THE NON-PARABOLICITY OF INFINITE VOLUME ENDS

M. P. CAVALCANTE, H. MIRANDOLA, AND F. VITÓRIO

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ABSTRACT. Let M^m , with $m \geq 3$, be an m -dimensional complete non-compact manifold isometrically immersed in a Hadamard manifold M . Assume that the mean curvature vector has finite L^p -norm, for some $2 \leq p \leq m$. We prove that each end of M must either have finite volume or be non-parabolic.

1. INTRODUCTION

Let $(M^m, \langle \cdot, \cdot \rangle)$ be a complete non-compact Riemannian manifold without boundary. We recall that M is *parabolic* if it does not admit a non-constant positive superharmonic function. Otherwise, it is said to be *non-parabolic*. There exist equivalent definitions for parabolic manifolds (see for instance Theorem 5.1 of [8]). Let $E \subset M$ be an *end* of M , that is, an unbounded connected component of $M - \Omega$, for some compact subset $\Omega \subset M$. The property of parabolicity can be localized on each end of M . Namely, we say that an end E is *parabolic* (see Definition 2.4 of [10]) if it does not admit a harmonic function $f : E \rightarrow \mathbb{R}$ satisfying:

- (1) $f|_{\partial E} = 1$;
- (2) $\liminf_{\substack{y \rightarrow \infty \\ y \in E}} f(y) < 1$.

Otherwise, we say that E is a *non-parabolic* end of M . It is well known that M is non-parabolic if and only if it admits a non-parabolic end. Furthermore, ends with finite volume are parabolic (see for instance Section 14.4 of [8]). In this direction we recall the following result due to Li and Wang:

Theorem A (Corollary 4 of [12] and Corollary 2.9 of [10]). *Let E be an end of a complete manifold. Suppose that, for some constants $\nu \geq 1$ and $C > 0$, E satisfies a Sobolev-type inequality of the form*

$$(1.1) \quad \left(\int_E |u|^{2\nu} \right)^{\frac{1}{\nu}} \leq C \int_E |\nabla u|^2,$$

for all compactly supported Sobolev function $u \in W_c^{1,2}(E)$. Then E must either have finite volume or be non-parabolic. Moreover, in the case $\nu > 1$, E must be non-parabolic.

Note that if a complete manifold M that satisfies a Sobolev inequality as in Theorem A with $\nu = 1$ (that is just the Dirichlet Poincaré inequality), then the first eigenvalue $\lambda_1(M)$ of the Laplace-Beltrami operator is positive; hence M must

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be non-parabolic (see Proposition 10.1 of [8]). Example 4.1 below exhibits a complete manifold that contains a finite volume end and that also satisfies a Sobolev inequality as in Theorem A with $\nu = 1$.

Cao, Shen and Zhu [2] showed that if M^m , with $m \geq 3$, is a complete manifold, then each end of M is non-parabolic provided that M can be realized as a minimal submanifold in a Euclidean space $\mathbb{R}^n$. The same conclusion also was obtained by Fu and Xu [7] provided that there exists an isometric immersion of M in a Hadamard manifold $\bar{M}$ with finite total mean curvature, that is, the mean curvature vector field H of the immersion satisfies $\|H\|_{L^m(M)} < \infty$. In both cases, they observed that M admits a Sobolev-type inequality as in Theorem A with $\nu > 1$.

Our main result states the following:

Theorem 1.1. *Let $x : M^m \rightarrow \bar{M}$, with $m \geq 3$, be an isometric immersion of a complete non-compact manifold M in a Hadamard manifold $\bar{M}$. Let E be an end of M such that the mean curvature vector satisfies $\|H\|_{L^p(E)} < \infty$, for some $2 \leq p \leq m$. Then E must either have finite volume or be non-parabolic.*

Example 4.3 below exhibits an example of a complete non-compact hypersurface M^m in $\mathbb{R}^{m+1}$, with $m \geq 3$, of finite volume and mean curvature vector with finite L^p -norm, for all $2 \leq p < m - 1$. This example shows that Theorem 1.1 is not a consequence of Theorem A (except when $p = m$). Note also that the catenoids in $\mathbb{R}^3$ are parabolic minimal surfaces whose ends have infinite area, which shows that the hypothesis $m \geq 3$ is essential.

