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A NOTE ON ZERO-SETS OF FRACTIONAL SOBOLEV FUNCTIONS WITH NEGATIVE POWER OF INTEGRABILITY

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Abstract. We extend a Poincaré-type inequality for functions with large zero-sets by Jiang and Lin to fractional Sobolev spaces. As a consequence, we obtain a Hausdorff dimension estimate on the size of zero-sets for fractional Sobolev functions whose inverse is integrable. Also, for a suboptimal Hausdorff dimension estimate, we give a completely elementary proof based on a pointwise Poincaré-style inequality.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. For functions $u : \Omega \to \mathbb{R}^n$ we are interested in the size of the zero set Σ,

$$\Sigma := \{x \in \Omega : \quad \lim_{r\to 0} \fint_{B_r(x)} |f| = 0\},$$

under the condition that for some $\alpha > 0$,

$$\int_\Omega |f|^{-\alpha} < \infty. \tag{1.1}$$

Here and henceforth, for a measurable set $A \subset \mathbb{R}^n$ we denote the mean value integral

$$\fint_A f \equiv (f)_A := |A|^{-1} \int_A f.$$

In [8] Jiang and Lin showed that if $f \in W^{1,p}(\Omega)$, then

$$\mathcal{H}^s(\Sigma) = 0 \quad \text{where } s = \max\{0, n - \tfrac{p\alpha}{p+\alpha}\}.$$

They were motivated by the analysis of rupture sets of thin films, which is described by a singular elliptic equation. We do not go into the details of this; instead, for applications, we refer to, e.g., [2, 3, 6, 7].

In this note, we extend Jiang and Lin's result to fractional Sobolev spaces and obtain

Theorem 1.1. *For $\sigma \in (0, 1]$ and for any $f \in W^{\sigma,p}(\Omega)$ satisfying (1.1), $\mathcal{H}^s(\Sigma) = 0$, where $s = \max\{0, n - \sigma\frac{p\alpha}{p+\alpha}\}$.*

Here, we use the following definitions for the (fractional) Sobolev space. For more on these we refer to, e.g., [1, 4, 10].

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Definition 1.2. The homogeneous $W^{\sigma,p}$-norms are defined as follows:

$$[f]_{\dot{W}^{1,p}(\Omega)} := \|\nabla f\|_{L^p(\Omega)}.$$

For $\sigma \in (0,1)$ we define the Slobodeckij-norm,

$$[f]_{\dot{W}^{\sigma,p}(\Omega)} := \begin{cases} \left(\int\limits_\Omega \int\limits_\Omega \left(\frac{|f(x)-f(y)|}{|x-y|^\sigma} \right)^p \frac{dx\ dy}{|x-y|^n} \right)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \\ \sup\limits_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^\sigma} & \text{if } p = \infty. \end{cases}$$

The respective Sobolev space $W^{\sigma,p}$, $\sigma \in (0,1]$, $p \in [1,\infty]$ is then the collection of functions $f : \Omega \to \mathbb{R}$ with finite Sobolev norms $\|f\|_{W^{\alpha,p}(\Omega)}$,

$$\|f\|_{W^{\alpha,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{\dot{W}^{\alpha,p}(\Omega)}.$$

To prove Theorem 1.1, the case $p \leq n/\sigma$ is the relevant one, since for the other cases we can use the embedding into the Hölder spaces; see [8]. We have the following extension to fractional Sobolev spaces of a Poincaré-type inequality from [8].

Theorem 1.3. *For any $\theta > 0$, $\sigma \in (0,1]$, $p \in (1, n/\sigma]$, $s \in (n - \sigma p, n]$, there is a constant $C > 0$ such that the following holds for any $R > 0$:*

Let B_R be any ball in $\mathbb{R}^n$ with radius R, $f \in W^{\sigma,p}(B_R)$ and assume that there is a closed *set $T \subset B_R$ such that*

$$T \subset \{x \in B_R : \quad \limsup_{r \to 0} \fint_{B_r} |f| = 0\},$$

$$\mathcal{H}^s(T) > \frac{1}{\theta}\ R^s, \tag{1.2}$$

and for any ball B_r with some radius $r > 0$,

$$\mathcal{H}^s(T \cap B_r) \leq \theta r^s. \tag{1.3}$$

Then,

$$\|f\|_{L^p(B_R)} \leq C\ R^\sigma\ [f]_{\dot{W}^{\sigma,p}(B_R)}.$$

In [8] this was proven for the classical Sobolev space $W^{1,p}$, using an argument based on the p-Laplace equation with measures and the Wolff potential. Our argument, on the other hand, is completely elementary and adapts the classical blow-up proof of the Poincaré inequality; see Section 2.

