THE SET OF CHARACTER DEGREES OF A FINITE GROUP DOES NOT DETERMINE ITS SOLVABILITY

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ABSTRACT. We find a pair of groups, one solvable and the other non-solvable, with the same set of character degrees.

1. INTRODUCTION

Despite the abundant literature on the set of character degrees of finite groups, a basic question has remained unsettled until now. Suppose that G is a finite group, let $\text{Irr}(G)$ be the set of the irreducible complex characters of G , and write

$$\text{cd}(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$$

for the set of character degrees of G .

Theorem A. *There exist two finite groups G and H such that $\text{cd}(G) = \text{cd}(H)$, with G solvable and H non-solvable.*

This solves Problem (8.4) in [3], for instance. The non-solvable group H in Theorem A is not simple. In fact, B. Huppert has conjectured that if H is simple and $\text{cd}(G) = \text{cd}(H)$, then $G = H \times A$ for some abelian group A [2]. Theorem A is also closely related to R. Brauer's Problem 2 in [1]. Brauer asked what kind of properties the group algebra $\mathbb{C}G$ knows about the group G . Even if we now know that $\text{cd}(G)$ does not determine the solvability of G , perhaps the group algebra does.

2. THE GROUPS

For our non-solvable group, we let H be the perfect group of order $37500 = 2^2 \cdot 3 \cdot 5^5$, which is an extension of the alternating group A_5 of degree 5 by an elementary abelian 5-group of order 5^4 . This is a group with set of character degrees $\mathcal{A} = \{1, 3, 4, 5, 10, 12, 15, 20, 30, 60\}$.

Now, we construct a solvable group G such that $\text{cd}(G) = \mathcal{A}$. Let $V = GF(3^4)$ be the additive group of the field of dimension 4 over $GF(3)$. Let σ be a generator of the Galois group $\text{Gal}(GF(3^4)/GF(3))$ of order 4, let U be the subgroup of order 5 inside the multiplicative group $GF(3^4)^\times$, and let $X = VD$ be the semidirect product of order $2^2 \cdot 3^4 \cdot 5$, where $D = U\langle\sigma\rangle$. The action of D on $\text{Irr}(V)$ has 8 orbits with representatives $\mathcal{B} = \{1_V, \lambda_i, \epsilon_j, \rho_k\}$, $i = 1, 2$, $j = 1, 2$, and $k = 1, 2, 3$. The D -orbit of λ_i has size 5 for $i = 1, 2$; the D -orbit of ϵ_j has size 20 for $j = 1, 2$;

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and the D -orbit of ρ_k has size 10 for $k = 1, 2, 3$. Every irreducible character of V extends to its stabilizer. Hence, it is easy to check that $\text{cd}(X) = \{1, 4, 5, 10, 20\}$.

Now, let $Y = A_4$, the alternating group of degree 4. Hence $\text{cd}(Y) = \{1, 3\}$. We now let $G = X \times Y$, and we have that

$$\text{cd}(G) = \{nm \mid n \in \text{cd}(X), m \in \text{cd}(Y)\} = \mathcal{A}.$$

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