In the present paper we also give a unified proof of the following fact:

Theorem B. *Let $x : M \rightarrow \bar{M}$ be an isometric immersion of a complete non-compact manifold M in a manifold $\bar{M}$ with bounded geometry (i.e., $\bar{M}$ has sectional curvature bounded from above and injectivity radius bounded from below by a positive constant). Let E be an end of M and assume that the mean curvature vector of x satisfies $\|H\|_{L^p(E)} < \infty$, for some $m \leq p \leq \infty$. Then E must have infinite volume.*

The fact above was proved by Frensel [4] and by do Carmo, Wang and Xia [3] for the case where the mean curvature vector field is bounded in norm (the case $p = \infty$), by Fu and Xu [7] for the case where the total mean curvature is finite (the case $p = m$) and by Cheung and Leung [1] for the case where the mean curvature vector has finite L^p -norm for some $p > m$. Since the cylinders of the form $M^m = \mathbb{S}^{m-1} \times \mathbb{R}$, where $\mathbb{S}^{m-1}$ is the unit Euclidean $(m - 1)$ -dimensional sphere, are examples of complete parabolic hypersurfaces in $\mathbb{R}^{m+1}$ we conclude that boundedness of the mean curvature vector does not imply that M admits a Sobolev-type inequality. Furthermore, for all $m \geq 3$, we exhibit an example of a parabolic complete non-compact hypersurface M^m in $\mathbb{R}^{m+1}$ such that the mean curvature vector has finite L^p -norm, for all $p > 2(m - 1)$. These examples show that Theorem B is not a consequence of Theorem A.

Two questions arise in this paper: is there an example of a complete non-compact submanifold M^m , with $m \geq 3$, in a Euclidean space satisfying one of the conditions below?

- (1) M has finite volume and $\|H\|_{L^p(M)} < \infty$, for some $m - 1 \leq p < m$;
- (2) M is parabolic and $\|H\|_{L^p(M)} < \infty$, for some $m < p \leq 2(m - 1)$.

2. PROOF OF THEOREM 1.1

Choose $r_0 > 0$ so that the geodesic ball $B_{r_0} \subset M$ of radius r_0 and center at some point $\xi_0 \in M$ satisfies $\partial E \subset B_{r_0}$. For each $r > r_0$, consider $E_r = E \cap B_r$ and let $f_r : \overline{E_r} \rightarrow \mathbb{R}$ be a solution of the Dirichlet Problem:

$$\begin{cases} \Delta_M f_r = 0 & \text{in } E_r, \\ f_r = 1 & \text{in } \partial E, \\ f_r = 0 & \text{on } E \cap \partial B_r. \end{cases}$$

It follows from the maximum principle that $0 < f_r \leq f_s < 1$ in E_r , for all $s \geq r$. Hence, by standard gradient estimates it follows that $\{f_r\}$ is an equicontinuous family which converges uniformly on compact subsets, when r goes to infinity, to a function $f : E \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} \Delta_M f = 0 & \text{in } E, \\ 0 \leq f \leq 1 & \text{in } E, \\ f = 1 & \text{on } \partial E. \end{cases}$$

If $f \not\equiv 1$, then it follows from the maximum principle that $\liminf_{x \rightarrow E(\infty)} f(x) < 1$, which shows that E is non-parabolic. Furthermore, it is well known that an end of finite volume is parabolic (see section 14.4 of [8]). Hence, to prove Theorem 1.1, it is sufficient to show the following:

Claim 2.1. Either $f \not\equiv 1$ or $\text{vol}(E)$ is finite.

Suppose, by contradiction, that $f \equiv 1$ and $\text{vol}(E)$ is infinite. This implies that, given any $L > 1$, there exists $r_1 > r_0$ such that $\text{vol}(E_{r_1} - E_{r_0}) > 2L$. Since $f_r \rightarrow 1$ uniformly on compact subsets, there exists $r_2 > r_1$ such that $f_r^{\frac{2m}{m-2}} > \frac{1}{2}$ everywhere in E_{r_1} , for all $r > r_2$. Thus, defining $h(r) = \int_{E_r - E_{r_0}} f_r^{\frac{2m}{m-2}}$, with $r > r_0$, we obtain

$$(2.1) \quad h(r) \geq \int_{E_{r_1} - E_{r_0}} f_r^{\frac{2m}{m-2}} > L,$$

for all $r > r_2$. In particular, we have that $\lim_{r \rightarrow \infty} h(r) = \infty$.