Once Theorem 1.3 is established, one can follow the arguments in [8] to obtain Theorem 1.1. These rely heavily on the theory of Sousslin sets, [9], to find the closed set $T \subset \Sigma$ with the condition (1.2) and (1.3) satisfied. Those arguments are by no means elementary, but we were unable to remove them in order to show that $\mathcal{H}^s(\Sigma) = 0$. However, if one is satisfied in showing that $\mathcal{H}^t(\Sigma) = 0$ for any $t > s$, then there is a completely elementary argument, the details of which we will present in Section 3. There, we prove the following "pointwise" Poincaré-style inequality, from which the suboptimal Hausdorff dimension estimate easily follows; see Corollary 3.1.

Lemma 1.4. *For any $\varepsilon > 0$, $p \in [1,\infty)$, there exists $C > 0$, such that the following holds. Let $f \in L^p_{loc}$, and assume $x \in \mathbb{R}^n$, such that*

$$\lim_{r\to 0} \fint_{B_r(x)} |f| = 0. \tag{1.4}$$

Then for any $R > 0$, there exists $\rho \in (0,R)$ such that

$$\int_{B_\rho(x)} |f|^p \le C \left(\frac{R}{\rho}\right)^\varepsilon \int_{B_\rho(x)} ||f| - (|f|)_{B_\rho}|^p.$$

2. Poincaré inequality: Proof of Theorem 1.3

By a scaling argument, Theorem 1.3 follows from Lemma 2.1.

Lemma 2.1. *For any $\theta > 0$, $\sigma \in (0,1]$, $p \in (1, n/\sigma]$, $s \in (n - \sigma p, n]$, there is a constant $C > 0$ such that the following holds:*

Let $f \in W^{\sigma,p}(B_1,[0,\infty))$ and assume that there is a closed *set $T \subset B_1$ such that*

$$T \subset \{x \in B_1 : \quad \limsup_{r\to 0} \fint_{B_r} f = 0\}$$

and

$$\mathcal{H}^s(T) > \frac{1}{\theta},$$

as well as

$$\mathcal{H}^s(T \cap B_r) \le \theta r^s \quad \textit{for any ball } B_r \textit{ with radius } r > 0.$$

Then,

$$\|f\|_{L^p(B_1)} \le C\ [f]_{\dot W^{\sigma,p}(B_1)}.$$

Proof. We proceed by the usual blow-up proof of the Poincaré inequality: Assume the claim is false, and that for fixed θ, p, s, σ for any $k \in \mathbb{N}$ there are $f_k \in W^{\sigma,p}(B_1,[0,\infty))$ such that

$$T_k \subset \{x \in B_1 : \quad \limsup_{r\to 0} \fint_{B_r} f_k = 0\},$$

$$\mathcal{H}^s(T_k) > \frac{1}{\theta}, \quad \mathcal{H}^s(T_k \cap B_r) \le \theta r^s\ \forall B_r,$$

and

$$\|f_k\|_{L^p(B_1)} > k\ [f_k]_{\dot W^{\sigma,p}(B_1)}.$$

Replacing f_k by $\frac{f_k}{\|f_k\|_p}$ (note that this does not change the definition and size of T_k), we can assume w.l.o.g.

