{t^2}{2} \exp\left(\frac{\xi_1^2}{m^2} + \xi_2^2 + \dots + \xi_n^2\right)$$

gives us

$$(5.4) \quad \begin{aligned} \langle (\|\cdot\|^{-p})''_{x_1}, \phi_m \rangle &= -2 \int_{S^{n-1}} \left(\int_0^\infty t^{p+1} \widehat{\phi}_m(t\xi) dt \right) \xi_1^2 d\mu(\xi) \\ &= -2^{(p+1)/2} \Gamma\left(\frac{p+2}{2}\right) \int_{S^{n-1}} \xi_1^2 \left(\frac{\xi_1^2}{m^2} + \xi_2^2 + \dots + \xi_n^2 \right) d\mu(\xi) \\ &\leq -2^{(p+1)/2} \Gamma\left(\frac{p+2}{2}\right) \int_{S^{n-1}} \xi_1^2 d\mu(\xi), \end{aligned}$$

where the last inequality follows because $\xi \in S^{n-1}$. By Lemma 4.1, we can make the left side of (5.4) as close to zero as we like by taking m large enough. Γ is positive on $(0, \infty)$, so the integral on the right side of (5.4) is non-positive; hence

$$(5.5) \quad \int_{S^{n-1}} \xi_1^2 d\mu(\xi) = 0.$$

This implies μ is supported on the intersection of the hyperplane $\xi_1 = 0$ and the sphere S^{n-1} . By (5.2) we have $(\|x\|^{-p})''_{x_1}(x) = 0$ on each A_σ , which implies for each $\sigma \in S_n$ that there exist two functions $g_{1,\sigma}(x_2, \dots, x_n)$ and $g_{1,\sigma}(x_2, \dots, x_n)$ such that $\|x\|^{-p} = g_{1,\sigma}(x_2, \dots, x_n) + x_1 g_{2,\sigma}(x_2, \dots, x_n)$ on A_σ . This is a contradiction because we know $\|x\|^{-p} \rightarrow 0$ as $x_1 \rightarrow \infty$ for all fixed $(x_2, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$, but this new representation of $\|x\|^{-p}$ as a linear function of x_1 shows $\|x\|^{-p} \rightarrow \infty$ on each A_σ as $x_1 \rightarrow \infty$. This completes the proof of the main theorem. $\square$

6. ACKNOWLEDGEMENTS

The author would like to thank his advisor, Alexander Koldobsky, for valuable feedback and guidance throughout the research and preparation of this article. He would also like to thank the referee for useful comments which have improved this article. Lastly, the author would like to thank the University of Missouri for support during his studies.

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