Now, for each $r > r_0$, let $\varphi = \varphi_r \in C_0^\infty(E)$ be a cut-off function satisfying:

- (1) $0 \leq \varphi \leq 1$ everywhere in E ;
- (2) $\varphi \equiv 1$ in $E_r - E_{r_0}$.

By the Hoffmann-Spruck Inequality [9] we have

$$S^{-1} \left(\int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq \int_{E_r} |\nabla(\varphi f_r)|^2 + \int_{E_r} (\varphi f_r)^2 |H|^2,$$

where S is a positive constant.

Note that

$$|\nabla(\varphi f_r)|^2 = f_r^2 |\nabla \varphi|^2 + \varphi^2 |\nabla f_r|^2 + \frac{1}{2} \langle \nabla \varphi^2, \nabla f_r^2 \rangle$$

and

$$\varphi^2 |\nabla f_r|^2 = \text{div}_M((f_r \varphi^2) \nabla f_r) - \frac{1}{2} \langle \nabla \varphi^2, \nabla f_r^2 \rangle,$$

since f_r is harmonic. Using that $f_r\varphi$ vanishes on ∂E_r we obtain

$$\begin{aligned} S^{-1} \left(\int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} &\leq \int_{E_r} f_r^2 |\nabla \varphi|^2 + \int_{E_r} \operatorname{div}_M((f_r \varphi^2) \nabla f_r) \\ &\quad + \int_{E_r} (\varphi f_r)^2 |H|^2 \\ &= \int_{E_r} f_r^2 |\nabla \varphi|^2 + \int_{E_r} (\varphi f_r)^2 |H|^2. \end{aligned}$$

Thus, since $0 \leq \varphi \leq 1$ in E and $\varphi \equiv 1$ in $E_r - E_{r_0}$, we obtain

$$(2.2) \quad S^{-1} h(r)^{\frac{m-2}{m}} \leq S^{-1} \left(\int_{E_r} (\varphi f_r)^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} f_r^2 |\nabla \varphi|^2 + \int_{E_r} f_r^2 |H|^2.$$

First, assume that $\|H\|_{L^2(E)}$ is finite. Then, since $0 \leq f_r \leq 1$, we have

$$S^{-1} h(r)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} |\nabla \varphi|^2 + \int_M |H|^2.$$

Thus, $\lim_{r \rightarrow \infty} h(r) < \infty$, which is a contradiction. Now, assume that $\|H\|_{L^p(E)}$ is finite, for some $2 < p \leq m$. Note that $\frac{m}{m-2} \leq \frac{p}{p-2}$. Since $0 \leq f_r \leq 1$ and $h(r) > 1$, for all $r > r_2$, we have:

- (1) $f_r^{\frac{2p}{p-2}} \leq f_r^{\frac{2m}{m-2}}$;
- (2) $h(r)^{\frac{p-2}{p}} \leq h(r)^{\frac{m-2}{m}}$, for all $r > r_2$.

Thus, using the Hölder Inequality, we have

$$\begin{aligned} (2.3) \quad \int_{E_r - E_{r_0}} f_r^2 |H|^2 &\leq \|H\|_{L^p(E_r - E_{r_0})}^2 \left(\int_{E_r - E_{r_0}} f_r^{\frac{2p}{p-2}} \right)^{\frac{p-2}{p}} \\ &\leq \|H\|_{L^p(E - E_{r_0})}^2 h(r)^{\frac{m-2}{m}}, \end{aligned}$$

for all $r > r_2$.