$$\|f_k\|_{L^p} \equiv 1$$

and

$$[f_k]_{\dot W^{\sigma,p}(B_1)} \xrightarrow{k\to\infty} 0.$$

In particular, f_k is uniformly bounded in $W^{\sigma,p}$, and by the Rellich-Kondrachov theorem, up to taking a subsequence, f_k converges strongly in L^p, and weakly in $W^{\sigma,p}$ to some $f \in W^{\sigma,p}$, with $[f]_{\dot W^{\sigma,p}(B_1)} \equiv 0$, $\|f\|_{L^p} = 1$. Thus,

$$f \equiv |B_1|^{-\frac{1}{p}},$$

and setting $g_k := |B_1|^{\frac{1}{p}} f_k$, we have found a sequence such that

$$g_k \to 1 \quad \text{in } W^{\sigma,p}(B_1),$$

$$\mathcal{H}^s(T_k) > \frac{1}{\theta},$$

and

$$\mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \quad \text{for any ball } B_r.$$

This is a contradiction to Lemma 2.2. □

We used the following lemma, which essentially quantifies the intuition, that a function approximating 1 in $W^{\sigma,p}$ cannot be zero on a large set.

Lemma 2.2. *Let $\sigma \in (0,1]$, $s \in (n-\sigma p, n]$, $f_k \in W^{\sigma,p}(B_1,[0,\infty))$, and assume that*

$$\|f_k - 1\|_{W^{\sigma,p}(B_1)} \xrightarrow{k\to\infty} 0.$$

Then, for any $T_k \subset B_1$ closed and

$$T_k \subset \{x \in B_1 : \quad \limsup_{r\to 0} \fint_{B_r} f_k = 0\},$$

as well as for some $\theta > 0$,

$$\mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \quad \textit{for any } B_r, \textit{ for all } k \tag{2.1}$$

we have

$$\lim_{k\to\infty} \mathcal{H}^s(T_k) = 0.$$

Proof. By the subsequence principle, it suffices to show

$$\liminf_{k\to\infty} \mathcal{H}^s(T_k) = 0.$$

By extension, we also can assume that $f_k - 1 \to 0$ in $W^{\sigma,p}(\mathbb{R}^n)$, and $f_k \equiv 1$ on $\mathbb{R}^n \backslash B_2$.

On the one hand, we have

$$[f_k]_{\dot{W}^{\sigma,p}(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.$$

On the other hand, up to picking a subsequence, we can assume the existence of $R_k \in (0,1)$, for $k \in \mathbb{N}$, and $\lim_{k\to\infty} R_k = 0$, such that

$$\inf_{r>R_k, x\in B_1} \fint_{B_r(x)} f_k \geq \frac{9}{10}.$$

Since for any point $x \in T_k$ we have that $\lim_{t\to 0} \fint_{B_r} f_k(x) = 0$, we expect the average (fractional) gradient around x to be fairly large. More precisely, we have the following

Claim. There is a uniform constant $c_{s,\sigma,p} > 0$, such that the following holds: For any $x \in T_k$, there exists $\rho = \rho_{k,x} \in (0, R_k)$ such that

$$c_{s,\sigma,p}\ \rho^s \leq \rho^{-\sigma p} \int_{B_\rho} |f_k - (f_k)_{B_\rho}|^p \leq C\ [f_k]^p_{\dot{W}^{\sigma,p}(B_\rho)}. \tag{2.2}$$

Of course, we only have to show the first inequality; the second inequality is the classical Poincaré inequality.

For the proof let us write f instead of f_k. Then, since for $x \in T$,

$$\lim_{l\to\infty} \fint_{B_{2^{-l-1}R_k(x)}} f = 0,$$

we have that

$$\begin{aligned}\frac{9}{10} &\leq \sum_{l=0}^{\infty} \left(\fint_{B_{2^{-l}R_k}(x)} f - \fint_{B_{2^{-l-1}R_k(x)}} f \right) \\ &\leq C \sum_{l=0}^{\infty} \left((2^{-l}R_k)^{-n} \int_{B_{2^{-l}R_k}} |f - (f)_{B_{2^{-l}R_k}}| \right).\end{aligned}$$

Consequently, for any $\varepsilon > 0$, there has to be some $c_\varepsilon > 0$ and some $l \in \mathbb{N}$ such that

$$\left((2^{-l}R_k)^{-n} \int_{B_{2^{-l}R_k}} |f - (f)_{B_{2^{-l}R_k}}| \right) \geq c_\varepsilon \left(2^{-l}R_k\right)^{\varepsilon},$$

because if the opposite inequality was true for all $l \in \mathbb{N}$ we would have

$$\frac{9}{10} \leq C\ c_\varepsilon R_k^\varepsilon \sum_{l\in\mathbb{N}} 2^{-\varepsilon l} \leq C\ c_\varepsilon \sum_{l\in\mathbb{N}} 2^{-\varepsilon l},$$

which is false for c_ε small enough.