Choose $r_0 > 0$ large enough so that $\|H\|_{L^p(E - E_{r_0})}^2 < \frac{1}{2S}$. Using (2.2) and (2.3) we obtain the following:

$$S^{-1} h(r)^{\frac{m-2}{m}} \leq \int_{E_{r_0}} |\nabla \varphi|^2 + \int_{E_{r_0}} |H|^2 + \frac{S^{-1}}{2} h(r)^{\frac{m-2}{m}}.$$

This shows that $\lim_{r \rightarrow \infty} h(r) < \infty$, which is a contradiction. Therefore, Claim 2.1 and Theorem 1.1 are proved.

3. PROOF OF THEOREM B

Since $\bar{M}$ has bounded geometry, the sectional curvature $\bar{K}$ and the injectivity radius $i(\bar{M})$ of $\bar{M}$ satisfy:

$$(3.1) \quad \bar{K} < b^2 \text{ and } i(\bar{M}) > r_0,$$

for some positive constants b and r_0 . Let E be an end of M and assume that $\|H\|_{L^p(E)}$ is finite, for some $m \leq p \leq \infty$. Fix $0 < R_0 < \min\{r_0, \frac{\pi}{2b}\}$, take $\xi_0 \in M$ and consider $B_R = B_R(\xi_0)$, for all $R > 0$. Choose $R_1 > R_0$, sufficiently large, so that $\partial E \subset B_{R_1}$ and the distance $d_M(\partial E, x) > R_0$, for all x in $E - B_{R_1}$. Let $q \in \tilde{E} = E - B_{2R_1}$ and $0 < R < R_0$. Since $B_R(q) \subset E - B_{R_1}(\xi_0)$ we obtain, by

the isoperimetric inequality for submanifolds (Theorem 2.2 of [9]) and the Hölder Inequality, the following:

$$(3.2) \quad S \operatorname{vol}(B_R(q))^{\frac{m-1}{m}} \leq \operatorname{vol}(\partial B_R(q)) + \|H\|_{L^p(E-B_{R_1}(\xi_0))} \operatorname{vol}(B_R(q))^{\frac{p-1}{p}}$$

where $S > 0$ is a constant that depends only on m .

Assume, by contradiction, that $\operatorname{vol}(E)$ is finite. Take $R_1 > 0$ sufficiently large so that

$$(3.3) \quad \operatorname{vol}(E - B_{R_1}) < 1 \text{ and } \|H\|_{L^p(E-B_{R_1})} < \frac{S}{2}.$$

Since $p \geq m$ we have that $\frac{p-1}{p} \geq \frac{m-1}{m}$. By (3.3) and using that $B_R(q) \subset E - B_{R_1}$ we have that $\operatorname{vol}(B_R(q))^{\frac{p-1}{p}} \leq \operatorname{vol}(B_R(q))^{\frac{m-1}{m}}$. Thus, using (3.2) and (3.3), we obtain

$$(3.4) \quad \frac{S}{2} \operatorname{vol}(B_R(q))^{\frac{m-1}{m}} \leq \operatorname{vol}(\partial B_R(q)).$$

By the coarea formula, we have that $\operatorname{vol}(\partial B_R(q)) = \frac{d}{dR} \operatorname{vol}(B_R(q))$. Using (3.4), we obtain $\frac{d}{dR} (\operatorname{vol}(B_R(q)))^{\frac{1}{m}} \geq \frac{S}{2m}$. This implies that

$$(3.5) \quad \operatorname{vol}(B_R(q)) \geq \frac{S}{2m} R^m,$$

for all $q \in E_1$ and $0 < R < R_0$.

Since M is complete and $E \subset M$ is connected and non-compact, there exists a sequence $p_2, p_3, \dots$ in E such that

$$(3.6) \quad p_k \in E \cap (B_{2kR_1} - B_{(2k-1)R_1}).$$

Note that $B_{R_0/2}(p_k) \subset E - B_{R_1}$ and $B_{R_0/2}(p_k) \cap B_{R_0/2}(p_{k'}) = \emptyset$, for all $k \neq k'$. Since

$$(3.7) \quad \operatorname{vol}(E) \geq \operatorname{vol}(E - B_{R_1}) \geq \sum_{k=2}^{\infty} \operatorname{vol}(B_{R_0/2}(p_k)),$$

it follows from (3.5) that $\operatorname{vol}(E)$ is infinite. Theorem B is proved.

4. EXAMPLES

Example 4.1. Consider the warped product manifold $M^m = \mathbb{R} \times_{e^t} P$, where P is any complete $(m-1)$ -dimensional manifold with finite volume. The metric of M is complete and the end $E = (-\infty, 0) \times P \subset M$ has finite volume given by

$$\operatorname{vol}(E) = \int_{-\infty}^0 \int_P e^{m-1} dt dP = \frac{\operatorname{vol}(P)}{m-1}.$$

Fix $k \in \mathbb{R}$ and let $h_k : M \rightarrow \mathbb{R}$ be the function defined by $h_k(t, x) = \kappa t$. The gradient vector field of h_k satisfies

$$(4.1) \quad \nabla h_k = \kappa \frac{\partial}{\partial t},$$

where $\frac{\partial}{\partial t}(t, x) = \frac{d}{ds} \big|_{s=t} (s, x) \in T_{(t,x)}M$. It is simple to show that $\nabla_Z \frac{\partial}{\partial t} = Z - \langle Z, \frac{\partial}{\partial t} \rangle \frac{\partial}{\partial t}$. This implies that the Laplacian of h_k satisfies

$$(4.2) \quad \Delta h_k = \kappa \operatorname{div} \left(\frac{\partial}{\partial t} \right) = \kappa (m-1).$$

Fix $\eta \in C_0^\infty(M)$. Using (4.1) and (4.2) we obtain

$$\begin{aligned} \kappa(m-1) \int_M \eta^2 &= \int_M \eta^2 \Delta h_\kappa = \int_M (\operatorname{div}(\eta^2 \nabla h_\kappa) - 2\eta \langle \nabla \eta, \nabla h_\kappa \rangle) \\ &= -2 \int_M \langle \nabla \eta, \eta \nabla h_\kappa \rangle \geq - \int_M |\nabla \eta|^2 - \eta^2 |\nabla h_\kappa|^2 \\ &= - \int_M |\nabla \eta|^2 - \kappa^2 |\eta|^2. \end{aligned}$$

Thus, it holds that $\int_M |\nabla \eta|^2 + \kappa(\kappa + (m-1))\eta^2 \geq 0$, for all $k \in \mathbb{R}$. In particular, if we take $\kappa = -\frac{m-1}{2}$ we obtain

$$\int_M |\nabla \eta|^2 - \frac{(m-1)^2}{4} \eta^2 \geq 0.$$

Hence M satisfies a Sobolev inequality as in Theorem A with $\nu = 1$.

Example 4.2. Let $f : (-\infty, \infty) \rightarrow (0, \infty)$ be a positive smooth function satisfying that $f(t) = f(-t)$ and $f(t) = t^{\frac{1}{m-1}}$, for all $t \geq 1$. Consider the immersion $x : \mathbb{S}^{m-1} \times \mathbb{R} \rightarrow \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$ given by $x(v, t) = (f(t)v, t)$. Consider M the product $\mathbb{S}^{m-1} \times \mathbb{R}$ endowed with the metric induced by x . The metric of M is given by

$$(4.3) \quad \langle \cdot, \cdot \rangle_{(v,t)} = (1 + f'(t)^2) dt^2 + f(t)^2 \langle \cdot, \cdot \rangle_v,$$

where $\langle \cdot, \cdot \rangle_v$ denotes the metric of $\mathbb{S}^{m-1}$. Note that M is a complete manifold with two ends.

We claim that M is parabolic. To do this, it is sufficient to prove that the following ends of M :

$$E_+ = (1, \infty) \times \mathbb{S}^{m-1} \text{ and } E_- = (-\infty, -1) \times \mathbb{S}^{m-1}$$

are parabolic (see Proposition 14.1 of [8]). In fact, we define:

$$V_+(s) = \operatorname{vol}_M(\{q \in E_+ \mid d(q, \partial E_+) \leq s\})$$

and

$$V_-(s) = \operatorname{vol}_M(\{q \in E_- \mid d(q, \partial E_-) \leq s\}).$$

Using (4.3) and that $f(t) = t^{\frac{1}{m-1}}$, for all $|t| \geq 1$, we obtain that

$$V_+(s) = V_-(s) \leq Ds^2,$$

for some constant $D > 0$ and for all $s \geq 1$. In particular,

$$\int_1^\infty \frac{s}{V_+(s)} ds = \int_1^\infty \frac{s}{V_-(s)} ds = \infty.$$

This implies that M is parabolic (see section 14.4 of [8]).