Thus, for $\rho := 2^{-l}R_k \in (0, R_k)$,

$$\rho^{n-\sigma+\varepsilon} \leq C_\varepsilon \rho^{-\sigma} \int_{B_\rho} |f - (f)_{B_\rho}| \leq C_\varepsilon \left(\rho^{-\sigma p} \int_{B_\rho} |f - (f)_{B_\rho}|^p \right)^{\frac{1}{p}} \rho^{n-\frac{n}{p}},$$

that is,

$$\rho^{n-\sigma p+\varepsilon p} \leq C_\varepsilon\ \rho^{-\sigma p} \int_{B_\rho} |f - (f)_{B_\rho}|^p.$$

Setting $\varepsilon = \frac{s-(n-\sigma p)}{p} > 0$, we have shown for any $x \in T$ the existence of some $\rho \in (0, R_k)$ satisfying (2.2), and the claim is proven.

For any k we cover T_k by the family

$$\mathcal{F}_k := \{B_\rho(x), \quad x \in T,\ B_\rho(x) \text{ satisfies (2.2)}\}.$$

Since $T \subset B_2$ is closed and bounded, i.e. compact, we can find a finite subfamily still covering all of T_k, and then using Vitali's (finite) covering theorem, we find a subfamily $\tilde{\mathcal{F}}_k \subset \mathcal{F}_k$ of disjoint balls $B_\rho(x)$, so that the union of the $B_{5\rho}$ covers all of T_k. We use this $\tilde{\mathcal{F}}_k$ as a cover for an estimate of the Hausdorff measure:

$$\begin{aligned}\mathcal{H}^s(T_k) &\leq \sum_{B_\rho\in\tilde{\mathcal{F}}_k} \mathcal{H}^s(B_{5\rho} \cap T_k) \overset{(2.1)}{\leq} \theta\ 5^s \sum_{B_\rho\in\tilde{\mathcal{F}}_k} \rho^s \\ &\overset{(2.2)}{\leq} C_{\theta,s} \sum_{B_\rho\in\tilde{\mathcal{F}}_k} [f_k]^p_{\dot{W}^{\sigma,p}(B_\rho)} \leq C_{\theta,s}\ [f_k]^p_{\dot{W}^{\sigma,p}(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.\end{aligned}$$

□

3. An elementary proof for the suboptimal case

We start with the proof of the pointwise inequality, Lemma 1.4.

Proof. First, let us show the claim for $p = 1$:

Fix $R, \varepsilon > 0$, $f \in L^1_{loc}$ and assume $x = 0$. W.l.o.g., $f \geq 0$. Set

$$\tau = 2^{-n-1} \left(\sum_{l=-\infty}^{0} 2^{\varepsilon l} \right)^{-1} R^{-\varepsilon}, \tag{3.1}$$

and $C_\varepsilon := R^{-\varepsilon}\tau^{-1}$. Assume by contradiction that the claim was false, i.e. assume that for any $\rho \in (0, R)$,

$$\fint_{B_\rho} |f - (f)_{B_\rho}| < \tau\ \rho^\varepsilon \fint_{B_\rho} f. \tag{3.2}$$

Then for any $K \in \mathbb{N}$,

$$\begin{aligned}
\fint_{B_\rho} |f - (f)_{B_\rho}| &< \tau\ \rho^\varepsilon \sum_{k=-K}^{0} \fint_{B_{2^k\rho}} f - \fint_{B_{2^{k-1}\rho}} f \quad + \tau\rho^\varepsilon \fint_{B_{2^{-K-1}\rho}} f\\
&\leq 2^n \tau\ \rho^\varepsilon \sum_{k=-K}^{0} \fint_{B_{2^k\rho}} |f - (f)_{B_{2^k\rho}}| + \tau\rho^\varepsilon \fint_{B_{2^{-K-1}\rho}} f.
\end{aligned}$$

Setting now for $l \in \mathbb{Z}$,

$$a_l := \fint_{B_{2^l R}} |f - (f)_{B_{2^l R}}|,$$

$$b_l := \fint_{B_{2^l R}} f,$$

the above equation applied to $\rho = 2^l R$ reads as

$$a_l \leq 2^n R^\varepsilon\ \tau\ 2^{\varepsilon l} \sum_{k=-K}^{0} a_{k+l} + \tau\ (2^l R)^\varepsilon\ b_{-K+l-1} \quad \text{for any } K \in \mathbb{N},\ l \in -\mathbb{N}.$$

In particular for any $L \in \mathbb{N}$,

$$\begin{aligned}
\sum_{l=-L}^{0} a_l &\leq 2^n R^\varepsilon\ \tau \sum_{l=-L}^{0} 2^{\varepsilon l} \sum_{k=-K}^{0} a_{k+l} + \tau\ R^\varepsilon \sum_{l=-L}^{0} 2^{\varepsilon l}\ b_{-K+l-1}\\
&\leq 2^n R^\varepsilon\ \tau \sum_{l=-L}^{0} 2^{\varepsilon l} \sum_{k=-K+l}^{0} a_k + \tau\ R^\varepsilon\ (\sup_{j \leq -K} b_j) \sum_{l=-\infty}^{0} 2^{\varepsilon l}\\
&\leq 2^n R^\varepsilon\ \tau \sum_{k=-L-K}^{0} a_k \sum_{l=-L}^{k+K} 2^{\varepsilon l} + \tau\ R^\varepsilon\ (\sup_{j \leq -K} b_j) \sum_{l=-\infty}^{0} 2^{\varepsilon l}\\
&\overset{(3.1)}{\leq} \frac{1}{2} \sum_{k=-L-K}^{0} a_k + \frac{1}{2} \sup_{j \leq -K} b_j.
\end{aligned}$$

Under the additional assumption that

$$\sum_{l=-\infty}^{0} a_l < \infty, \tag{3.3}$$

letting $L, K \to \infty$, using that by (1.4) we have $\lim_{l\to\infty} b_l = 0$, the above estimate implies that $a_k = 0$ for all $k \leq 0$. This means that f is a constant on B_R, and in particular by (1.4), f is constantly zero in B_R. This contradicts the strict inequality (3.2).

To see (3.3), fix $K \in \mathbb{N}$ such that $\sup_{j\leq -K} b_j \leq 2$. Then for

$$c_L := \sum_{l=-L}^{0} a_l,$$

the above estimate becomes

$$c_L \leq \frac{1}{2} c_{L+K} + 1 \quad \text{for any } L \in \mathbb{N}.$$

In particular, for any $i \in \mathbb{N}$,

$$c_{L+iK} \leq 2^{-i} c_L + \sum_{j=0}^{i} 2^{-j}.$$

Since c_i is monotonically increasing,

$$\sup_{i\geq L+K} c_i \leq c_L + \sum_{j=0}^{\infty} 2^{-j} < \infty.$$

This proves Lemma 1.4 for $p = 1$.

If $p > 1$, we apply this to f^p, and obtain

$$\int\limits_{B_\rho(x)} f^p \leq C \left(\frac{R}{\rho}\right)^{\varepsilon} \int\limits_{B_\rho(x)} |f^p - (f^p)_{B_\rho}|. \tag{3.4}$$

We now need the following estimate, which holds for any $p \in [1, \infty)$, and $\delta \in (0, 1)$:

$$\big||a - b|^p - |a|^p - |b|^p\big| \leq \delta |a|^p + \frac{C_p}{\delta^p} |b|^p.$$

Since B_ρ is fixed, let us write (f) for $(f)_{B_\rho}$. First, for any $\delta \in (0, 1)$,

$$\big|f^p - (f^p)\big| \leq \big|f - (f)\big|^p + \big|(f)^p - (f^p)\big| + \frac{C}{\delta^p} |f - (f)|^p + \delta (f)^p.$$