We claim that the mean curvature vector H of the isometric immersion x has finite L^p -norm, for all $p > m$. In fact, a simple computation shows that

$$(4.4) \quad mH(x(v, t)) = \frac{(m-1)}{f(t)\sqrt{1+f'(t)^2}} - \frac{f''(t)}{(1+f'(t)^2)^{\frac{3}{2}}}.$$

Using that $f(t) = t^{\frac{1}{m-1}}$, for all $|t| \geq 1$, we obtain that $|H(x(v, t))| \leq Ct^{-\frac{1}{m-1}}$, for some $C > 0$ and for all $x(v, t) \in E_+ \cup E_-$. Thus, we obtain

$$\int_M |H|^p dM \leq D \int_1^\infty t^{1-\frac{p}{m-1}} dt,$$

for some $D > 0$. This implies that $\|H\|_{L^p(M)}$ is finite when $p > 2(m-1)$.

Example 4.3. Let $x : \mathbb{S}^{m-1} \times \mathbb{R} \rightarrow \mathbb{R}^{m+1}$ be the immersion given by $x(v, t) = (e^{-t^2}v, t)$ and consider $M = \mathbb{S}^{m-1} \times \mathbb{R}$ endowed with the metric induced by x . The metric of M is complete and the volume of M is given by

$$(4.5) \quad \text{vol}(M) = \omega_{m-1} \int_{-\infty}^{+\infty} (1 + 4t^2 e^{-2t^2})^{\frac{1}{2}} e^{-(m-1)t^2} dt,$$

where ω_{m-1} is the volume of $\mathbb{S}^{m-1}$. This implies that $\text{vol}(M)$ is finite, since the integral $\int_{-\infty}^{+\infty} e^{-(m-1)t^2} dt$ is finite and the function $t \in \mathbb{R} \mapsto 1 + 4t^2 e^{-2t^2}$ is bounded. In particular, M is parabolic since it has finite volume (see Theorem 7.3 of [8]).

The mean curvature vector H of the isometric immersion x is given by

$$(4.6) \quad H(x(v, t)) = h(t) = \frac{2e^{-t^2}(1 - 2t^2)}{m(4t^2 e^{-2t^2} + 1)^{\frac{3}{2}}} + \frac{(m-1)e^{t^2}}{m(4t^2 e^{-2t^2} + 1)^{\frac{1}{2}}}.$$

Using that $\lim_{t \rightarrow \infty} e^{-t^2}(1 - 2t^2) = \lim_{t \rightarrow \infty} 4t^2 e^{-2t^2} = 0$ we obtain that

$$(4.7) \quad \lim_{t \rightarrow \pm\infty} h(t)e^{-t^2} = \frac{m-1}{m}.$$

Thus the integral

$$(4.8) \quad \begin{aligned} \int_M |H|^p &= \omega_{n-1} \int_{-\infty}^{+\infty} \left(|h(t)|^p (1 + 4t^2 e^{-2t^2})^{\frac{1}{2}} e^{-(m-1)t^2} \right) dt \\ &= \omega_{n-1} \int_{-\infty}^{+\infty} \left((|h(t)|e^{-t^2})^p (1 + 4t^2 e^{-2t^2})^{\frac{1}{2}} \right) e^{(p-m+1)t^2} dt \end{aligned}$$

converges if $0 \leq p < m-1$ and diverges if $p \geq m-1$.

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INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE ALAGOAS, MACEIÓ, AL, CEP 57072-970, BRAZIL

E-mail address: marcos.petrucio@pq.cnpq.br

INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, RIO DE JANEIRO, RJ, CEP 21945-970, BRASIL

E-mail address: mirandola@im.ufrj.br

INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE ALAGOAS, MACEIÓ, AL, CEP 57072-970, BRAZIL

E-mail address: feliciano.vitorio@pq.cnpq.br