Plugging this into (3.4), for $\delta = \tilde{\delta}(R/\rho)^{-\varepsilon}$ small enough, we arrive at

$$\int\limits_{B_\rho(x)} f^p \leq C \left(\frac{R}{\rho}\right)^{(1+p)\varepsilon} \int\limits_{B_\rho(x)} |f - (f)|^p + C\, \rho^n \left(\frac{R}{\rho}\right)^{(1+p)\varepsilon} \big|(f)^p - (f^p)\big|. \tag{3.5}$$

Next,

$$\big|(f)^p - (f^p)\big| \leq \big(|(f)^p - f^p|\big) \leq (|f - (f)|^p) + \delta f^p + \frac{C}{\delta^p} (|f - (f)|^p).$$

Plugging this now for $\delta = \tilde{\delta}(R/\rho)^{-(1+p)\varepsilon}$ into (3.5), by absorbing we arrive at

$$\int\limits_{B_\rho(x)} f^p \leq C \left(\frac{R}{\rho}\right)^{\varepsilon c_p} \int\limits_{B_\rho(x)} |f-(f)|^p.$$

Since this holds for $\varepsilon > 0$ is arbitrarily small, this proves Lemma 1.4. □

Corollary 3.1. *For $\sigma \in (0,1]$ and for any $f \in W^{\sigma,p}(\Omega)$ satisfying* (1.1)*, $\mathcal{H}^t(\Sigma) = 0$, whenever $t > s = \max\{0, n - \sigma\frac{p\alpha}{p+\alpha}\}$.*

Proof. Let $\varepsilon > 0$, $R > 0$, and $x \in \Sigma$. Pick $\rho < R$ from Lemma 1.4, so that

$$\int\limits_{B_\rho(x)} |f|^p \leq C\ R^\varepsilon \rho^{\sigma p - \varepsilon}\ [f]^p_{\dot{W}^{\sigma,p}(B_\rho)}.$$

By Hölder and Young inequality, as in [8, Corollary 2.1],

$$\begin{aligned}\rho^{n+(2\varepsilon-\sigma p)\frac{\alpha}{p+\alpha}} &\leq C\ \rho^{2\varepsilon-\sigma p} \int\limits_{B_\rho(x)} |f|^p + C\rho^\varepsilon \int\limits_{B_\rho(x)} |f|^{-\alpha}\\ &\leq C\ R^{2\varepsilon}[f]^p_{\dot{W}^{\sigma,p}(B_\rho)} + C\ R^\varepsilon \int\limits_{B_\rho(x)} |f|^{-\alpha}.\end{aligned}$$

Now let $\varepsilon > 0$ such that $t > n + (2\varepsilon - \sigma p)\frac{\alpha}{p+\alpha}$. Then what we have shown is that for any $R > 0$ and any $x \in \Sigma$ there exists $\rho \in (0, R)$ such that

$$\rho^t \leq C\ R^\varepsilon [f]^p_{\dot{W}^{\sigma,p}(B_\rho)} + C \int\limits_{B_\rho(x)} |f|^{-\alpha}. \tag{3.6}$$

Now let

$$\mathcal{V}_R := \{B_\rho(x) :\ x \in \Sigma,\ \rho < R, (3.6) \text{ holds}\}.$$

Any countable disjoint subclass $\mathcal{U}_R \subset \mathcal{V}_R$ satisfies

$$\sum_{B_\rho \subset \mathcal{U}_R} \rho^t \leq C\ R^\varepsilon [f]^p_{\dot{W}^{\sigma,p}(\Omega)} + CR^\varepsilon \int\limits_\Omega |f|^{-\alpha}.$$

By the Besicovitch covering theorem, as in, e.g., [5, Theorem 18.1], we find for any R a countable subclass $\mathcal{U}_R \subset \mathcal{V}_R$, such that any point of Σ is covered at least once, and at most a fixed number of times. Thus,

$$\mathcal{H}^t(\Sigma) = \lim_{R\to 0} \mathcal{H}^t_R(\Sigma) \leq C\ \lim_{R\to 0} \sum_{B_\rho \subset \mathcal{U}_R} \rho^t \leq C_f \lim_{R\to 0} R^\varepsilon = 0.$$

□

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